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VOLUME ESTIMATES AND NEARLY EUCLIDEAN DECOMPOSITIONS FOR NORMED SPACES

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The purpose of this talk is to present a new isomorphic invariant of a finite dimensional normed space, so called "volume ratio" (introduced in [8]). We set

$$vr(E) = \left(\frac{\text{vol } B_E}{\text{vol } \mathcal{E}} \right)^{1/n} ,$$

where B_E is the unit ball of an n-dimensional real normed space E, $\mathcal{E}$ -the ellipsoid of maximal volume contained in B_E (so called "John's ellipsoid of E) and vol A stands for volume of a set A.
It follows directly from the definition that

$$vr(E) \leq vr(E)\ d(E,F) \quad , \tag{1}$$

where E and F are normed spaces of the same dimension, d -the Banach-Mazur distance.

To explain the motivation for introducing such an invariant let us mention the following :

<u>Theorem 1</u> (Kashin [6]) : There is a universal constant C such that, given n, there exist two n-dimensional subspaces E_1, E_2 of L^1_{2n}, orthogonal (in ℓ^2_{2n}) satisfying

$$d(E_i, \ell^2_n) \leq C \quad \text{for } i = 1,2 \quad . \qquad \blacksquare$$

Theorem 1 solved some problems from the approximation theory and was used later (see [3]) to construct an n-dimensional space, whose constant of local unconditional structure is of order $\sqrt{n}$ (the largest possible). However, Kashin's original proof was very complicated.
A simple proof of Th. 1 appeared in [7]. It depends essentially on the following two observations.

<u>Proposition 2</u> : $vr(\ell^1_n) \leq \sqrt{2\, e/\pi}$ for $n = 1,2,\ldots$ $\blacksquare$

<u>Proposition 3</u> : Let C and $\theta < 1$ be positive constants. Then, for any normed space E with $vr(E) < C$ and positive integer $k \leq \theta \dim E$, "most of" k-dimensional subspaces F of E satisfy

$$d(F, \ell^2_k) \leq C' \quad , \tag{2}$$

where C' depends only on C and θ. More precisely : if $G = G(k,n)$ is the Grassmann manifold of k-dimensional subspaces of E, μ -a normalized invariant measure on G, generated by the John's ellipsoid. Then

$$\mu(\{F \in G : F \text{ satisfies } (2)\}) > \frac{1}{2} \quad . \quad \blacksquare \tag{3}$$

Deducing Th. 1 from Prop. 2 and Prop. 3 is immediate, one must only remember that the map $F \mapsto F^{\perp}$ (the orthogonal complement of F), acting on $G(n,2n)$, is measure-preserving.

Proof of Prop. 2 : By direct computation. ■

Proof of Prop. 3 : Let $E = (R^n, \|\cdot\|)$. We may assume that the John's ellipsoid of E is equal to the Euclidean unit ball $B^n = \{\|x\|_2 \le 1\}$. Denote by m the normalized Haar measure on S^{n-1}. Then

$$C^n > vr(E)^n = \int_{S^{n-1}} \|x\|^{-n} \, m(dx) \tag{4}$$

(one gets the equality by representing vol A as $\int_{R^n} \chi_A$ and passing to polar coordinates).

Given $r \in (0,1)$ define $A_r = \{x \in S^{n-1} : \|x\| < r\}$. Then one gets from (4) that

$$m(A_r) < (Cr)^n \quad .$$

On the other hand, we have

$$m(A_r) = \int_{S^{n-1}} \chi_{A_r} \, dm = \int_G \mu(dF) \int_{S_F} \chi_{A_r \cap F} \, dm_F =$$

$$= \int_G m_F(A_r \cap F) \, \mu(dF) \quad ,$$

where m_F is the normalized Haar measure on $S_F = F \cap S^{n-1}$. The last two formulae show that

$$\mu(\{F \in G : m_F(A_r \cap F) < 2(Cr)^n\}) > \frac{1}{2} \quad ;$$

in other words, for "most of" $F \in G$ we have

$$m_F(\{x \in S_F : \|x\| < r\}) < 2(Cr)^n \le (2\,Cr)^n \quad .$$

We show that every such F is "close" to ℓ_k^2 in the Banach-Mazur sense, thus proving Prop. 3.

