SÉMINAIRE D'ANALYSE FONCTIONNELLE
ÉCOLE POLYTECHNIQUE

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(Appendice n°1) Sufficiently rich sets of stopping times, measurable cluster points and submartingales

Séminaire d'analyse fonctionnelle (Polytechnique) (1977-1978), p. 1-11
<http://www.numdam.org/item?id=SAF_1977-1978____A22_0>

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1977-1978

SUFFICIENTLY RICH SETS OF STOPPING TIMES, MEASURABLE CLUSTER POINTS AND SUBMARTINGALES

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Appendice No 1 17 Mars 1978

Sufficiently rich sets of stopping times, measurable cluster points and submartingales

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Let $(\Omega,\mathcal{F},P)$ be a fixed probability space. We denote by N the set of positive integers; $\bar{N} = N \cup \{+\infty\}$. We shall assume in what follows that:

$(\mathcal{F}_n)_{n \in N}$ is an increasing sequence of sub-σ-fields of $\mathcal{F}$, i.e., $\mathcal{F}_m \subset \mathcal{F}_n \subset \mathcal{F}$ for $m \leq n$ and we let

$$\mathcal{F}_\infty = \sigma(\bigcup_{n \in N} \mathcal{F}_n);$$

that is, $\mathcal{F}_\infty$ is the σ-field spanned by $\bigcup_{n \in N} \mathcal{F}_n$.

A mapping $\theta\colon \Omega \to \bar{N}$ is called a stopping time (relative to $(\mathcal{F}_n)_{n \in N}$) if $\{\theta = n\} \in \mathcal{F}_n$ for each $n \in N$. We associate with θ the σ-field $\mathcal{F}_\theta$ defined by

$$\mathcal{F}_\theta = \{A \in \mathcal{F}_\infty | A \cap \{\theta = n\} \in \mathcal{F}_n \text{ for each } n \in N\};$$

$\mathcal{F}_\theta$ is "the σ-field of events prior to time θ."

We denote by $\mathbf{T}_f$ the set of all stopping times σ that are finite a.s., that is, such that $P(\{\sigma < +\infty\}) = 1$. We denote by $\mathbf{T}$ the set of all bounded stopping times, that is, the set of all stopping times $\sigma\colon\Omega \to N$, assuming only finitely many values. Clearly $\mathbf{T}$ is a proper subset of $\mathbf{T}_f$. We recall also that if σ, τ belong to $\mathbf{T}_f$, the relation $\sigma \leq \tau$ implies $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$.

Let now $\mathbf{S}$ be a subset of $\mathbf{T}_f$. For each $\tau \in \mathbf{T}_f$ we define

$$\mathbf{S}(\tau) = \{\sigma \in \mathbf{S} | \sigma \geq \tau\};$$

in particular, for each $n \in N$

$$\mathbf{S}(n) = \{\sigma \in \mathbf{S} | \sigma \geq n\}.$$

For $X \in L^1 = L^1_R(\Omega,\mathcal{F},P)$ we write

*Research supported in part by the National Science Foundation (U.S.A.).

$$\|X\|_1 = \int_\Omega |X(\omega)| dP(\omega).$$

We say that a sequence $(X_n)_{n \in N}$ of elements of L^1 is L^1-bounded if

$$\sup_{n \in N} \|X_n\|_1 < +\infty.$$

If $\mathcal{G} \subset \mathcal{F}$ is a sub-σ-field of $\mathcal{F}$, we denote by $E^{\mathcal{G}}$ the conditional expectation operator in L^1.

Below whenever we speak of r.v.'s we shall always mean real-valued random variables.

A sequence $(X_n)_{n \in N}$ of r.v.'s is called adapted (relative to $(\mathcal{F}_n)_{n \in N}$) if each X_n is $\mathcal{F}_n$-measurable. If $(X_n)_{n \in N}$ is an adapted sequence of r.v.'s and if $\tau \in \mathbf{T}_f$, then X_τ denotes the r.v. defined by $(X_\tau)(\omega) = X_{\tau(\omega)}(\omega)$ if $\omega \in \{\tau < +\infty\}$, and $(X_\tau)(\omega) = 0$ otherwise. Note that X_τ is always $\mathcal{F}_\tau$-measurable.

