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GEOMETRY OF NUCLEAR SPACES

III - SPACES OF HOLOMORPHIC FUNCTIONS

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After the works of L. Schwartz, G. Köthe, A. Grothendieck, and I. Gelfand and G. Shilov at the beginning of the fifties, the spaces of holomorphic functions of different types became an important class of linear topological spaces. In [1] we have seen how these spaces motivated the construction of new linear topological invariants.

1. Let G be a domain of holomorphy in $\mathbb{C}^n$. What is the connection between the geometry of this domain and the linear topological type of the space H(G) ?

If G is a bounded Reinhardt domain in $\mathbb{C}^n$ then H(G) is isomorphic to $H(\mathcal{D}^n)$ and such an isomorphism $T : H(G) \to H(\mathcal{D}^n)$ can be chosen [2] easily :

$$T : z^k \to z^k \cdot d_k(G) \quad ,$$

where $\quad d_k(G) = \max\{|z^k| : z \in G\}, \; z^k = z_1^{k_1} \dots z_n^{k_n}, \; k \in \mathbb{Z}_+^n \quad .$

The analysis of Köthe spaces of Taylor coefficients of functions in $H(\mathcal{D}^m \times \mathbb{C}^n)$ shows (P. Djakov [3], V. Zaharyuta [4]) that these spaces are isomorphic for $m,n \geq 1$ if the sum $m + n$ is the same, and because of the diametral dimension they are not isomorphic for different $m + n$. We discussed above, exposé II, the more difficult case of general unbounded Reinhardt domains and we get a continuum of pairwise spaces $H(G_\gamma)$ for this class of domains.

At last we mention that for strongly pseudoconvex domain G, i.e.

$$G = \{z \in \mathbb{C}^n : \rho(z) < 0\} \quad \text{where} \quad \rho \in C^2_{real} \quad ,$$

$$\Sigma \frac{\partial^2 \rho}{\partial z_i \, \partial \bar{z}_j} \xi_i \overline{\xi_j} > 0, \; \forall \; \xi \neq 0, \text{ and } \rho = 0 \Rightarrow \partial \rho \neq 0 \quad ,$$

the space H(G) is isomorphic to $H(\mathcal{D}^n)$ also but this case need the more complicated $\bar{\partial}$-technique beside of the general methods of Hilbert scales and linear topological invariants (see details in Mityagin-Henkin [5]).

2. Recently we considered [6] the case of an algebraic variety $V \subset \mathbb{C}^n$ and proved that the space $H(V)$ is isomorphic to $H(\mathbb{C}^k)$, $k = \dim V$ (see [7] also), and that the ideal

$$J(V) = \{f \in H(\mathbb{C}^n) : f|V = 0\}$$

is a complemented subspace in $H(\mathbb{C}^n)$.

More general consideration is possible. Let $\{Q_i(z)\}_1^p$ be a finite set of polynomials, and

$$J(Q) = \{f \in H(\mathbb{C}^n) : f = \sum_{i=1}^{p} g_i\, Q_i\,,\ \ g_i \in H(\mathbb{C}^n),\ 1 \le i \le p\}$$

be the ideal of this system in $H(\mathbb{C}^n)$, and $I(Q)$ be the ideal of this system in the algebra of all polynomials.

We introduce an ordering in $\mathbb{Z}_+^n$ by the following way : $\alpha \prec \alpha'$ iff $|\alpha| < |\alpha'|$, or $|\alpha| = |\alpha'|$, $\alpha_j = \alpha'_j$, $j = k+1,\dots,n$; $\alpha_k \prec \alpha'_k$, and for some k, $2 \le k \le n$, and for any ideal I in the algebra of all polynomials put

$$T = T(I) = \{\alpha \in \mathbb{Z}_+^n \ : \ z^\alpha \in [z^\beta, \beta \prec \alpha] + I\}$$

where the brackets denote the linear hull of corresponding vectors.

