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N. J. NIELSEN

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ÉCOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES

91128 PALAISEAU CEDEX - FRANCE

Tél. : (1) 941.82.00 - Poste N°
Télex : ECOLEX 691596 F

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BOUNDED OPERATORS ON TENSOR PRODUCTS OF BANACH LATTICES

N.J. NIELSEN
(Université d ' Odense)

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0. INTRODUCTION AND NOTATION.

In this note we shall investigate for which Banach spaces E and Banach lattices X the m-tensor product (defined below) has the property that for every bounded operator T on X $I \otimes T$ (I denoting the identity on E) is a bounded operator on $E \otimes_m X$. We shall then apply it to the question when spaces of absolutely summing operators have the uniform approximation property.

All the results of this note will appear in [7] to which we refer for further information and detailed proofs.

We shall use the notation and terminology commonly used in Banach space theory as it appears in [4].

If E and F are Banach spaces $B(E,F)$ denotes the space of all bounded operators from E to F equipped with the operator norm and we write $B(E,E) = B(E)$. If $1 \leq p \leq \infty$ $N_p(E,F)$ denotes the space of all p-nuclear operators from E to F with the p-nuclear norm n_p, $I_p(E,F)$ the space of all p-integral operators from E to F with the p-integral norm i_p and $\Pi_p(E,F)$ the space of all p-summing operators from E to F with the p-summing norm π_p. Finally $\Gamma_\infty(E,F)$ is the space of operators from E to F, which factor through an L_∞-space equipped with the factorization norm γ_∞. If $\mathfrak{A}(E,F)$ is one of the operator ideals above $\mathfrak{A}^f(E,F)$ denotes the closure of $E^* \otimes F$ in $\mathfrak{A}(E,F)$.

If $1 \leq p \leq \infty$ then we write $E \hookrightarrow L_p$, respectively $E \hookrightarrow QL_p$, if there is a measure μ so that E is isomorphic to a subspace of $L_p(\mu)$, respectively a subspace of a quotient of $L_p(\mu)$.

Throughout the paper we let E and F denote Banach spaces and X a Banach lattice. p_X and q_X are defined by

$$p_X = \sup\{p \mid X \text{ is p-convex}\} \quad .$$

$$q_X = \inf\{q \mid X \text{ is q-concave}\} \quad .$$

1. THE TENSOR PRODUCT $E\otimes_m X$.

Let us recall that a linear operator $T : E \to X$ is called order bounded if there exists a $z \in X$, $z \geq 0$ so that

(1) $$|Tx| \leq \|x\| z \quad \text{for all} \quad x \in E ,$$

and we define the order bounded norm $\|T\|_m$ of T by

$$\|T\|_m = \inf\{ \|z\| \mid z \text{ satisfies (1)}\} .$$

$\|\cdot\|_m$ is a norm on the space $\mathcal{B}(E,X)$ of all order bounded operators from E to X turning it into a Banach space [6].

1.1 Definition

The m-tensor product $E\otimes_m X$ is defined to be the closure in $\|\cdot\|_m$ of $E\otimes X$ in $\mathcal{B}(E^*,X)$.

This tensor product was originally introduced by Schaefer [11]. Further investigation of the geometric properties of $E\otimes_m X$ e.g. concerning the uniform approximation property, can be found in [2]. The tensor product is a generalization of spaces of vector-valued functions. Indeed, in [2] it was proved that if X is an order continuous Köthe function space on a probability space $(\Omega,\mathcal{S},\mu)$, then $E\otimes_m X$ can be identified in a canonical manner with the space $X(E)$ consisting of all measurable functions $f : \Omega \to E$ with $\|f(\cdot)\|_E \in X$.

