Séminaire d'analyse fonctionnelle
École Polytechnique

J. Lindenstrauss

Uniqueness of some unconditional bases II

Séminaire d'analyse fonctionnelle (Polytechnique) (1980-1981), exp. nº 10, p. 1-10
<http://www.numdam.org/item?id=SAF_1980-1981____A10_0>

ÉCOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES

91128 PALAISEAU CEDEX - FRANCE

Tél. : (1) 941.82.00 - Poste N°
Télex : ECOLEX 691596 F

S E M I N A I R E

D ' A N A L Y S E   F O N C T I O N N E L L E

1980-1981

UNIQUENESS OF SOME UNCONDITIONAL BASES II

J. LINDENSTRAUSS
(Université de Jérusalem)

Exposé No X   15 mai 1981

In this exposé we present some results from a joint paper with J. Bourgain P.G. Casazza and L. Tzafriri (in preparation). It is a continuation of exposé no IV of this seminar [1] in which another part of this paper was presented.

It is well known that ℓ_2, ℓ_1 and c_0 are the only Banach spaces which have up to equivalence a unique normalized unconditional basis. If we consider spaces which have a unique normalized unconditional basis up to equivalence <u>and a permutation</u> we get a larger class of spaces whose extent is not clear at present. Edelstein and Wojtaszczyk proved in [2] that the spaces $\ell_1 \oplus c_0$, $\ell_1 \oplus \ell_2$, $c_0 \oplus \ell_2$ and $c_0 \oplus \ell_1 \oplus \ell_2$ belong to this class. We shall present below (cf. Proposition 5) a simple result concerning unconditional bases in direct sums of two Banach spaces which gives in particular a simple proof of the result of Edelstein and Wojtaszczyk and allows us to handle also some other direct sums which cannot be handled by the methods of [2].

The main purpose of this exposé is however to treat infinite direct sums. If we consider the most simple infinite direct sums of the three spaces c_0, ℓ_1 and ℓ_2 then there are up to duality three such spaces namely $(\Sigma \oplus \ell_2)_0$, $(\Sigma \oplus \ell_1)_0$, and $(\Sigma \oplus \ell_1)_2$. Surprisingly these three spaces exhibit different behaviour in connection with the problem of uniqueness of unconditional bases.

<u>Theorem 1</u> : The space $(\Sigma \oplus \ell_2)_0$ has up to equivalence and permutation a unique normalized unconditional basis. More precisely : if $\{e_i\}_{i=1}^\infty$ is the natural unit vector basis of $(\Sigma \oplus \ell_2)_0$ and if $\{\mathcal{U}_i\}_{i=1}^\infty$ is another normalized unconditional basis of this space with unconditionality constant λ then there is a permutation π of the integers so that

$$(1) \qquad f(\lambda)^{-1}\left\| \sum_{i=1}^\infty a_i\, e_i\right\| \le \left\| \sum_{i=1}^\infty a_i\, \mathcal{U}_i\right\| \le f(\lambda)\left\| \sum_{i=1}^\infty a_i\, e_i\right\|$$

for all choices of scalars $\{a_i\}_{i=1}^\infty$, where $f(\lambda) = c\lambda^n$ for some $c > 0$ and integer n.

Theorem 2 : The spaces $(\Sigma \oplus c_o)_1$ has up to equivalence and permutation a unique normalized unconditional basis. However, in this case any function $f(\lambda)$ for which (1) holds cannot be of polynomial growth. The function $f(\lambda)$ has to satisfy $f(\lambda) \geq e^{c\lambda^2}$ for some $c > 0$,

Theorem 3 : The space $(\Sigma \oplus \ell_1)_2$ fails to have a unique normalized unconditional basis up to equivalence and permutation.

Theorem 1 follows by a standard compactness argument from the following proposition

Proposition 4 : There are constants c, α, $\beta > 0$ having the following property. Let $\{x_i\}_{i=1}^n$ be a finite normalized sequence in $(\Sigma \oplus \ell_2)_o$ with unconditional constant λ. Let P be a projection from $(\Sigma \oplus \ell_2)_o$ onto $[x_i]_{i=1}^n$. Then there is a partition of $\{1,2,\dots,n\}$ into disjoint sets $\{\tau_s\}_{s=1}^t$ so that for all scalars $\{a_i\}_{i=1}^n$

$$(2)\quad K^{-1} \max_s \Big(\sum_{i\in\tau_s} |a_i|^2 \Big)^{1/2} \leq \Big\| \sum_{i=1}^n a_i x_i \Big\| \leq K \max_s \Big(\sum_{i\in\tau_s} |a_i|^2 \Big)^{1/2}$$

where $K = K(\|P\|,\lambda) = C\|P\|^{\alpha} \lambda^{\beta}$.