<u>Theorem LD</u> (on linear decomposition) : Let $\mathbb{T} = \mathbb{T}(I(Q))$, $S = \mathbb{Z}_+^n \setminus \mathbb{T}$; then there exists continuous linear operators L_j, $0 \le j \le p$, $L_j : H(\mathbb{C}^n) \to H(\mathbb{C}^n)$, such that

(a) $$F = L_o F + \sum_{j=1}^{p} L_j F \cdot Q_j \quad , \quad \forall\, F \in H(\mathbb{C}^n) \quad ,$$

(b) $$\operatorname{Im} L_o = H_S(\mathbb{C}^n) = \{F = \sum F_\alpha z^\alpha \in H(\mathbb{C}^n) : F_\alpha = 0,\ \alpha \notin S\}$$

$$\operatorname{Ker} L_o = J(Q) \quad , \quad L_o^2 = L_o \quad ;$$

(c) for some vector $b = (b_1,\dots,b_n) > 0$ and $C > 0$, $\beta_1 > 0$,

$$\|L_j F\|_{tb} \le C t^{\beta_1} \|F\|_{2tb} \quad , \quad 0 \le j \le p,\ t \ge t_o(V) \quad ,$$

where $$\|G\|_{at} = \sup\{|G(z)| : |z_i| \le t a_i \quad , \quad 1 \le i \le n\} \quad ,$$

and

(d) $$\deg L_j P \le \deg P + d \quad , \quad d = d(Q)$$

and deg P denotes the degree of the polynomial P.

The proof of this theorem contains some tricks and it is not simple (see [6]) but it is completely elementary, i.e. it involves the estimates of Taylor coefficients of F and LF and we define the operators L_j, $0 \le j \le p$, by linearity ($L_j\, z^\alpha = \ell_j^\alpha$) after the appropriate individual decomposition

$$z^\alpha = \ell_o^\alpha + \sum_{j=1}^{p} \ell_j^\alpha \cdot Q_j \quad , \quad \alpha \in \mathbb{Z}_+^n$$

of all monomials. The special inductive construction gives polynomials $\{\ell_j^\alpha\}$ with the desired estimates.

3. Now we discuss some consequences of Theorem LD.

If f is a holomorphic function on V, i.e. $f \in H(V)$, then by Oka-Cartan theorem

(1) $$f = F|V \quad \text{for some } F \in H(\mathbb{C}^n) \quad .$$

This choice is not unique or linear but the image LF is the same by (a) and (b) for any extension F so the mapping

(2) $$E : f \to L_o F \quad , \quad \text{where } F|V = f \quad ,$$

be a linear extension operator $E : H(V) \to H(\mathbb{C}^n)$.

<u>Corollary</u> : For any algebraic variety $V \subset \mathbb{C}^n$ there exists a linear extension operator $E_V : H(V) \to H(\mathbb{C}^n)$. By other words, the ideal J(V) is a complemented subspace in $H(\mathbb{C}^n)$.

For contrast recall that for arbitrary closed submanifold (subvariety) M in $\mathbb{C}^n$ such a linear extension operator could not exist. For example, if $M = \psi\mathcal{D}^1$ or $\psi\mathcal{D}^k$ where $\psi : \mathcal{D}^k \to \mathbb{C}^n$ is any biholomorphic embedding of the unit polydisc to $\mathbb{C}^n$ there does not exist ([5], Sect. 5) a linear extension operator $E : H(M) \to H(\mathbb{C}^n)$.

The property (c) of Theorem states in particular that the operators L_j, $0 \le j \le p$, are continuous in the wide class of spaces of entire functions with estimates of their growth for $|z| \to \infty$. For example, it is true for the space of entire functions of the order $\rho > 0$ and minimal type

$$H_\rho(\mathbb{C}^n) = \{F \in H(\mathbb{C}^n) : |F(z)| \le C_\varepsilon \cdot e^{\varepsilon|z|^\rho} \quad , \forall \ \varepsilon > 0\}$$

so the statement of Theorem LD holds in this case also.

Moreover it is true for the spaces $H(t\mathcal{D}^n)$; more exactly, by (c) the operators L_j, $0 \le j \le p$, are continuous in Banach spaces

$$L : H^\infty(t\mathcal{D}^n) \to H^\infty(\gamma t\mathcal{D}^n) \quad , \quad \gamma = \frac{1}{3} \min_{1 \le i,k \le n} \frac{b_i}{b_k}$$

and for any $F \in H(2t\mathcal{D}^n)$ the decomposition

$$F(z) = (L_0 F)(z) + \sum_{j=1}^{p} (L_j F)(z) \cdot Q_j(z) \quad , \quad z \in \gamma t\mathcal{D}^n \quad ,$$

holds on the polydisc $\gamma t\mathcal{D}^n$ for $t \ge t_1(V)$.

We could repeat the argument above to get a linear extension operator

$$S : H_\rho(V) \to H_\rho(\mathbb{C}^n) \quad ,$$

where $\quad H_\rho(V) = \{f \in H(V) : |f(w)| \le A_\varepsilon \cdot e^{\varepsilon|w|^\rho} , \ w \in V \ ; \forall \ \varepsilon > 0\} \quad ,$

but we need an analogue of Oka-Cartan theorem with the estimates (a global functions F in (1) has to be of the same exponential growth). This analogue can be proved.