We now wish to comment a little on the computation of norms in $E\otimes_m X$. If $e_1, e_2, \ldots, e_n \in E$ then the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(t_1, t_2, \ldots, t_n) = \| \sum_{j=1}^{n} t_j e_j \| \qquad (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$$

is a continuous function, homogeneous of degree one. Therefore the Krivine calculus of 1-homogeneous expressions in Banach lattices (see [4]) gives that $f(x_1, x_2, \ldots, x_n)$ can be given a unique meaning as an element in X for all $x_1, x_2, \ldots, x_n \in X$, and we denote that element by $\| \sum_{j=1}^{n} x_j e_j \|_E$. It is readily verified that

$$(2) \qquad \| \sum_{j=1}^{n} x_j e_j \|_E = \sup\{ | \sum_{j=1}^{n} e^*(e_j) x_j \,|\, e^* \in E^*, \|e^*\| \leq 1\} \quad .$$

Hence if $T = \sum_{j=1}^{n} e_j \otimes x_j \in E\otimes X$ then

$$(3) \qquad \|T\|_m = \| \; \| \sum_{j=1}^{n} x_j e_j \|_E \|_X \quad .$$

2. THE IDEAL PROPERTY OF (E,X) .

2.1 Definition

The pair (E,X) is said to have the ideal property (I.P.), if for every operator $T \in E\otimes_m X$ and every operator $S \in B(X)$ $ST \in E\otimes_m X$. In other words if $I\otimes S \in B(E\otimes_m X)$ for every $S \in B(X)$, where I is the identity operator on E .

The following result was proved in [6]

2.2 Proposition

Let X be weakly sequentially complete. If $T \in B(E,F)$ with $T^* \in \Pi_1(F^*, E^*)$ then $ST \in \mathfrak{B}(E,X)$ for all $S \in B(F,X)$ and $\|ST\|_m \leq \|S\| \, \pi_1(T^*)$. If X is arbitrary the result holds for all finite dimensional $T \in B(E,F)$.

It was proved by Kwapien [3] that $(E,L_p(\mu))$, $1\leq p\leq\infty$ has the I.P. if and only if $E \hookrightarrow QL_p$. Grothendieck's inequality gives together with Proposition 2.2 that (ℓ_2,X) has the I.P. for all Banach lattices X . In fact the following was noted in [7].

2.3 Proposition

If $T \in \ell_2 \otimes_m X$ then

$$K_G^{-1}\pi_1(T^*) \leq \|T\|_m \leq \pi_1(T^*) \quad .$$

If X is weakly sequentially complete, then

$$T^* \in \Pi_1(X^*,\ell_2) \Rightarrow T \in \ell_2 \otimes_m X \quad .$$

In the sequal we shall need the following lemma:

2.4 Lemma

If $2<p<q$ and $E \hookrightarrow QL_p$ then $B(\ell_1,E) = \Pi_q(\ell_1,E)$.

Proof

It is readily verified that it suffices to prove the statement when E is a quotient of $L_p(0,1)$ so let us assume that.

Since E is of cotype p it follows from [5] that $\pi_1(E,G) = \pi_{q'}(E,G)$ for every Banach space G . By assumption E^* is a subspace of $L_{p'}(0,1)$ and hence isomorphic to a subspace of $L_1(0,1)$. Hence if $T \in \Pi_1(E,\ell_1)$ then $T^* \in \Pi_1(\ell_\infty,E^*)$ by a result of Kwapien [3] and therefore $T \in \mathfrak{B}(E,\ell_1)$. Clearly $\mathfrak{B}(E,\ell_1) = E^*\otimes_m \ell_1 = E^*\otimes_\pi \ell_1 = N_1(E,\ell_1)$. Combining this with the above we get $N_1(E,\ell_1) = \Pi_{q'}(E,\ell_1) = N_{q'}(E,\ell_1)$ and therefore by duality $B(\ell_1,E) = \Pi_q(\ell_1,E)$.

q.e.d.