We present now the proof of proposition 4. It is similar in spirit to the proof of the main result in [4].

We can assume without loss of generality that each x_i has only a finite number of components i.e. $x_i = \sum_{j=1}^m x_{i,j}$; $1 \leq i \leq n$ where $x_{i,j} \in \ell_2$ for every i and j

Consider now the vectors

$$\hat{x}_i = \sum_{j=1}^m \sum_{k=1}^{2^n} \theta_i^k x_{i,j} \in (\Sigma \oplus \ell_2)_o \ , \ 1 \leq i \leq n$$

where $\{\theta_1^k,\dots,\theta_n^k\}$, $k = 1,2,\dots,2^n$ are all the possible n-tuples of signs $\pm$ 1 and for fixed i each $\theta_i^k x_{i,j}$ is considered as an element of a different copy of ℓ_2 . Obviously $\{\hat{x}_i\}_{i=1}^n$ is 1-unconditional and λ equivalent to $\{x_i\}_{i=1}^n$. Indeed

$$\|\sum_{i=1}^{n} a_i \hat{x}_i\| = \sup_{j,k} \|\sum_{i=1}^{n} a_i \theta_i^k x_{i,j}\|_2 = \sup_k \|\sum_{i=1}^{n} a_i \theta_i^k x_i\| \leq \lambda \|\sum_{i=1}^{n} a_i x_i\|.$$

Let the projection P be given by

$$Px = \sum_{i=1}^{n} x_i^*(x)\, x_i$$

where $x_i^* = \sum_{j=1}^{m} x_{i,j}^* \in (\Sigma \oplus \ell_2)_1$. Put

$$\hat{x}_i^* = \sum_{j=1}^{m} \sum_{k=1}^{2^n} \theta_i^k x_{i,j}^*/2^n \in (\Sigma \oplus \ell_2)_1,\ 1 \leq i \leq n.$$

Notice that $\hat{x}_i^*(\hat{x}_h) = \delta_{i,h}$ since

(3) $$\sum_{k=1}^{2^n} \theta_i^k \theta_h^k = 2^n \delta_{i,h}.$$

Therefore $Q u = \sum_{i=1}^{n} \hat{x}_i^*(u)\, \hat{x}_i$ is a projection from $(\Sigma \oplus \ell_2)_o$ onto $[\hat{x}_i]_{i=1}^n$. A direct verification shows that $\|Q\| \leq \lambda \|P\|$. Put, for $1 \leq i \leq n$,

$$\sigma_i = \{j \,;\, \|x_{i,j}\|_2 \geq 1/2\, \|Q\|\}$$

and

$$v_i = \sum_{j \in \sigma_i} \sum_{k=1}^{2^n} \theta_i^k x_{i,j}.$$

The sequence $\{v_i\}_{i=1}^n$ is 1-unconditional and

$$\|\sum_{i=1}^{n} a_i \hat{x}_i\| = \sup_{k,j} \|\sum_{i=1}^{n} a_i \theta_i^k x_{i,j}\|_2 \geq$$

$$\geq \sup_{k,j} \|\sum_{\substack{i=1 \\ j \in \sigma_i}}^{n} a_i \theta_i^k x_{i,j}\|_2 = \|\sum_{i=1}^{n} a_i v_i\|.$$

On the other hand by (3) $\hat{x}_i^*(v_h) = 0$ for $i \neq h$ and

$$\hat{x}_i^*(v_i) = 1 - \hat{x}_i^*(\hat{x}_i - v_i) \geq 1 - \|Q\|\,\|\hat{x}_i - v_i\| \geq \frac{1}{2}$$

and thus

$$\|\sum_{i=1}^{n} a_i \hat{x}_i\| \leq 2\|\sum_{i=1}^{n} a_i \hat{x}_i^*(v_i)\hat{x}_i\| = 2\|\sum_{i=1}^{n} a_i Q(v_i)\| \leq 2\|Q\|\,\|\sum_{i=1}^{n} a_i v_i\|.$$

Hence $\{\hat{\chi}_i\}_{i=1}^n$ is $2\|Q\|$ equivalent to $\{v_i\}_{i=1}^n$ and

$$R\,u = \sum_{i=1}^{n} \frac{\hat{\chi}_i(u)}{\hat{\chi}_i^*(v_i)}\, v_i$$

is a projection onto $[v_i]_{i=1}^n$ with $\|R\| \le 2\,\|Q\|$.