Indeed, since, for given $x_o \in S_F$ and $\delta \le \frac{1}{2}$,

$$m_F(\{x \in S_F : \|x - x_o\|_2 \le \delta\}) \ge (\frac{\delta}{4})^k \quad ,$$

the previous estimate shows (remember that $k \le \theta n$) that $S_F \setminus A_r$ is an r/2-net (in ℓ_n^2 metric) for S_F, provided $r = r(\theta, C)$ is small enough (precisely, if $r \le (2^{3\theta+1} C)^{1/(1-\theta)}$. Fix such r. Then, for any $y \in S_F$, there is a $y_o \in S_F \setminus A_r$ (i.e. $\|y_o\| \ge r$) such that $\|y - y_o\|_2 \le r/2$. Since (by $B^n \subset B_E$) $\|x\|_2 \ge \|x\|$ for all $x \in E$, we have also $\|y - y_o\| \le \frac{r}{2}$. Therefore

$$\|y\| \ge \|y_o\| - \|y - y_o\| \ge r - \frac{r}{2} = \frac{r}{2} \quad .$$

So, by homogeneity,

$$\frac{r}{2} \|y\|_2 \le \|y\| \le \|y\|_2$$

for all $y \in F$. Hence $d(F, \ell_k^2) \le 2r^{-1} = 2\, r(\theta, C)^{-1}$.
This ends the proof of Prop. 3. ∎

In the sequel, we shall frequently use the following concepts. We say that (e_i) is an unconditional basis of a B-space E provided

$$\mathrm{ubc}(e_i) \overset{\mathrm{def}}{=\!=} \sup_{|\varepsilon_i| \le 1, \|\sum_i t_i e_i\| \le 1} \|\sum_i \varepsilon_i\, t_i\, e_i\| < \infty \quad .$$

We say that a B-space E is of cotype q $(q \ge 2)$ if there is a constant K such that, for every finite sequence $x_1, x_2, \ldots \in E$, we have

$$\int \|\sum_i r_i\, x_i\| \ge K^{-1} (\sum_i \|x_i\|^q)^{1/q} \quad ,$$

where (r_i) is the sequence of Rademacher functions. The smallest such constant K is called the cotype q constant of E and denoted by $K_q(E)$.

It was proved in [4] that given K there exist C, $\theta > 0$ such that, for every finite dimensional E with $K_2(E) \le K$, one can find a subspace of E, say F, with $\dim F = k \ge \theta \dim E$ and $d(F, \ell_k^2) \le C$. Prop. 2 and Prop. 3 strengthen this result in the special case $E = \ell_n^1$. This raises the following problems :

Problem 4 : Given $\theta \in (0,1)$, does every normed space E contain a $[\theta \dim E]$-dimensional subspace F with $d(\ell^2_{\dim F}, F) < C$, where C depends only on $K_2(E)$? ■

Problem 5 : Does there exist a function C(.) such that $vr(E) \le C(K_2(E))$ for every E ? ■

Of course a positive solution of Problem 5 implies a positive solution of Problem 4. We have two partial results in this direction.

Theorem 6 [8] : Let E be a finite dimensional space, (e_i) -its basis. Then

$$vr(E) \le C\, K_2(E)\, ubc(e_i)$$

where C is a universal constant. ■

Theorem 7 [8] : There is a universal constant C such that

$$vr(\ell^2_n \hat{\otimes} \ell^2_n) \le C \quad \text{for all } n. \quad ■$$

Recall that $\ell^2_n \hat{\otimes} \ell^2_n$ is the tensor product $\ell^2_n \otimes \ell^2_n$ equipped with the largest tensor norm (in other words : the space of nuclear operators on ℓ^2_n). It is known that $ubc(\omega_i)$ is of order $\sqrt{n}$ for every basis (ω_i) of $\ell^2_n \hat{\otimes} \ell^2_n$, while $K_2(\ell^2_n \hat{\otimes} \ell^2_n) \le K$, where K does not depend on n.

Theorem 7 can be generalized to a large class of tensor products and unitary ideals. In particular, a unitary ideal $\mathfrak{A}$ on ℓ^2_n has "small" volume ratio if the associated n-dimension symmetric space $\ell_{\mathfrak{A}}$ has (in the case of Th. 7 we have $\ell_{\mathfrak{A}} = \ell^1_n$; see e.g. [5] for definitions).

Now I present a sketch of the proof of Th. 6. We shall need two lemmas.