We are now ready to show

2.5 Theorem

1^o. If $1<q\leq 2$, X is q-concave and $B(\ell_1,E^*) = \Pi_{q'}(\ell_1,E^*)$ then

(i) $$T \in E\otimes_m X \Leftrightarrow T \in I_q(E^*,X) \Leftrightarrow T^* \in \Pi_1(X^*,E) \quad .$$

Dually

2^o. If $2\leq p<\infty$, X is p-convex and $B(\ell_1,E) = \Pi_p(\ell_1,E)$ then

(ii) $$T \in E\otimes_m X \Leftrightarrow T^* \in \Pi_p^f(X^*,E) \Leftrightarrow T \in \Gamma_\infty(E^*,X) \quad .$$

If furthermore X is weakly sequentially complete, the superscripts "f" can be removed in (ii).

Proof

We shall only prove 1^o. (ii) in 2^o can be obtained from 1^o using duality theory and the second statement in 2^o follows from Theorem 1.3 in [2].

Note that the assumptions in 1^o imply that $E \hookrightarrow QL_q$ so that E is reflexive.

Let $T \in E\otimes_m X$. Then there exists a compact Hausdorff space S and operators $T_1 : E^* \to C(S)$, $T_2 : C(S) \to X$ so that $T = T_2T_1$ and $\|T_1\| \leq 1$, $T_2 \geq 0$, $\|T\|_m = \|T_2\|$. Since X is q-concave T_2 is q-integral by [4] and hence T is q-integral as well.

Assume next that $T \in I_q(E^*,X)$. Let μ be a measure so that there is a quotient map S of $L_1(\mu)$ onto E^* . Since S is q'-summing by assumption it follows from [9] that TS and hence also S^*T^* are 1-integral. S^* is an isometry and therefore $T^* \in \Pi_1(X^*,E)$.

If $T^* \in \Pi_1(X^*,E)$, then Proposition 2.2 gives that $T \in \mathfrak{B}(E^*,X)$ (X is weakly sequentially complete), but the reflexivity of E implies that $\mathfrak{B}(E^*,X) = E\otimes_m X$ [2] .

q.e.d.

The result corresponding to Theorem 2.5 in case $q=1$ is wellknown. Indeed if $X = L_1(\mu)$ for some measure μ then by a result of Grothendieck [1] we have for all Banach spaces E : $E\otimes_m L_1(\mu) = E\otimes_\pi L_1(\mu) = N_1(E^*,L_1(\mu))$.

The next theorem gives a necessary and sufficient condition for a pair (E,X) to have the I.P. in certain cases.

2.6 Theorem

(i) If $p_X<q_X<2$ and X is q_X-concave then (E,X) has the I.P. if and only if $B(\ell_1,E^*) = \Pi_{q'_X}(\ell_1,E^*)$.

Dually if $2<p_X<q_X\leq\infty$ and X is p_X-convex then (E,X) has the I.P. if and only if $B(\ell_1,E) = \Pi_{p_X}(\ell_1,E)$.

(ii) If (E,X) has the I.P. and X is p_X-convex with $p_X<q_X<2$ (resp. X is q_X-concave and $2<p_X<q_X\leq\infty$) then $B(\ell_1,E^*) = \Pi_{q'_X}(\ell_1,E^*)$ (resp. $B(\ell_1,E) = \Pi_{p_X}(\ell_1,E)$).

(iii) If $p_X\leq 2\leq q_X$ and either p_X or q_X is attained or X contains (ℓ_2^n) uniformly complemented on disjoint blocks then (E,X) has the I.P. if and only if E is isomorphic to a Hilbert space.

Proof

The "if" part of (i) follows from Proposition 2.3 and Theorem 2.5. The "only if" parts of (i)-(iii) are based on the following argument:

Assume that (E,X) has the I.P. and that X is q_X-concave or $q_X = \infty$.