§1. Sufficiently rich sets of stopping times and measurable cluster points

We begin with the following definition:

Definition 1. We say that a set $\mathbf{S} \subset \mathbf{T}_f$ is sufficiently rich if:

a) For each $n \in N$, $\mathbf{S}(n) \neq \emptyset$;

b) (Localization) For each finite family $(\tau_j)_{j \in J}$ of stopping times with $\tau_j \in \mathbf{S}$ (for $j \in J$) and finite partition of Ω, $(A_j)_{j \in J}$ with $A_j \in \mathcal{F}_{\tau_j}$ (for $j \in J$), if we set $\tau(\omega) = \tau_j(\omega)$ for $\omega \in A_j$ ($j \in J$), then $\tau \in \mathbf{S}$.

Remark. If $\mathbf{S} \subset \mathbf{T}_f$ is sufficiently rich, then for any $\sigma \in \mathbf{S}, \tau \in \mathbf{S}$, the stopping times $\sigma \vee \tau$ and $\sigma \wedge \tau$ belong to $\mathbf{S}$ (note that the set $\{\sigma \leq \tau\}$ belongs both to $\mathcal{F}_\sigma$ and $\mathcal{F}_\tau$).

Examples. 1) The sets $\mathbf{T}$ and $\mathbf{T}_f$ clearly are sufficiently rich.

2) If $\mathbf{S} \subset \mathbf{T}$ is sufficiently rich and if $\mathbf{S}$ contains the constants, then $\mathbf{S} = \mathbf{T}$.

3) Let $(X_n)_{n \in N}$ be an adapted sequence of r.v.'s and let $B \subset R$ be a Borel set which is recurrent for $(X_n)_{n \in N}$; this means that a.s. for $\omega \in \Omega$, the sequence $(X_n(\omega))_{n \in N}$ visits the set B infinitely many times. Let $\mathbf{S}$ be the set of all $\tau \in \mathbf{T}_f$ with the property that $P(\{X_\tau \in B\}) = 1$. Then the set $\mathbf{S}$ is sufficiently rich.

Definition 2. Let $(X_n)_{n \in N}$ be an adapted sequence of r.v.'s and let $\mathbf{S} \subset \mathbf{T}_f$ be a sufficiently rich set of stopping times. We say that a r.v. Y is a measurable cluster point of the sequence $(X_n)_{n \in N}$ relative to $\mathbf{S}$ and we write $Y \in \mathcal{MC}[(X_n)_{n \in N};\mathbf{S}]$ if: there is a sequence $(\tau_n)_{n \in N}$ with $\tau_n \in \mathbf{S}(n)$ such that $X_{\tau_n} \to Y$ a.s.

Remarks. 1) Suppose $\mathbf{S} = \mathbf{T}$. In this case every r.v. Y which coincides a.s. with an $\mathcal{F}_\infty$-measurable one and having the property that a.s. for $\omega \in \Omega$, $Y(\omega)$ is a cluster value of the sequence $(X_n(\omega))_{n \in N}$, belongs to $\mathcal{MC}[(X_n)_{n \in N};\mathbf{T}]$ (see for instance Theorem 1 in [4]). We write $\mathcal{MC}[(X_n)_{n \in N}] = \mathcal{MC}[(X_n)_{n \in N};\mathbf{T}]$ and we speak of the elements of $\mathcal{MC}[(X_n)_{n \in N}]$ as the measurable cluster points of the sequence $(X_n)_{n \in N}$.

