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S E M I N A I R E

D ' A N A L Y S E   F O N C T I O N N E L L E

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GEOMETRY OF NUCLEAR SPACES

II - LINEAR TOPOLOGICAL INVARIANTS

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0. The spaces of holomorphic functions $H(\mathcal{D}^k)$ and $H(\mathbb{C}^N)$ were the stimulating examples for Kolmogoroff [1] and Pełczynski [2] to construct the linear topological invariants, so-called approximative and diametral dimensions, on the class of Schwartz metric spaces. After the observation that

$$(1) \qquad H(\mathcal{D}^k) \simeq K(c) \quad , \quad c_{np} = \exp(-\frac{1}{p} n^{1/k}) \quad , \quad n \in \mathbb{Z}_+ \quad ,$$

and

$$(2) \qquad H(\mathbb{C}^k) \simeq K(d) \quad , \quad d_{np} = \exp(p \; |n|) \quad , \quad n \in \mathbb{Z}_+^k \quad ,$$

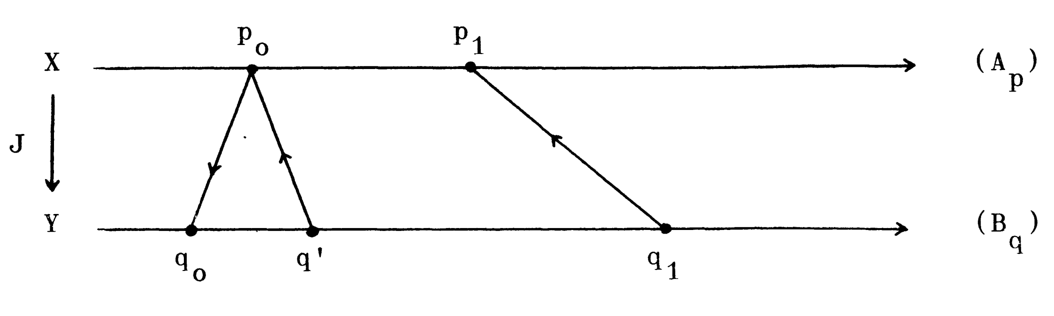
or

$$\simeq K(a) \quad , \quad a_{np} = \exp(p \; n^{1/k}) \quad , \quad n \in \mathbb{Z}_+ \quad ,$$

these invariants show in particular that all the spaces of two series (1) and (2) are pairwise-nonisomorphic (see details on [3], [4], [5]).

Recently the author [6], [7] and V. Zaharyuta [8], [9] give the further series of more general invariants and the core of this talk is to present these invariants and some concrete examples of its applications.

1. Recall that a nuclear Fréchet space (with a continuous norm) is the common domain of (monotone systems of) self-adjoint operators $\{A_p\}_0^\infty$ in a Hilbert space H_o, i.e. $1 = A_o \le A_1^2 \le A_2^2 \le \dots$, and $X = \bigcap_{p>0} \mathcal{D}(A_p)$. (We will denote this series of operators A_p, or norms $|x|_p = |A_p x|$, or Hilbert spaces $H_p = \mathcal{D}(A_p)$, graphically as the line with points $\{p\}$.) If two such spaces X and Y are isomorphic



,

i.e. $J : X \to Y$ is a linear topological isomorphism then

$\forall\, q\ \exists\, p,\ C' \mid \|Jx\|_q \le C' \ |x|_p$

$\forall\, p\ \exists\, q', C'' \mid |J^{-1}y|_p \le C'' \ \|y\|_{q'}$, so $\frac{1}{C}\, T_{q'} \subset J\ S_p \subset C\ T_q$, where

$C = \max\,\{C', C''\}$, and $S_p = \{x \in X : |A_p x| \le 1\}$, $T_q = \{y \in Y : \|B_q y\| \le 1\}$, or

$$C\ T_{q_o} \supset J\ S_{p_o} \supset J\ S_{p_1} \supset \frac{1}{C}\ T_{q_1} \quad . \tag{3}$$

Hence,

$$s_k(T_{q_1}, T_{q_o}) \le C^2 \cdot s_k(S_{p_1}, S_{p_2})$$

where $\{s_k(\mathcal{E}_1, \mathcal{E}_o)\}$ denotes the sequence of s-numbers of the identity operator $1 : H_{\mathcal{E}_1} \to H_{\mathcal{E}_o}$ of two Hilbert spaces with the unit balls $\mathcal{E}_1$ and $\mathcal{E}_o$ correspondingly. If we put $N_A(p_o, p_1; t) = |\{k : s_k(S_{p_1}, S_{p_o}) \ge 1/t\}|$, then by (3)

$$\forall\, q_o\ \exists\, p_o,\ \forall\, p_1 > p_o\ \exists\, q_1,\ C \mid N_B(q_o, q_1; t) \le N_A(p_o, p_1; \tfrac{t}{C}) \tag{4}$$

and analoguous condition (4') holds if we change the places of A and B.