All these considerations show us that we could assume from the beginning that for every i we have $\|\chi_{i,j}\| = 1$ if $j \in \sigma_i$ and $\|\chi_{i,j}\| = 0$ if $j \notin \sigma_i$, that the $\{\chi_i\}_{i=1}^n$ are exchangeable, and that $\lambda = 1$. We do this and return to the original notation of the vectors $\{\chi_i\}_{i=1}^n$ and the projection P. We put $\mu = \|P\|$.

In order to obtain the partition required in Proposition 4 we introduce a notion of "friendship" between integers :

The integers i and h are friends if

$$\chi_i^*(\chi_i|_{\sigma_h}) \ge \varphi(\mu) \text{ and } \chi_h^*(\chi_h|_{\sigma_i}) \ge \varphi(\mu)$$

where $\varphi(\mu)$ is a function of μ to be determined later and $\chi_i|_{\sigma_h}$ denotes $\sum_{j\in\sigma_h} \chi_{i,j}$.

We partition now the integers $\{1,2,\dots n\}$ into disjoint subsets $\{\tau_s\}_{s=1}^t$ so that in each τ_s there is a representative $i(s)$ satisfying :

(a) Every $i \in \tau_s$ is a friend of $i(s)$

(b) For $s_1 \ne s_2$, $i(s_i)$ is not a friend of $i(s_2)$.

We claim that with this partition (2) holds.

Fix some $1 \le s \le t$. Since $\{\chi_i\}_{i\in\tau_s}$ are unconditional and their span complemented we get for some constant A

$$(4) \qquad A^{-1}\mu^{-1}\Big\|\sum_{i\in\tau_s} a_i\,\chi_i\Big\| \le \Big\|\Big(\sum_{i\in\tau_s} |a_i\,\chi_i|^2\Big)^{1/2}\Big\| \le A\Big\|\sum_{i\in\tau_s} a_i\,\chi_i\Big\|.$$

Hence, if we put $\delta_j = \{i; j \in \sigma_i\}$ we get that

$$A^2\|\sum_{i\in\tau_s} a_i\, \chi_i\|^2 \geq \sup_j \|(\sum_{i\in\tau_s\cap\delta_j} |a_i\, \chi_{i,j}|^2)^{1/2}\|_2^2 = \sup_j \sum_{i\in\tau_s\cap\delta_j} |a_i|^2.$$

Since

$$\mu \geq \|\chi^*_{i(s)}\| = \sum_{j=1}^m \|\chi^*_{i(s),j}\|_2$$

we deduce that

$$A^2\mu\|\sum_{i\in\tau_s} a_i\, \chi_i\|^2 \geq \sum_{j=1}^m \sum_{i\in\tau_s\cap\delta_j} |a_i|^2 \|\chi^*_{i(s),j}\|_2$$

$$= \sum_{i\in\tau_s} |a_i|^2 \sum_{j\in\sigma_i} \|\chi^*_{i(s),j}\|_2 = \sum_{i\in\tau_s} |a_i|^2 \|\chi^*_{i(s)}|\sigma_i\| .$$

Since every $i\in\tau_s$ is a friend of $i(s)$ it follows that

(5) $$\|\sum_{i\in\tau_s} a_i\, \chi_i\|^2 \geq A^{-2}\, \mu^{-1}\, \varphi(\mu) \sum_{i\in\tau_s} |a_i|^2$$

and hence

$$\|\sum_{i=1}^n a_i\, \chi_i\| \geq \max_s \|\sum_{i\in\tau_s} a_i\, \chi_i\| \geq A\, \mu^{-1/2}\, \varphi(\mu)^{1/2} \max_s (\sum_{i\in\tau_s} |a_i|^2)^{1/2}$$

which is the left half of (2)

In order to prove the second inequality of (2) we put for $i\in\tau_s$, $1 \leq s \leq \ell$, $y_i = \chi_i|\sigma_{i(s)}$. By the definition of the notion of friends we have $\chi^*_i(y_i) \geq \varphi(\mu)$ and by the assumption that the $\{\chi_i\}$ are exchangeable in signs we get that $\chi^*_i(y_h) = 0$ for $i \neq h$. Hence

$$\|\sum_{i=1}^n a_i\, \chi_i\|\varphi(\mu) \leq \|\sum_{i=1}^n a_i\, \chi^*_i(y_i)\chi_i\| = \|\sum_{i=1}^n P\, a_i y_i\| \leq \mu\|\sum_{i=1}^n a_i\, y_i\| .$$