2) Let $(X_n)_{n \in N}$ be an adapted sequence of r.v.'s and for each $k \in N$ let $P(k)$ be a measurable property that the process $(X_n)_{n \in N}$ might satisfy. We assume that: i) For each $k \in N$, the set

$$\{\omega \in \Omega \mid \text{the process } (X_n)_{n \in N} \text{ satisfies } P(k)\}$$

belongs to $\mathcal{F}_k$. ii) For almost every $\omega \in \Omega$, the process $(X_n)_{n \in N}$ satisfies $P(k)$ for all k large enough, that is, for all $k \geq k_\omega$ (here the integer k_ω may depend on ω). Let $\mathbf{S}$ be the set of all $\tau \in \mathbf{T}_f$ satisfying: on the set $\{\tau = k\}$ the process $(X_n)_{n \in N}$ satisfies $P(k)$. Then the set $\mathbf{S}$ is sufficiently rich and it is easily seen that $\mathcal{MC}[(X_n);\mathbf{S}] = \mathcal{MC}[(X_n)_{n \in N}]$.

§2. The submartingales associated with X, the sequence $(X_n)_{n \in N}$ and the set $\mathbf{S}$

From now on, through the rest of the paper we shall assume that:

$(X_n)_{n \in N}$ *is an adapted sequence of elements of* L^1, *and* $\mathbf{S} \subset \mathbf{T}_f$ *is a sufficiently rich set of stopping times such that* $X_\tau \in L^1$ *for each* $\tau \in \mathbf{S}$.

Our starting point is an idea proposed by Baxter (see [2]; see also [4]) which we expand as follows:

Proposition 1. *Let* $X \in L^1$. *For each* $n \in N$ *define* $\mu_n : \mathcal{F}_n \to R_+$ *by*

$$\mu_n(A) = \inf\{\int_A |X - X_\tau| dP \mid \tau \in \mathbf{S}(n)\}, \quad \textit{for } A \in \mathcal{F}_n.$$

There is then a positive submartingale $(S_n)_{n \in N}$ (relative to $(\mathcal{F}_n)_{n \in N}$ of course) *such that for each* $n \in N$

$$\mu_n(A) = \int_A S_n dP, \quad \textit{for all } A \in \mathcal{F}_n.$$

Proof: The fact that

(1) $\mu_n : \mathcal{F}_n \to R_+$ is finitely additive is an immediate consequence of the "localization" property b) of $\mathbf{S}$. Note also that if we fix $\tau(n) \in \mathbf{S}(n)$ then

(2) $\mu_n(A) \leq \int_A |X - X_{\tau(n)}| dP$, for all $A \in \mathcal{F}_n$.

Properties (1) and (2) imply in particular that μ_n is countably additive and absolutely continuous with respect to the restriction $P|\mathcal{F}_n$. This yields the existence of $S_n \in L^1(\mathcal{F}_n)$, $S_n \geq 0$ satisfying

$$\mu_n(A) = \int_A S_n dP, \quad \text{for all } A \in \mathcal{F}_n.$$

It is clear that the sequence $(S_n)_{n \in N}$ satisfies the submartingale property relative to $(\mathcal{F}_n)_{n \in N}$ (for the definition and basic properties of submartingales; see for instance Chap. IV in [8]).

Definition 3. We call the sequence $(S_n)_{n \in N}$ of Proposition 1 the submartingale of type (I) associated with X, the sequence $(X_n)_{n \in N}$ and the set $\mathbf{S}$.

With the notation of Proposition 1 we have:

Corollary 1. The submartingale $(S_n)_{n \in N}$ is L^1-bounded if and only if there is a sequence $(\tau(n))_{n \in N}$ with $\tau(n) \in \mathbf{S}(n)$ such that $(X_{\tau(n)})_{n \in N}$ is L^1-bounded. In particular, this is the case if $\mathbf{S}$ contains the constants and

$$\liminf_n \|X_n\|_1 < \infty.$$

Proof: Immediate consequence of the definition of μ_n and S_n.