It follows from [2], Proposition 1.6 that every $T \in E\otimes_m X$ has p_X-summing adjoint. Since q_X is attained we get from [5] and [10] that for every n there is a sublattice F_n of X spanned by n mutually disjoint positive vectors, 2-equivalent to the unit vector basis of $\ell^n_{q_X}$ and so that the F_n's are uniformly complemented in X. Together with the above this shows that there is a constant K_1 so that for every n and every $T \in E\otimes_m \ell^n_{q_X}$ we have

$$(1)\qquad \pi_{p_X}(T^*) \leq K_1 \|T\|_m \ .$$

An approximation argument yields that (1) holds for every $T \in E\otimes_m \ell_{q_X}$.

<u>Proof of (i)</u>

If $p_X<q_X<2$ then by [5] there is a constant K_2 so that $\pi_1(S) \leq K_2\pi_{p_X}(S)$ for all $S \in \pi_{p_X}(\ell_{q'_X},E)$. Combining this with (1) and Proposition 2.2 we conclude that $T \in E\otimes_m \ell_{q_X}$ if and only if $T^* \in \pi_1(\ell_{q'_X},E)$. In particular (E,ℓ_{q_X}) has the I.P. and therefore $E \hookrightarrow QL_{q_X}$ by Kwapien's result so that E is reflexive. By duality we get that

$$(2)\qquad T \in E^*\otimes_m \ell_{q'_X} \Leftrightarrow T \in \Gamma_\infty(E,\ell_{q'_X}) \ .$$

Let K_3 be a constant so that $\|T\|_m \leq K_3\gamma_\infty(T)$ for all $T \in \Gamma_\infty(E,\ell_{q'_X})$.

Now let $S \in B(\ell_1,E^*)$ and $V \in B(\ell_\infty,\ell_{q'_X})$. Then $VS^* \in \Gamma_\infty(E,\ell_{q'_X})$ and hence

$$(3)\qquad \|VS^*\|_m \leq K_3\gamma_\infty(VS^*) \leq K_3 \|V\| \ \|S\| \ .$$

Since (3) holds for all $V \in B(\ell_\infty,\ell_{q'_X})$ it follows from [6] that S is q'_X-summing with $\pi_{q'_X}(S) \leq K_3 \|S\|$. This shows the first part of (i). The second part follows from the above by duality.

We shall not prove (ii) and (iii) here, but let us just mention that to obtain (ii) (1) is used to show that for every $S \in B(\ell_1,E)$ and every $V \in B(\ell_\infty,\ell_{p_X})$ $VS^* \in E\otimes_m \ell_{p_X}$ so that S is p_X-summing. To get (iii) we observe that (1) implies that if $r = q_X$, if q_X is attained and $r=2$ if X contains uniformly complemented copies of ℓ_2^n on disjoint blocks then every element in $E\otimes_m L_r(0,1)$ has 2-summing adjoint. It is easily verified that this implies that E is both of type 2 and of cotype 2 so that E is isomorphic to a Hilbert space. We refer to [7] for details concerning (ii) and (iii).

q.e.d.

We have not been able to extend Theorem 2.6 to the case where neither p_X nor q_X is attained and to the case where $p_X = q_X$ and at most one of them is attained.

The condition $B(\ell_1,E^*) = \Pi_{q'_X}(\ell_1,E)$ is not completely satisfactory. We can pose

2.7 Problem

If $1<p<2$ and $B(\ell_1,E^*) = \Pi_{p'}(\ell_1,E)$. Does $E \hookrightarrow QL_r$ for some r, $p<r\leq 2$.

If the answer to this question is affirmative then it follows from Theorem 2.5 that the condition in (ii) above is also sufficient for (E,X) to have the I.P.

It can be shown that the condition $B(\ell_1,E^*) = \Pi_{p'}(\ell_1,E^*)$ for some p, $1<p<2$ implies that $E \hookrightarrow QL_p$ and that E is of type p-stable. We may therefore ask

2.8 Problem

Let $1<p<2$, and let $E \hookrightarrow QL_p$ be of type p-stable. Does there exist an $r>p$ so that $E \hookrightarrow QL_r$?