Consequently,

$$\|\sum_{i=1}^n a_i\, \chi_i\| \leq \mu\, \varphi(\mu)^{-1} \sup_j \|\sum_{s=1}^{\ell} \sum_{\substack{i\in\tau_s \\ j\in\sigma_{i(s)}}} a_i\, \chi_{i,j}\|_2$$

$$\leq \mu\, \varphi(\mu)^{-1}\, M \sup_j \sup_s \|\sum_{\substack{i\in\tau_s \\ j\in\sigma_{i(s)}}} a_i\, \chi_{i,j}\|_2 \leq \mu\, \varphi(\mu)^{-1} M \sup_s \|\sum_{i\in\tau_s} a_i\, \chi_i\|,$$

Where

$$M = \max_j \text{ cardinality } \{1 \leq s \leq \ell\ ;\ j \in \sigma_{i(s)}\}$$

We shall show that if $\varphi(\mu) = \mu^{-2/9}$ then $M < 1/2\ \varphi(\mu)$ and this (in view also of (4)) will establish the second part of (2).

Assume that $M \geq 1/2\ \varphi(\mu)$. Then there is a j_o so that e.g. $\|\chi_{i(k),j_o}\|_2 = 1$ for $1 \leq k \leq 1/2\ \varphi(\mu)$. Put for each such k

$$\eta_k = \sigma_{i(k)} - \cup \{\sigma_{i(\ell)}\ ;\ 1 \leq \ell \leq 1/2\ \varphi(\mu);\ \chi^*_{i(k)}(\chi_{i(k)}|\sigma_{i(\ell)}) \leq \varphi(\mu)\}$$

and $z_k = \sum_{j\in\eta_k} \chi_{i(k),j}$. By condition (b) of the choice of the τ_s it follows that the sets η_k, $1 \leq k \leq 1/2\ \varphi(\mu)$ are mutually disjoint and hence $\|\sum_k z_k\| = 1$. On the other hand

$$\chi^*_{i(k)}(z_k) \geq 1 - \varphi(\mu)\ (1/2\ \varphi(\mu)) \geq 1/2$$

and by exchangeability $\chi^*_{i(k)}(z_\ell) = 0$ for $k \neq \ell$. Hence

$$\|\sum_k \chi_{i(k)}\| \leq 2\|\sum_k \chi^*_{i(k)}(z_k)\ \chi_{i(k)}\| \leq 2\|P \sum_k z_k\| \leq 2\ \mu\ .$$

On the other hand

$$\|\sum_{k=1}^{1/2\varphi(\mu)} \chi_{i(k)}\| = \sup_{j,\theta_k=\pm 1} \|\sum_k \theta_k\ \chi_{i(k),j}\|_2$$

$$\geq \sup_{\theta_k=\pm 1} \|\sum \theta_k \chi_{i(k),j_o}\|_2 = (2\ \varphi(\mu))^{-1/2}$$

i.e. $2\ \mu \geq (2\ \varphi(\mu))^{-1/2}$ and this contradicts our choice of $\varphi(\mu)$.

From Proposition 4 we get actually the following stronger version of Theorem 1.

<u>Theorem 1'</u> : Every normalized unconditional basic sequence in $(\Sigma\oplus\ell_2)_1$ whose span is complemented is equivalent to a permutation of the unit vector basis of one of the following 6 spaces

$$c_o,\ \ell_2,\ c_o \oplus \ell_2,\ (\sum_{n=1}^{\infty} \oplus \ell_2^n)_o,\ (\sum_{n=1}^{\infty} \oplus \ell_2^n)_o \oplus \ell_2,\ (\Sigma \oplus \ell_2)_o$$

A similar statement is clearly true for the dual space $(\Sigma \oplus \ell_2)_1$.

Corollary : The six spaces appearing in the statement of theorem 1' and their duals have up to equivalence and permutation a unique normalized unconditional basis.

The first statement in Theorem 2 is proved by showing an analogue of Proposition 4. Inequality (2) takes now the form

$$(6) \qquad K^{-1} \max_s \sum_{i\in\tau_s} |a_i| \le \left\| \sum_{i=1}^n a_i x_i \right\| \le K \max_s \sum_{i\in\tau_s} |a_i|$$

where $K = K(\lambda, \|P\|)$ is a more complicated function of λ and $\|P\|$ than the one appearing in Proposition 4.