Corollary 2. Suppose that there is a sequence $(\tau(n))_{n \in N}$ with $\tau(n) \in \mathbf{S}(n)$ such that $(X_{\tau(n)})_{n \in N}$ is uniformly integrable. Then the submartingale $(S_n)_{n \in N}$ is uniformly integrable. In particular this is the case if $\mathbf{S}$ contains the constants and if there is a subsequence of $(X_n)_{n \in N}$ which is uniformly integrable.

Proof: Corollary 2 follows easily from formula (2) (in the proof of Proposition 1) if we note that

$$0 \leq S_n \leq E^{\mathcal{F}_n}(|X|) + E^{\mathcal{F}_n}(|X_{\tau(n)}|), \quad \text{for } n \in N$$

and if we recall that whenever $\mathcal{H} \subset L^1$ is uniformly integrable, then the set

$$\{E^{\mathcal{G}}(Y) \mid Y \in \mathcal{H},\ \mathcal{G} \subset \mathcal{F} \text{ an arbitrary sub-}\sigma\text{-field}\}$$

is also uniformly integrable (for an elegant treatment of uniform integrability see [7], pp. 16-17).

Corollary 3. Assume that the submartingale $(S_n)_{n \in N}$ is L^1-bounded. Then: i) For each $\sigma \in \mathbf{T}_f$, S_σ is integrable; ii) if $\sigma \in \mathbf{T}_f$ is such that $\mathbf{S}(\sigma) \neq \emptyset$ then we also have

(3) $\int_A S_\sigma dP \leq \inf\{\int_A |X - X_\tau| dP \mid \tau \in \mathbf{S}(\sigma)\}$, <u>for</u> $A \in \mathcal{F}_\sigma$.

<u>Proof:</u> i) is an immediate consequence of the L^1-boundedness of $(S_n)_{n \in N}$ and the submartingale property.

ii) Let $\sigma \in \mathbf{T}_f$ such that $\mathbf{S}(\sigma) \neq \emptyset$ and let $A \in \mathcal{F}_\sigma$. Take <u>any</u> $\tau \in \mathbf{S}(\sigma)$. For each $n \in N$ let $A_n = A \cap \{\sigma = n\}$. Then $A_n \in \mathcal{F}_n$ and $A_n \in \mathcal{F}_\sigma \subset \mathcal{F}_\tau$; choose now $\sigma_n \in \mathbf{S}(n)$ and define

$$\tau_n(\omega) = \tau(\omega) \quad \text{for } \omega \in A_n, \quad \tau_n(\omega) = \sigma_n(\omega) \quad \text{for } \omega \in (A_n)^c.$$

Clearly $\tau_n \in \mathbf{S}(n)$ for each $n \in N$ and we have

$$\begin{aligned} \int_A S_\sigma dP &= \sum_{n \in N} \int_{A_n} S_n dP \\ &\leq \sum_{n \in N} \int_{A_n} |X - X_{\tau_n}| dP = \int_A |X - X_\tau| dP \end{aligned}$$

which proves (3).

<u>Remarks</u>. 1) If $\sigma \in \mathbf{S}$, then clearly $\mathbf{S}(\sigma) \neq \emptyset$.

2) With the notation of Corollary 3, if σ assumes only finitely many values, i.e. if $\sigma \in \mathbf{T}$, then as is easily seen, we actually have equality in (3).

We now show how one can associate a second type of submartingale with X, the sequence $(X_n)_{n \in N}$ and the set $\mathbf{S}$:

<u>Proposition 2</u>. <u>Let</u> $X \in L^1$. <u>For each</u> $n \in N$ <u>define</u> $\gamma_n : \mathcal{F}_n \to R_+$ <u>by</u>

$$\gamma_n(A) = \inf\{\int_A |X - (X_\sigma - X_\tau)| dP \mid \sigma, \tau \in \mathbf{S}(n)\},$$

<u>for</u> $A \in \mathcal{F}_n$. <u>There is then a positive submartingale</u> $(G_n)_{n \in N}$ (relative to $(\mathcal{F}_n)_{n \in N}$ of course) <u>such that for each</u> $n \in N$

$$\gamma_n(A) = \int_A G_n dP, \quad \underline{\text{for all}}\ A \in \mathcal{F}_n.$$

<u>The submartingale</u> $(G_n)_{n \in N}$ <u>is always</u> L^1-<u>bounded and even uniformly integrable</u>.