Lemma A : Let $(E, \|\cdot\|)$ be a B-space of cotype 2 with an unconditional basis (e_i). Then there exists a norm $\|\cdot\|^{(1)}$ such that

a) $\|x\| \le \|x\|^{(1)} \le C\, K_2(E)\, ubc(e_i)\, \|x\|$ for $x \in E$
(C is an absolute constant).

b) $ubc(e_i) = 1$ in $(E, \|\cdot\|^{(1)})$

c) the dual norm $\|\cdot\|$ on E^* is 2-convex ; in other words a functional defined by $|||(x_j)||| = (\|\sum_j \sqrt{|x_j|}\, e_j^*\|_*^{(1)})^2$ ((e_j^*) is a basic sequence E^* dual to (e_j)) is a norm (then, of course, unconditional). ■

Lemma A is well known (see e.g. [1]).

Lemma B : Let $(F,\|\cdot\|)$ be an n-dimensional normed space, (f_i) -its basis with $\mathrm{ubc}(f_i) = 1$. Then there exists a sequence of positive numbers $\beta_1,\beta_2,\dots,\beta_n$ such that, for all $\lambda_1,\lambda_2,\dots,\lambda_n \in R$,

(5) $$\frac{1}{n}\sum_i |\lambda_i| \le \|\sum_i \beta_i \lambda_i f_i\| \le \max_i |\lambda_i| \quad .$$ ■

Proof of lemma B : Some variants of lemma B are known in a more general setting of B-lattices. I present a proof, which is essentially due to T.K. Carne.

Consider $f : B_E \to R$ defined by $f(\sum_i b_i f_i) = \prod_i b_i$. Let $\beta = \sum_i \beta_i f_i$ be a point, where f attains its maximum. Of course one can choose β to satisfy $\beta_i \ge 0$ for $i = 1,2,\dots,n$. Clearly $\|\beta\| = 1$; this implies immediately the right hand inequality of (5), because $\mathrm{ubc}(f_i) = 1$. By the same reason, to prove the left hand inequality of (5) it is enough to show that the functional $\varphi : \sum_i \lambda_i \beta_i f_i \mapsto \frac{1}{n}\sum_i \lambda_i$ is of norm at most 1.
It is easy to see that φ is the only functional satisfying

(i) $\varphi(\beta) = 1$,

and (ii) $\varphi(x) \ge 1$ if $x \in Q \equiv \left\{x = \sum_i x_i f_i : x_i \ge 0 \text{ for } i = 1,2,\dots,n \text{ and } \prod_i x_i > \prod_i \beta_i\right\}$.

But it is clear that the functional ψ separating disjoint (by definition of β) and convex sets B_E and Q (i.e. $\psi(B_E) \le 1$, $\psi(Q) > 1$) satisfies (i) and (ii) ; hence $\varphi = \psi$ and $\varphi(B_E) \le 1$, in other words $\|\varphi\| \le 1$. This proves lemma B.

Now we shall derive th. 6 from lemmas A and B.

Clearly, by lemma A and (1), it is enough to prove that if $(E,\|\cdot\|^{(1)})$ satisfies conditions (b) and (c) of lemma A, then $\mathrm{vr}(E) \le C$, where C is a universal constant. On the other hand, this estimate will immediately follow from existence of a sequence (α_k) such that

(*) $$\sum_k |x_k| \le \|\sum_k \alpha_k x_k e_k\|^{(1)} \le \sqrt{n}\,(\sum_k |x_k|^2)^{1/2}$$

for all $x_1, x_2, \dots, x_n \in R$. Indeed, defining an ellipsoid

$$\mathcal{E} = \{x = \sum_k \alpha_k x_k e_k : \sqrt{n}\sqrt{\sum_k |x_k|^2} \le 1\}$$

we get $\mathcal{E} \subset B = B_{(E,\|\cdot\|^{(1)})} \subset B_{\ell_n^1}$. Hence

$$vr(E) \le \left(\frac{\text{vol } B}{\text{vol } \mathcal{E}}\right)^{1/n} \le \left(\frac{\text{vol } B_{\ell_n^1}}{\text{vol } \mathcal{E}}\right)^{1/n} = vr(\ell_n^1) \le \sqrt{\frac{2e}{\pi}}$$

by proposition 2.