So the system $\{N_A(p_o, p_1; t)\}_{0 \le p_o < p_1 < \infty}$ is a characteristics of the space, and the systems $\{N_A\}$ and $\{N_B\}$ are equivalent in the sense (4) - (4') if the spaces X and Y are isomorphic. Hence, any scalar parameter or any functional object generated by the class of equivalent systems of functions $\{N_A(P;t)\}$, $P = (p_o, p_1)$, would be a linear topological invariant.

For example, in the case (2) $N(p_o, p_1; t) \sim \left(\frac{\log t}{p_1 - p_o}\right)^k$ and the parameter

$$\gamma(p_o, p_1; A) = \limsup_{t \to \infty} \frac{\log N(p_o, p_1; t)}{\log \log t} \tag{5}$$

is the same (by occasion ?) for different p_o, p_1 and is equal to k. So the spaces (2) (and (1) also) are not isomorphic for different k. The parameters

$$\beta(p_o, p_1; A) = \limsup \frac{(N(p_o, p_1; t))^{1/k}}{\log t} = \frac{1}{p_1 - p_o} \tag{6}$$

show that $H(\mathcal{D}^k)$ and $H(\mathbb{C}^k)$ are not isomorphic for the same k.

2. Now we consider the more complicated invariants for the case of Köthe spaces

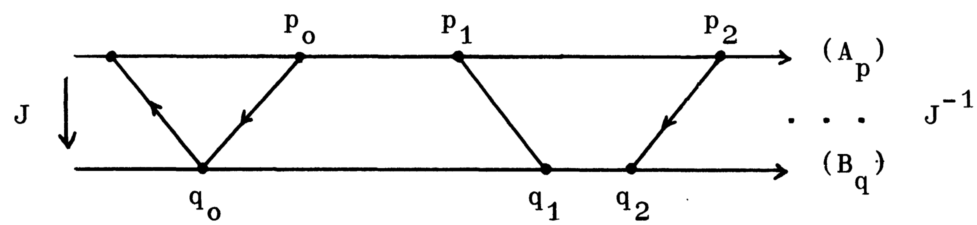
(7) $$K(a) = \{x = (x_n)_o^\infty : \Sigma\, a_{np}^2\, |x_n|^2 < \infty \ , \ \forall\, p\}$$

i.e. by [10], Theorem AB, for the case of nuclear Fréchet spaces with a basis.

Put $a_p(i) = a_{ip}$, $P = (p_o, p_1, p_2)$, and

(8) $$N_a(P;t_1,t_2) = \left| \left\{ \tau : \frac{a_{p_1}(i)}{a_{p_o}(i)} \geq t_1 \ , \ \frac{a_{p_2}(i)}{a_{p_1}(i)} \leq t_2 \right\} \right| \ .$$

If the spaces $X = K(a)$ and $Y = K(b)$ are isomorphic then

(9) 

and for any $x \in E = \text{Lin. Span}\{e_i : i \in I\}$, I being the set in the right side of (8), we have the following inequalities

$$|x|_{p_o} \leq \frac{1}{t_1} |x|_{p_1} \quad , \quad |x|_{p_2} \leq t_2\, |x|_{p_1}$$

and then for any $y = Jx \in L$, $L = JE \subset Y$, by (9) we have for some $C > 0$:

$$\|y\|_{q_o} \leq C\, |x|_{p_o} \leq \frac{C}{t_1} |x|_{p_1} \leq \frac{C^2}{t_1} \|y\|_{q_1}$$

(10)

$$\|y\|_{q_2} \leq C\, |x|_{p_2} \leq C\, t_2\, |x|_{p_1} \leq C^2\, t_2 \cdot \|y\|_{q_1} \quad .$$

<u>Lemma CE</u> : Let V, W_o, W_1 be coaxed ellipsoids in $\mathbb{C}^\infty$

$$V = \{\xi = (\xi_n) : \Sigma\, |\xi_n|^2 \leq 1\}$$

and $\quad W_\varepsilon = \{x \in \mathbb{C}^\infty : \Sigma\, |x_n|^2/w_{\varepsilon i}^2 \le 1\}$, $w_{\varepsilon i} > 0$, $\varepsilon = 0,1$,

and for some subspace L, dim $L = k$, the inequality

$$\|y\|_V \ge \|y\|_{W_\varepsilon} \quad , \quad \varepsilon = 0,1 \quad , \tag{11}$$

holds for any $y \in L$.