<u>Proof:</u> We note that (take $\sigma = \tau$)

(4) $\gamma_n(A) \leq \int_A |X| dP$, for all $A \in \mathcal{F}_n$.

The existence of the submartingale $(G_n)_{n \in N}$ follows by an argument similar to that used in the proof of Proposition 1. The L^1-boundedness of $(G_n)_{n \in N}$ and even the uniform integrability of $(G_n)_{n \in N}$ follow from inequality (4) (see the argument in the proof of Corollary 2 above).

Definition 4. We call the sequence $(G_n)_{n \in N}$ of Proposition 2 *the submartingale of type* (II) *associated with* X, *the sequence* $(X_n)_{n \in N}$ *and the set* **S**.

§3. The main result: Submartingale characterization of measurable cluster points.

The result is the following:

Theorem 1. *Suppose that there is a sequence* $(\tau(n))_{n \in N}$ *with* $\tau(n) \in \mathbf{S}(n)$ *such that* $(X_{\tau(n)})_{n \in N}$ *is* L^1-*bounded. Let* $Y \in L^1$ *and let* $(S_n)_{n \in N}$ *be the submartingale of type* (I) *associated with* Y, *the sequence* $(X_n)_{n \in N}$ *and the set* **S**. *Then the following assertions are equivalent*:

(i) *The r.v.* Y *is a measurable cluster point of the sequence* $(X_n)_{n \in N}$ *relative to* **S**, *that is*, $Y \in \mathcal{MC}[(X_n)_{n \in N}; \mathbf{S}]$.

(ii) *The submartingale* $(S_n)_{n \in N}$ *converges to zero a.s.*

Proof: (ii) => (i). By assumption $S_n \to 0$ in probability. Thus for each $n \in N$ we can find an integer $k(n) \geq n$ and a set $A(n) \in \mathcal{F}_{k(n)}$ such that

$$\mu_{k(n)}(A(n)) = \int_{A(n)} S_{k(n)} dP < \frac{1}{n} \quad \text{and} \quad P((A(n))^c) < \frac{1}{n}.$$

By the definition of $\mu_{k(n)}$ there is then $\tau_n \in \mathbf{S}(k(n))$ such that

$$\int_{A(n)} |Y - X_{\tau_n}| dP < \frac{1}{n} \quad \text{and of course} \quad P((A(n))^c) < \frac{1}{n}.$$

It is then clear that $\tau_n \in \mathbf{S}(n)$ for all $n \in N$ and that $X_{\tau_n} \to Y$ in probability.

Thus $Y \in \mathcal{MC}[(X_n)_{n \in N}; S]$.

(i) => (ii). Let $(\xi(n))_{n \in N}$ be a sequence with $\xi(n) \in S(n)$ such that $X_{\xi(n)} \to Y$ a.s.

By Corollary 1 in Section 2, the submartingale $(S_n)_{n \in N}$ is L^1-bounded and hence by the "Doob a.s. convergence theorem for submartingales" (see for instance [8], p. 63), $\lim_n S_n(\omega)$ exists a.s.; to identify the limit it suffices to show that for some sequence of stopping times $(\sigma_k)_{k \in N}$ with $\sigma_k \in S(k)$ we have

(1) $S_{\sigma_k} \to 0$ in probability.