The proof of the right half of (6) is identical to the proof we presented of the right half of (2). The proof of the left half of (2) shows also in the case of $(\Sigma \oplus \ell_1)_o$ that

$$\sup_s \Big(\sum_{i\in\tau_s} |a_i|^2 \Big)^{1/2} \le K_o \left\| \sum_{i=1}^n a_i x_i \right\| .$$

The fact that actually the stronger statement, appearing in the left half of (6), holds is the contents of the exposé [1]. Of course also in this case we get stronger statements namely the exact analogues of Theorem 1' and its corollary.

The second statement of Theorem 2 follows immediately from the fact, proved in [3], that for every n there is a subspace X_n of $(\Sigma \oplus \ell_1)_o$ so that $d(X_n, \ell_2^n) \le 2$ and so that there is a projection on X_n with norm $\le \sqrt{\lg n}$.

We pass to the proof of Theorem 3. It depends only on the following trivial remark. Let $\{\mathcal{F}_i\}_{i=1}^n$ be n independent finite algebras of subsets of $[0,1]$ (i.e. $\mu(A \cap B) = \mu(A)\,\mu(B)$ for every $A \in \mathcal{F}_i$, $B \in \mathcal{F}_j$, $i \ne j$). Let $\{E_i\}_{i=1}^n$ be the conditional expectation operators corresponding to $\{\mathcal{F}_i\}_{i=1}^n$. Then for every choice of functions f_i we have

$$(7) \qquad \left\| \sum_{i=1}^n |E_i f_i| \right\|_2 \le 2^{1/2} \left\| \Sigma |f_i| \right\|_2 .$$

Indeed,

$$\left\| \sum_i |E_i f_i| \right\|_2^2 = \sum_i \int |E_i f_i|^2 + \sum_{i\ne j} \int |E_i f_i|\,|E_j f_j|$$

$$= \sum_i \int |E_i f_i|^2 + \sum_{i \neq j} \int |E_i f_i| \int |E_j f_j| \leq \sum_i \|f_i\|_2^2 + \sum_{i \neq j} \|f_i\|_1 \|f_j\|_1$$

$$\leq \| \sum_i |f_i| \|_2^2 + \|\Sigma |f_i| \|_1^2 \leq 2 \|\Sigma |f_i| \|_2^2$$

For each integer n let now $\{\mathfrak{F}_i\}_{i=1}^n$ be independent algebras of subsets of $[0,1]$ each having n atoms $\{A_{i,j}\}_{j=1}^n$ with $\mu(A_{i,j}) = 1/n$ for all i and j. Let $\{e_i\}_{i=1}^n$ denote the unit vectors in ℓ_1^n and put

$$z_{i,j} = \sqrt{n}\ \chi_{A_{i,j}} \otimes e_i \in L_2([0,1], \ell_1^n).$$

Clearly $\{z_{i,j}\}_{i,j=1}^n$ is a normalized 1-unconditional basic sequence in $L_2([0,1],\ell_1^n)$ and by (7) there is a projection P with $\|P\| \leq \sqrt{2}$ onto $X_n = [z_{i,j}]_{i,j=1}^n$. Indeed put

$$P\left(\sum_{i=1}^n f_i(t) \otimes e_i\right) = \sum_{i=1}^n (E_i f_i) \otimes e_i.$$

Clearly we may consider X_n also as subspace of a space isometric to a finite direct sum of the form $(\Sigma \oplus \ell_1^n)_2$ in $L_2([0,1],\ell_1^n)$. The sequence $y_i = n^{-1/2} \sum_{j=1}^n z_{i,j}$, $1 \leq i \leq n$ is 1 isometric to the unite vector basis in ℓ_1^n and there is a projection of norm 1 from X_n onto $[y_i]_{i=1}^n$. Hence, by the decomposition method $(\sum_{n=1}^\infty \oplus X_n)_2$ is isomorphic to $(\sum_n \oplus \ell_1^n)_2$. The natural unit vector basis in $(\sum_n \oplus X_n)_2$ is **however not** equivalent to the unit vector basis in $(\sum_n \oplus \ell_1^n)_2$. This follows immediately from the following observation. For n large the sets $\{A_{i,j}\}_{i,j=1}^n$ are mutually almost disjoint in a sense that given k and $\varepsilon > 0$ then for $n \geq n(k,\varepsilon)$ every k of the vectors $\{z_{i,j}\}_{i,j=1}^n$ are $1+\varepsilon$ equivalent to the unit vector basis of ℓ_2^k.