Then there exists a coordinate k-dimensional subspace L^o such that $\|y\|_V \ge \frac{1}{2} \|y\|_{W_\varepsilon}$, $\varepsilon = 0,1$, for any $y \in L^o$, i.e. $|K| \ge k$, where $K = \{i : w_{\varepsilon i} \ge \frac{1}{2},\ \ \varepsilon = 0,1\}$.

Proof : Let us consider the coordinate subspace $\mathbb{C}^k = \{(\xi_i) : \xi_i = 0,\ i \notin K\}$ and the natural projection $\pi_K : \mathbb{C}^\infty \to \mathbb{C}^K$. Then $\mathrm{Ker}(\pi_K \mid L) = \{0\}$; indeed if $y \in L$ and $\pi_K y = 0$ then by (11)

$$\|y\|_V^2 \ge \sup_\varepsilon \|y\|_{W_\varepsilon}^2 \ge \frac{1}{2}\left(\Sigma \frac{|y_i|^2}{w_{oi}^2} + \Sigma \frac{|y_i|^2}{w_{1i}^2}\right)$$

$$\ge \frac{1}{2}\, \Sigma \frac{|y_i|^2}{r_i^2} = \frac{1}{2} \sum_{r_i < 1/2} \frac{|y_i|^2}{r_i^2} \ge \frac{1}{2}\, \Sigma\, 4 \cdot |y_i|^2 \equiv 2\, \|y\|_V^2$$

and $$\|y\| = 0 \quad .$$

Hence $\pi_K | L : L \to \mathbb{C}^K$ is a monomorphism and $|K| \ge \dim\mathrm{Im}(\pi_K | L) = \dim L = k$.

Now if we put $V = T_{q_1}$, $W_o = \frac{C^2}{t_1} T_{q_o}$, $W_1 = C^2\, t_2 \cdot T_{q_2}$, then by Lemma CE and by (10), $Q = (q_o, q_1, q_2)$,

$$N_b(Q; \frac{t_1}{2C^2}\ ,\ 2C^2\, t_2) \overset{\text{def}}{=} |\{j : b_{q_1}(j)/b_{q_o}(j) \ge \frac{t_1}{2C^2}\ ,$$

$$b_{q_2}(j)/b_{q_1}(j) \le 2C^2\, t_2\}| \ge N_a(P; t_1, t_2) \qquad \text{(see (8))}.$$

Hence the following statement is true.

Theorem IN : If the spaces X and Y in (9) are isomorphic then the systems (8) of functions $\{N_a(P,t)\}$ and $\{N_b(Q,t)\}$, $t \in \mathbb{R}_+^2$, are equivalent in the following sense :

(12) $\forall\, q_o \; \exists\, p_o, \; \forall\, p_1 > p_o \; \exists\, q_1, \; \forall\, q_2 > q_1 \; \exists\, p_2, \; C \ni$

$$N_b(Q;t') \geq N_a(P;t)$$

and

$$N_a(Q;t') \geq N_b(P;t) \;, \; t' = \left(\frac{t_1}{2C^2} \;;\; 2C^2 t_2\right) \;.$$

3. These invariants have been motivated by [6] where the particular cases of (7)

(13) a) $a_{ip} = a_i^{-1/p}$ and b) $a_{ip} = a_i^p$

have been considered in detail. Remark that in (13) there is no restriction to the sequence $(a_i)_o^\infty$ but $a_i \geq 1$, so it may have finite points of accumulation or take the same value infinitely many times. In the case (13.b)

$$N_a(P;t) = \left|\left\{i : t_1^{\frac{1}{p_1-p_o}} \leq a_i \leq t_2^{\frac{1}{p_2-p_1}}\right\}\right| =$$

$$= \left|\left\{i : \frac{1}{p_1-p_o} \log t_1 \leq \log a_i \leq \frac{1}{p_2-p_1} \log t_2\right\}\right| \;.$$

<u>Example</u> : The space $H(\mathbb{C}^k;V)$ of all entire vector-valued functions, V be a Hilbert space, dim $V = \infty$, is isomorphic to the generalized Köthe space

(14) $K(a;V) = \{x = (x_n)_o^\infty, \; x_n \in V : \Sigma\, a_n^{2p} \, \|x_n\|^2 < \infty, \; \forall\, p\}$,

$$\log a_n = (1+n)^{1/k} \;, \; n \in \mathbb{Z}_+ \;.$$

In this case for $t_s = \exp \tau_s$, $s = 1,2$,

$$N_a(P;t) = \infty \;\text{ if }\; \left[\left(\frac{\tau_1}{p_1-p_o}\right)^k , \left(\frac{\tau_2}{p_2-p_1}\right)^k\right] \cap \mathbb{Z}_+ \neq \emptyset$$

$$= 0 \;\text{ otherwise} \;.$$

The spaces $H(\mathbb{C}^k;V)$, $\dim V=\infty$, are isomorphic for all $k\in\mathbb{Z}_+$, $k>0$.