4. However the proof is not elementary. As usually after Hörmander's monography [8] it uses $\bar\partial$-technique, and the local analysis of algebraic singularities also.

<u>Theorem EP</u> : Let $g \in H(V \cap 3t\mathcal{D}^n)$ where $t \ge t_2(V)$. Then there does exist such a bounded function $G \in H(t\mathcal{D}^n)$ that

$$G|V \cap t\mathcal{D}^n = g \mid V \cap t\mathcal{D}^n \quad , \quad \text{and}$$

$$\sup\{|G(z)| : z \in t\mathcal{D}^n\} \leq Ct^{\beta_2} \cdot \sup\{|g(v)| : v \in V \cap 2t\mathcal{D}^n\}$$

where C and β_2 do not depend on t and g but on V.

Theorems EP and LD together give an extension (and linear extension operator also !) of H_ρ-functions. Indeed, for

$$f \in H_\rho(V) \quad \text{put} \quad g_t = f \mid V \cap 3t\mathcal{D}^n \quad .$$

By Theorem EP for some $G_t \in H(t\mathcal{D}^n)$

$$G_t \mid V \cap t\mathcal{D}^n = g_t \mid V \cap t\mathcal{D}^n = f \mid V \cap t\mathcal{D}^n \quad , \quad \text{and}$$

$$\|G_t\|_t \leq C\ t^{\beta_2}\ \|g_t\|_{2t} = C\ t^{\beta_2}\ \|f\|_{2t} \leq$$

$$\leq C \cdot A_\varepsilon\ t^{\beta_2} \exp(\varepsilon(2nt)^\rho) \quad .$$

5. The proof of Theorem EP contains four steps.

<u>1 step</u>. <u>Local extension</u>.

<u>Lemma 1</u> : For any point $z \in V$ there exists a pair of polydiscs $z + 2r_1\mathcal{D}^n$ and $z + r_2\mathcal{D}^n$ such that for any $h \in H(V \cap (z + 2r_1\mathcal{D}^n))$ one can choose such a bounded function $\tilde{h} \in H^\infty(z + r_2\mathcal{D}^n)$ that

$$\tilde{h}|V \cap (z + r_2\mathcal{D}^n) = h|V \cap (z + r_2\mathcal{D}^n)$$

and

$$\sup\{|\tilde{h}(w)| : |w - z| \leq r_2\} \leq C \cdot (1 + |z|^2)^{\beta_3} \times$$

$$\times \sup\{|h(w)| : |w - z| \leq r_1 \ , \ w \in V\} \quad .$$

The parameters $r_\delta = r_\delta(z)$, $\delta = 1, 2$, do not depend on a function h but they depend on z and can be chosen sufficiently large, i.e. $r_1(z) > r_2(z) \geq a(|z| + 1)^{-\beta_4}$ for some $a = a(V) > 0$ and $\beta_4 = \beta_4(V) > 0$.

This statement gives the possibility to get a finite covering $\{U_i\}$ of a neighborhood of $V \cap 3t\mathcal{D}^n$ and a system $\tilde{h}_i \in H(U_i)$ of functions such that

$$\tilde{h}_i | U_i \cap V = h | U_i \cap V \quad \text{for all } i \quad ,$$

so
$$\tilde{h}_{ij} = \tilde{h}_i (U_i \cap U_j) - \tilde{h}_j | (U_i \cap U_j)$$

vanishes on $(U_i \cap U_j) \cap V$ for any pair (i,j).

Step 2. Division.

Any function $h = \tilde{h}_{ij}$ has to be represented as

$$(3) \qquad h = g_o(z) + \sum_{\pi=1}^{p} g_\pi(z) \cdot Q_\pi(z) \quad , \quad z \in U \subset U_{ij}$$

if we want to use the $\bar{\partial}$-cohomology technique. This is possible (see [8], Prop. 7.6.5) but now we have to get good estimates of the norms of g's in (3), and the size of U from below. The inequality (7.6.5) in [8] gives such estimates for norms of g's but we have to repeat carefully its proof to get "large" neighborhood U. More precisely,