We turn now to the proposition on unconditional bases in direct sums of two spaces mentioned in the beginning.

Proposition 5 : Let X and Y be Banach spaces and let $1 \leq p, r \leq \infty$. Assume that $\{z_i\}_{i=1}^n$ is a λ unconditional basic sequence in $X \oplus Y$ on whose span there is a projection P. Then there exists a subset

$\sigma \subset \{1,2,\ldots,n\}$ so that $\{z_i\}_{i\in\sigma}$ is $M = M(\|P\|,\lambda)$ equivalent to an M complemented 1 - unconditional sequence in $(X \oplus X \oplus \ldots \oplus X)_p$ and $\{z_i\}_{i\notin\sigma}$ is M-equivalent to a similar sequence in $(Y \oplus Y \oplus \ldots \oplus Y)_r$.

The proof is similar to the first step of the proof of Proposition 4. Put $z_i = x_i + y_i$ and

$$Pz = \sum_{i=1}^{n} z_i^*(z)\, z_i \;, \quad z_i^* = x_i^* + y_i^* \subset X^* \oplus Y^* .$$

Let $\hat{X}$ be the ℓ_p sum of 2^n copies of X and $\hat{Y}$ the ℓ_r sum of 2^n copies of Y. Put

$$\hat{x}_i = (\theta_i^1\, x_i/2^{n/p},\ldots,\theta_i^{2^n}\, x_i/2^{n/p}) \in \hat{X} ,$$

$$\hat{x}_i^* = (\theta_i^1\, x_i^*/2^{n/p'},\ldots,\theta_i^{2^n}\, x_i/2^{n/p'}) \in \hat{X}^*$$

where $\{\theta_i^j\}_{j=1}^{2^n}$ is the collection of all n-tuples of signs, and p' is the adjoint exponent of p. The vectors $\hat{y}_i$ and $\hat{y}_i^*$ are defined similarly with p replaced by r.

Then $\{\hat{x}_i\}_{i=1}^n$, $\{\hat{y}_i\}_{i=1}^n$ and $\{\hat{z}_i = \hat{x}_i + \hat{y}_i\}_{i=1}^n$ are all 1-unconditional, the latter one being λ-equivalent to $\{z_i\}_{i=1}^n$. Put $\sigma = \{i;\ z_i^*\, Px_i \geq 1/2\}$. A simple computation similar to that done in the beginning of the proof of Proposition 4 shows that $\{\hat{z}_i\}_{i\in\sigma}$ is $2\,\lambda\,\|P\|$ equivalent to $\{\hat{x}_i\}_{i\in\sigma}$ and that

$$Q\,\hat{x} = \sum_{i\in\sigma} \frac{\hat{x}_i^*(\hat{x})}{\hat{x}_i^*(\hat{x}_i)}\ \hat{x}_i$$

is a projection from $\hat{X}$ onto $[\hat{x}_i]_{i\in\sigma}$ of norm $\leq 2\,\lambda\,\|P\|$.

It follows e.g. from Theorems 1 and 2 and Proposition 5 that $(\Sigma \oplus c_o)_1 \oplus (\Sigma \oplus \ell_2)_1$ has up to equivalence and permutation a unique normalized unconditional basis. (The methods of [2] do not apply here since $(\Sigma \oplus c_o)_1$ and $(\Sigma \oplus \ell_2)_1$ are not totally incomparable).

The methods of this exposé and [1] seem to enable a complete classification of those spaces obtainable from R by taking iterated direct sums in the ℓ_1 ℓ_2 and c_o sense, which have up to equivalence and permutation a unique normalized unconditional basis. It is however unclear at present whether there exist completely different spaces (from those obtainable as c_o, ℓ_1 or ℓ_2 direct sums) which have a unique normalized unconditional basis up to equivalence and permutation.

References :

[1] J. Bourgain : Unicité de certaines bases inconditionnelles, séminaire d'analyse fonctionnelle 80/81 Exposé IV.

[2] I.S. Edelstein and P. Wojtaszczyk : On projections and unconditional bases in direct sums of Banach spaces Studia Math. 56 (1976) 263 - 276.

[3] T. Figiel, J. Lindenstrauss and V. Milman : The dimension of almost spherical sections of convex bodies, Acta Math 139 (1977) 53-94.

[4] J. Lindenstrauss and L. Tzafriri : On the isomorphic classification of injective Banach lattices, Advances in Math. (supplementary studies) 1981, to appear.

* * *
*