However if we consider the spaces (14) for the sequences (a_n)

$$\log a_n = \lambda_n^\gamma \quad , \quad \text{where } \lambda_{n+1}/\lambda_n \to \infty \quad , \quad 0<\gamma<\infty \tag{15}$$

then the spaces $K_\gamma(a;V)$ defined by (14), (15) are pairwise-nonisomorphic for a continuum $\Gamma\subset\mathbb{R}_+$. Indeed

$$N_{a(\gamma)}(P;t) = \infty \quad \text{if} \quad \left\{ i : \frac{\tau_1}{p_1-p_o} \le \lambda_i^\gamma \le \frac{\tau_2}{p_2-p_1} \right\} \ne \emptyset$$

$$= 0 \text{ otherwise,}$$

and one can prove the following statement.

<u>Lemma RI</u> : If the systems of functions $\{N_{a(\gamma)}\}$ are equivalent in the sense (12) then

$$\lim \frac{1}{n} \log\log \lambda_n = \ell \ne 0 \tag{16}$$

does exist, and $\ell/(\log \frac{\gamma}{\delta})$ is a rational number.

Hence there are two possibilities : 1^o. the spaces $K_\gamma(a;V)$ are pairwise-noniosmorphic for all $\gamma>0$; 2^o. for a pair (γ,δ) the spaces K_γ and K_δ are isomorphic and then (16) holds. In the second case we choose a continuum $\Gamma\subset\mathbb{R}_+$ by such a way that for any $\gamma_1,\gamma_2\in\Gamma$ the number $\frac{1}{\ell}\log\frac{\gamma_1}{\gamma_2}$ is irrational.

4. The invariants (8) and (12) of Theorem IN can be extended essentially. Let us define for any $n\ge 1$ the system of functions

$$N^n(P;x,y) = |\{ i : a_{p_{2j+1}}(i)/a_{p_{2j}}(i) \ge e^{x_{j+1}} ,$$

$$a_{p_{2j+2}}(i)/a_{p_{2j+1}}(i) \le e^{y_{j+1}} , \ j=0,1,\dots,n-1 \}| \ , \tag{17}$$

where $P=(p_o,p_1,\dots,p_{2n})$; $x,y\in\mathbb{R}^n$.

If the spaces X and Y in (9) are isomorphic then the systems (17) $\{N_a^n\}$ and $\{N_b^n\}$, $P \in \mathbb{Z}_+^{2n+1}$, $x,y \in \mathbb{R}^n$, are equivalent in the following sense :

(18_n) $\forall\, q_o \ \exists\, p_o,\ \forall\, p_1 \ \exists\, q_1 \ \dots \ \forall\, q_{2n} \ \exists\, p_{2n}$, T and S subdiagonal matrices

$$\ni N_a^n(P;x,y) \le N_b^n(Q;x - TX - Sy,\ y + TX + Sy), \quad \forall\, x,y \in \mathbb{R}_+^n \ .$$

This is the more general statement than Theorem IN above ; the relations (17), (18_n) give the invariant I_n for any $n \ge 0$.

Theorem SM : For any $n \ge 0$ one can construct such a pair of (nuclear) Köthe spaces E_n and F_n that the systems of functions $N_{E_n}^k$ and $N_{F_n}^k$ are equivalent in the sense (18_k) for $0 \le k \le n$, and are not equivalent for $k = n+1$.

If N_a^{n+1} and N_b^{n+1} are equivalent, then N_a^k and N_b^k are (18_k)-equivalent, $0 \le k \le n$, so by Theorem SM $\{I_n\}_o^\infty$ is a strongly monotone system of invariants on the class of Köthe spaces.

Analoguous system $\{I'_n\}$ can be constructed for multiindices P, $|P|$ be even.

5. The spaces E and F in Theorem SM need the special construction. What "natural" spaces can be considered with the help of these invariants ?