To show (*) consider its dual version

$$(**) \qquad \max_k |y_k| \ge \|\sum_k \frac{y_k}{\alpha_k} e_k^*\|_*^{(1)} \ge \frac{1}{\sqrt{n}} (\sum_k |y_k|^2)^{1/2} .$$

Of course it is enough to prove (**) for nonnegative sequences (y_k) only. Substituting $y_k = \sqrt{\lambda_k}$ and $\alpha_k = 1/\sqrt{\beta_k}$ one gets

$$(***) \qquad \frac{1}{n} \sum_k \lambda_k \le (\|\sum_k \sqrt{\beta_k \lambda_k} e_k^*\|_*^{(1)})^2 \le \max_k \lambda_k .$$

Now existence of (β_k) satisfying (***) follows immediately from condition (c) of lemma A (i.e. the fact that the term in the centre of (***) is equal to $|||(\beta_k \lambda_k)|||$ for some unconditional norm $|||\cdot|||$) and lemma B. ■

Let us introduce another invariant :

$$hvr(E) \overset{\text{def}}{=\!=} \sup_{F \subset E, \dim F < \infty} vr(E) ,$$

where E is a Banach space, not necessarily of finite dimension. Using some methods from [4], one can easily derive from Prop. 3 the following :

<u>Theorem 8</u> : If $hvr(E) < \infty$, then E is of cotype $2 + \varepsilon$ for every $\varepsilon > 0$. In general ε cannot be omitted. ■

Finally I am going to present :

<u>Theorem 9</u> : There exists a function $(0,1) \ni \theta \to C(\theta)$ such that for any k-dimensional subspace E of ℓ_n^∞ we have

$$(+) \qquad d(E, \ell_k^2) > C(k/n)\sqrt{k} . \quad ■$$

Remark : Our proof gives $C(\theta) = \sqrt{\pi/2\, e^3}\ \theta$.
Recently Figiel and Johnson proved th. 9 with $C(\theta) = \sqrt{\theta/2}$. ■

Proof of theorem 9 : Since $d(E,\ell_k^2) = d(E^*,\ell_k^2)$, it is enough to prove (+) with E replaced by E^*.

To say that E is a subspace of ℓ_n^∞ is the same as to say that the unit ball of E^* has at most 2n extreme points, say $x_1, x_2, \dots, x_n$, $-x_1, -x_2, \dots, -x_n$. Let $\mathcal{E}$ be an ellipsoid contained in the unit ball of E^*. We must show that, for some i, $x_i \notin C(k/n)\sqrt{k}\,\mathcal{E}$. Thus the proof reduces to the following fact :

Let $B = \text{abs conv}(y_i)_{i=1}^n \subset R^k$ and let the Euclidean unit ball $B^k = \{x \in R^k : \|x\|_2 \le 1\}$ be contained in B. Then $\max_{1\le i\le n} \|y_i\|_2 \ge \sqrt{k}\ C(k/n)$.

To see the above consider all sets of the form
$B_A = \text{abs conv}(y_i)_{i\in A}$, $A \subset \{1,2,\dots,n\}$, card $A = k$.
Clearly $\bigcup_A B_A = B$. Choose A so that vol B_A is maximal. Then

$$\binom{n}{k} \text{vol } B_A \ge \text{vol } B \ge \text{vol } B^k .$$

On the other hand

$$\text{vol } B_A \le \prod_{i\in A} \|y_i\|_2 \ \text{vol } B_{\ell_k^1} .$$

Combining these two estimates one gets

$$\prod_{i\in A} \|y_i\|_2 \ge \binom{n}{k}^{-1} \frac{\text{vol } B^k}{\text{vol } B_{\ell_k^1}}) = \binom{n}{k}^{-1} (\sqrt{k})^k \frac{\text{vol } B^k}{\text{vol}(\sqrt{k}\ B_{\ell_k^1})} =$$

$$= \binom{n}{k}^{-1} (\sqrt{k})^k \left[vr(\ell_k^1)\right]^{-k} .$$

Hence

$$\max_{i\in A} \|y_i\|_2 \ge \left[\binom{n}{k}^{1/k} vr(\ell_k^1) \right]^{-1} \sqrt{k} \ge \frac{k}{en} \sqrt{\frac{\pi}{2e}} \sqrt{k} .$$

This ends the proof of theorem 9. ■

Let us mention finally some easy observations, which may indicate another application of concepts introduced here. Namely, we have

$$[vr(E)]^\theta\ [vr(F)]^{1-\theta} = vr(E \oplus_{\ell^2} F)$$

where $\theta = \dim E/(\dim E + \dim F)$ and $E \oplus_{\ell^2} F$ is a direct sum of E and F in the sense of ℓ_2^2.

One can hope that this may help in investigating complemented susbpaces of a normed space.

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