It is wellknown that the answer to 2.8 is affirmative if either $E \hookrightarrow L_p$ (follows from a result of Rosenthal) or if E is a quotient of L_p (in which case E is isomorphic to a Hilbert space).

If F is a subspace of X then we put $E\bar{\otimes}_m F = \overline{E\otimes F}^{E\otimes_m X}$.
We have the following theorem:

2.9 Theorem

Let $1\leq q<p<2$ or $1\leq q<\infty$ and $p=2$, X q-concave and $E \hookrightarrow L_p$, and let $F \subseteq X$ be a subspace.

1^o. For every $T \in B(E^*,F)$ we have

(i) $\quad T \in E\bar{\otimes}_m F \;\Leftrightarrow\; T^* \in \Pi_1^f(F^*,E) \;\Leftrightarrow$

$\quad T \in \Pi_q^f(E^*,F) \quad (\;\Leftrightarrow\; T \in \Pi_1^f(E,F) \;$ if $\; 1<q<p\leq 2 \;)$

2^o. The superscript "f" can be removed if either

(ii) F is complemented in X

(iii) E or F and X have the bounded approximation property.

Sketch of proof

(i) If $1\leq q<p\leq 2$ then it follows from [5] that $\Pi_q(E^*,F) = \Pi_1(E^*,F)$ and since $E \hookrightarrow L_1$ $T \in \Pi_1(E^*,F)$ implies that $T^* \in \Pi_1(F^*,E)$. Further if E is isomorphic to a Hilbert space then clearly $\Pi_1(E,F) = \Pi_2(E,F)$ and if $q\geq 2$ then it follows from [3] that $T \in \pi_q(E^*,F)$ implies $T^* \in \pi_1(F^*,E)$.

Combining this with Theorem 2.5 we obtain (i).

2^o: It follows directly from the arguments above that "f" can be removed in case $F = X$. To prove 2^o in general it is therefore enough to show that if $T \in E\otimes_m X$ and $T(E^*) \subseteq F$ then $T \in E\bar{\otimes}_m F$.

(ii): Assume that there is a projection P of X onto F. Since (E,X) has the I.P. there is a constant K, so that $\|PS\|_m \leq K\|P\|\,\|S\|_m$ for all $S \in E\otimes_m X$.

If now $T \in E\otimes_m X$ with $T(E^*) \subseteq F$ and $(T_n) \subseteq E\otimes X$ converges to T in the m-norm then $(PT_n) \subseteq E\otimes F$ and by the inequality above it is a Cauchy sequence in the m-norm. Clearly its limit has to be T.

That the "f" can be removed under the assumptions in (iii) follows from the fact that if G_1 and G_2 are arbitrary Banach spaces so that either G^* or G_2 has the bounded approximation property then an operator $T \in B(G_1,G_2)$ belongs to $\Pi_q^f(G_1,G_2)$ if and only if it is quasi-q-nuclear.

q.e.d.

We have omitted the well-known case $p=q=1$ in Theorem 2.9.

Let $F \subseteq X$ be a subspace and let $K \geq 1$ be a constant. We shall say that the pair (E,F) has the I.P. with constant K relative to X, if for every $T \in E\bar{\otimes}_m F$ and every $S \in B(F,X)$ $ST \in E\otimes_m X$ with $\|ST\|_m \leq K\|S\|\,\|T\|_m$.

Using arguments similar to the ones in the proof of Theorem 2.6 we get

<u>2.10 Theorem</u>

1^o. If $p_X<q_X<2$ then the following statements are equivalent

(i) $\exists r \quad q_X<r\leq 2 \quad E \hookrightarrow L_r$

(ii) $\exists K \geq 1$, so that (E,F) has the I.P. with constant K relative to X for all subspaces $F \subseteq X$.