V. Zaharyuta [9] studied the spaces H(G) of holomorphic functions in Reinhardt domains G, $G \subset \mathbb{C}^n$, $n \ge 2$. By the definition

$$z \in G \ ; \ |w_i| < |z_i| \ , \ 1 \le i \le n \Rightarrow w \in G \quad ,$$

and G is a domain of holomorphy, so this domain is determined by the support function

$$h_G(\omega) = \sup\{(x,\omega) : x_i = \log |z_i|,\ 1 \le i \le n \ , z \in G\} \quad ,$$

$$\omega \in \sigma^{n-1} = \{y \in \mathbb{R}_+^n : \Sigma\, y_i = 1\} \quad .$$

Modified invariants of type (8), (12) are defined by the functions

$$M_a^1(P,\tau) = |\{i : a_{p_1}(i)/a_{p_o}(i) \le e^{\tau_1} ; \ a_{p_2}(i)/a_{p_1}(i) \ge e^{\tau_2}\}|$$

$$P = (p_o,p_1,p_2) \ , \quad \tau \in \mathbb{R}^2 \ ,$$

and by analoguous functions M_a^n of several variables P and τ.

As in (5) or (6) one can define the "functional" parameter

$$\delta(P;v) = \lim_{p_3\to\infty} \lim_{\substack{\alpha\to\infty \\ \beta/\alpha\to v \\ \gamma/\alpha\to 1}} \frac{M^1(p_o,p_1,p_3;\alpha,\beta)}{N(p_1,p_2;e^\gamma)} \ , \tag{19}$$

$$P = (p_o,p_1,p_2) \ , \ v \in \mathbb{R}^1_+ \ .$$

It happens that the properties of the support function h_G can be described in terms of the invariant (19). Namely,

<u>Lemma Z</u> ([9], p. 29) : $\forall \ p_o \ \exists \ p_1, \ \forall \ p_2 \ \exists \ C \ni$

$$\frac{1}{C} \ell_G(ct) \le \delta(P,t) \le C \ \ell_G \left(\frac{t}{C}\right) \ , \ t \ge t_o$$

where $\ell(n) = \text{mes}\{\omega \in \delta^{n-1} : u \le h_G(\omega) < \infty\}$.

It implies that if for any C_1, $C_2 > 0$ the function $\ell_{G_1}(c_1t)/\ell_{G_2}(c_2t)$ is unbounded, then the spaces $H(G_1)$ and $H(G_2)$ are not isomorphic.

<u>Corollary</u> : For any $n \ge 2$ there exists a continuum G_γ of domains of holomorphy in $\mathbb{C}^n$ such that the spaces $H(G_\gamma)$ are pairwise-nonisomorphic.

For example, one can choose

$$G_\gamma = \{z \in \mathbb{C}^n : |z_i| < 1, \ 1 \le i \le n-1, \ |z_n| < \exp(\log \frac{1}{|z|_1})^\gamma\} \ , \ 0 < \gamma < 1 \ .$$

6. It should be mentioned that the general problem of quasi-equivalence of bases in a nuclear Fréchet space with a base motivated the construction of new invariants. Recall that two bases (x_n) and (f_n) in E are quasiequivalent if there is a bijection $\rho : \mathbb{N} \to \mathbb{N}$ of the positive integers and a sequence of nonzero scalars (r_n) such that the operator T :

$$Tf_n = r_n \, x_{\rho(n)} \quad , \quad n \in \mathbb{N} \quad ,$$

is an automorphism of the space E.

For any unconditional basis (x_n) in a Fréchet space E one can define the group

$$G(x) = \{\sigma : \mathbb{N} \to \mathbb{N} \mid \exists\, (r_n),\ r_n \neq 0 \ ;\ T \in \text{Auto } E \ni$$
$$Tx_n = r_n \, x_{\sigma(n)} \, , \ \forall\, n \in \mathbb{N} \}$$

of rearrangements of $\mathbb{N}$.

If the bases (x_n) and (f_n) are quasiequivalent, then subgroups G(x) and G(f) are isomorphic :

$$\rho^* : G(x) \to G(f) \quad , \quad \rho^* : \sigma \mapsto \rho^{-1} \circ \sigma \circ \rho \quad .$$

Hence, if the space E has QEP (quasiequivalence property), i.e. any two bases in E are quasiequivalent, then the groups G(x) are isomorphic, x be a basis, so this group is an invariant in the class of nuclear spaces with a basis and QEP.

The wide class of nuclear Fréchet spaces with regular basis has QEP (see [11] and references there). Nevertheless we do not know whether any Fréchet space with a base has QEP. But invariants described above are "characteristics" of non-invariant or invariant (respect to the base) group G(x) and these characteristics are invariants.

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