Lemma 5.2 : One can choose such B, $\beta_4, \beta_5 > 0$ that for any $v \in V$ if $h \in H(v + r_4(v)\mathcal{D}^n)$, $r_4(v) = B \cdot (1 + |v|^2)^{-\beta_4}$, and $h|V \equiv 0$ in this polydisc then there exists such a system $(g_\pi)_o^p \subset H^\infty(v + r_5(v) \cdot \mathcal{D}^n)$ that

$$(4) \qquad h = \sum_{\pi=1}^{p} g_\pi(z)\, Q_\pi(z) \quad , \quad z \in r_5(v)\mathcal{D}^n + v \quad ,$$

and

$$\sup_{1 \le \pi \le p} \sup\{|g_\pi(z)| : |z - v| \le r_5(v)\} \le \sup\{|h(v)| : |z - v| \le r_4(v)\} \quad .$$

Of course, (4) has the same form as (a) of Theorem LD but we could not use "global" operators L_π, $0 \le \pi \le p$, of Theorem LD -they are good for large polydiscs $(t \ge t_o(v))$ only. However, instead of Hörmander technique we can repeat our proof of Theorem LD in the local version to get the parametrized family $L^{(v)}$ of operators for Lemma 5.2 such that

$$h = L_o h + \sum_{\pi=1}^{p} L_\pi h \circ Q_\pi$$

and the statement of Lemma 5.2 is true.

Step 3. $\bar{\partial}$-equation and estimates of $\bar{\partial}$-solution.

The accurate estimates of the size of polydiscs $v + r_\varepsilon(v)\mathcal{D}^n$, $\varepsilon = 1,\dots,5$ on Steps 1, 2 give a finite covering of $3t\mathcal{D}^n$ and the decomposition of the identity such that

$$1 = \sum_{\gamma\in\Gamma} e_\gamma(z) \quad\text{and}\quad |\bar{\partial}e_\gamma(z)| \le B_5(1+|z|^2)^{\beta_5}$$

so any vector-valued C^∞-function of the form

$$g^\delta(z) = \sum_\gamma g^{\delta\gamma}(z)\, e_\gamma(z) \quad ,$$

where $g^{\delta\gamma}$ corresponds by (3) to the functions $\tilde{h}_{\delta\gamma}$ (or their restrictions for smaller polydiscs) of Step 1, has the estimate

$$|\bar{\partial}g^\delta(z)| \le B_6(1+|z|^2)^{\beta_6} \quad , \quad z\in 2t\mathcal{D}^n \quad .$$

The next lemma is well-known and elementary.

Lemma 5.3 : Let $\psi\in C^\infty$ be (p,q)-form on $2t\mathcal{D}^n$ and $\bar{\partial}\psi = 0$. Then there does exist such a $(p,q-1)$-form φ that

$$\bar{\partial}\varphi = \psi \quad\text{on } t\mathcal{D}^n \quad\text{and}\quad \|\varphi\|_{L^\infty(t\mathcal{D}^n)} \le B_7\|\psi\|_{L^\infty(2t\mathcal{D}^n)} \quad .$$

Step 4.

Cohomology techniqne as usually gives the possibility to get a global extension G in Theorem EP with good (of polynomial growth) estimates of constants.

This is the scheme of the proof of the following statement.

Theorem LDE : The operator $E_V : H(V)\to H(\mathbb{C}^n)$ of Corollary is a linear continuous extension operator in the spaces of functions of exponential growth, i.e. if $f\in H(V)$ and

$$|f(v)| \le C\exp(c|v|^\rho) \quad , \quad v\in V \quad ,$$

than

$$E_V f\in H_\rho(V) \Leftrightarrow |(E_V f)(z)| \le C_1\exp(c_1|z|^\rho) \quad , \quad z\in\mathbb{C}^n \quad .$$

For applications to the theory of partial differential equations (with constant coefficients) other classes of spaces of holomorphic functions are more essential. In our statement the estimates of growth are isotropic but sometimes the classes with different estimates respect to the real and imaginary part of z are important. For example, the Fourier transformation of Schwartz space $\mathcal{D}$ is the space

$$\mathcal{E} = \{F \in H(\mathbb{C}^n) : |F(z)| \le C_F(1+|z|^2)^{N_F} \exp C_F |\operatorname{Im} z|, \quad z \in \mathbb{C}^n\} \quad ,$$

and Gelfand-Shilov spaces are

$$S_\alpha^\beta = \{F \in H(\mathbb{C}^n) : |F(z)| \le C_F \exp(-a_F |\operatorname{Re} z|^\alpha + b_F |\operatorname{Im} z|^\beta)\} \quad .$$

I guess than in these cases for some algebraic varieties (or even manifolds) a linear extension does not exist.

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