2^o. If $2\leq q_X<\infty$ and $p_X<q_X$ unless $q_X=2$ then (ii) above holds if and only if E is isomorphic to a Hilbert space.

3. APPLICATIONS TO THE UNIFORM APPROXIMATION PROPERTY.

Let us recall the following definition

3.1 Definition

Let $\lambda \geq 1$ $\varphi : \mathbb{N} \to \mathbb{N}$. E is said to have the (λ,φ)-uniform approximation property ((λ,φ)-u.a.p.) if for every n-dimensional subspace $F \subseteq E$ there is an operator $T \in B(E)$ with $Tx=x$ for all $x \in F$, $\|T\| \leq \lambda$ and $rk(T) \leq \varphi(n)$.

We shall say that E has the λ-u.a.p. if it has the (λ,φ)-u.a.p. for some function φ . The u.a.p. was first introduced by Pelczynski and Rosenthal [8] and has since been studied by various authors. In [2] the u.a.p. of m-tensor products and Banach lattices was studied and we wish to apply the results there to the situation of this note.

It follows from [2], Theorem 3.7 that if E and X both have the u.a.p. and X is superreflexive then $E \otimes_m X$ has the u.a.p. Combining this with Theorem 2.5 and the fact that the u.a.p. is a self-dual property we obtain immediately

3.2 Theorem

Let $2 \leq p < \infty$ and assume that X is p-convex and superreflexive and that $B(\ell_1,E) = \Pi_p(\ell_1,E)$. If E and X have the u.a.p. then $\Pi_1(X,E)$ has the u.a.p.

If E has the u.a.p. and X is a general Banach lattice, then [2] gives that $E \otimes_m X$ has the u.a.p., provided X has the order u.a.p. (which means that the operators in the definition of the u.a.p. can be chosen with "controlled" modulus). However since for every p , $1 \leq p \leq \infty$ the class of L_p-spaces is closed under ultraproducts the proof of [2], Theorem 3.7 easily yields:

3.3 Theorem

Let $1\le p\le\infty$ and assume that (ℓ_p,X) has the I.P. and that $E \hookrightarrow L_p$.

If E and X have the u.a.p. so does $E\otimes_m X$.

As a corollary we obtain:

3.4 Corollary

1^o. Let $1<q<p\le 2$ or $q=1$ and $1\le p\le 2$ and let $E \hookrightarrow L_p$, X q-concave. If E and X have the u.a.p. and F is a complemented subspace of X, then $\Pi_1(E^*,F)$ and $\Pi_1(F^*,E^{**})$ have the u.a.p.

2^o. If F^* is isomorphic to a complemented subspace of a weakly sequentially complete Banach lattice X with the u.a.p., then $\Pi_1(F,\ell_2)$ has the u.a.p.

Proof

Assume first $1\le q<p\le 2$. By Theorem 2.9 it suffices to show that $E\bar{\otimes}_m F$ has the u.a.p. Since (E,X) has the I.P. $E\bar{\otimes}_m F$ is complemented in $E\otimes_m X$, which has the u.a.p. by Theorem 3.3.
If $p=q=1$, then $\pi_1(E^*,X) = \mathcal{B}(E^*,X)$ and the latter space has the u.a.p., [2]. $\Pi_1(E^*,F)$ is clearly complemented in $\pi_1(E^*,F)$.
This finishes the proof of 1^o.
2^o follows from Theorem 2.3.

q.e.d.

3.5 Corollary

Let $1\le s\le 2\le r\le\infty$ and $r\neq s'$ unless $s=1,2$. If E is an $\mathcal{L}_r$-space and F is an $\mathcal{L}_s$-space complemented in F^{**}, then $\pi_1(E,F)$ has the u.a.p.

We conjecture that for all p, $1\le p\le 2$ $\pi_1(\ell_{p'},\ell_p)$ has the u.a.p. but we were not able to verify it using the methods of this note.

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