By assumption Y is integrable and Y coincides a.s. with an $\mathcal{F}_\infty$-measurable r.v.; hence if we let $Y_n = E^{\mathcal{F}_n}(Y)$, then $\|Y - Y_n\|_1 \to 0$ (see for instance [8], pp. 103-104). In particular then $Y_n - X_{\xi(n)} \to 0$ in probability. Choose now an increasing sequence of integers (n_k) such that

$$(2) \qquad \begin{cases} \|Y - Y_{n_k}\|_1 \le \dfrac{1}{k} \\ P(\{|Y_{n_k} - X_{\xi(n_k)}| \ge \dfrac{1}{k}\}) \le \dfrac{1}{k}. \end{cases}$$

Since Y_{n_k} is $\mathcal{F}_{n_k}$-measurable and $n_k \le \xi(n_k)$, the set $B(k) = \{|Y_{n_k} - X_{\xi(n_k)}| < \frac{1}{k}\}$ belongs to $\mathcal{F}_{\xi(n_k)}$. Using Corollary 3 in Section 2 and (2) above we deduce

$$\int_{B(k)} S_{\xi(n_k)} dP \le \int_{B(k)} |Y - X_{\xi(n_k)}| dP \le \frac{1}{k} + \int_{B(k)} |Y_{n_k} - X_{\xi(n_k)}| dP$$

$$\le \frac{1}{k} + \frac{1}{k} = \frac{2}{k}$$

and of course $P((B(k))^c) \le 1/k$. Setting $\sigma_k = \xi(n_k)$ yields (1) and thus finishes the proof.

<u>Remark</u>. The above theorem gives (under suitable assumptions) a characterization of the integrable elements $Y \in \mathcal{MC}[(X_n)_{n \in N}; S]$. This extends Theorem 1 of [4].

§4. Consequences

From Theorem 1 we easily obtain the following result which generalizes a theorem of Baxter [2] (see also Theorem 2 of [4]):

Theorem 2. Suppose that there is a sequence $(\tau(n))_{n \in N}$ *with* $\tau(n) \in \mathbf{S}(n)$ *such that* $(X_{\tau(n)})_{n \in N}$ *is* L^1*-bounded. Let* Y *and* Z *be integrable elements of* $\mathfrak{M}\mathcal{C}[(X_n)_{n \in N};\mathbf{S}]$. *Then the submartingale of type* (II) *associated with* $X = Y - Z$, *the sequence* $(X_n)_{n \in N}$ *and the set* $\mathbf{S}$ *is identically zero and hence there are sequences* $(\sigma'(k))_{k \in N}$ *and* $(\sigma''(k))_{k \in N}$ *with* $\sigma'(k) \in \mathbf{S}(k)$, $\sigma''(k) \in \mathbf{S}(k)$ *such that*

$$\lim_k \|(Y - Z) - (X_{\sigma'(k)} - X_{\sigma''(k)})\|_1 = 0.$$

Proof: Let $(S_n)_{n \in N}$ — respectively $(T_n)_{n \in N}$ — be the submartingales of type (I) associated with Y, the sequence $(X_n)_{n \in N}$ and the set $\mathbf{S}$ — respectively with Z, the sequence $(X_n)_{n \in N}$ and the set $\mathbf{S}$. Let $(G_n)_{n \in N}$ be the submartingale of type (II) associated with $X = Y - Z$, the sequence $(X_n)_{n \in N}$ and the set $\mathbf{S}$. Now S_n, T_n, G_n correspond respectively to the set functions μ_n, ν_n and γ_n defined on $\mathcal{F}_n$. From the obvious inequality $\gamma_n \leq \mu_n + \nu_n$ follows that $0 \leq G_n \leq S_n + T_n$ for each $n \in N$. By Theorem 1 in Section 3, $\lim_n S_n(\omega) = \lim_n T_n(\omega) = 0$ a.s. We deduce that

$$\lim_n G_n(\omega) = 0 \quad \text{a.s.}$$

But $(G_n)_{n \in N}$ is uniformly integrable by Proposition 2 in Section 2; as the sequence $\left(\int G_n dP\right)_{n \in N}$ increases and must converge to zero, we deduce the desired conclusion: $G_n = 0$ a.s. for all $n \in N$.

We shall need two more observations which we state in the form of lemmas:

Lemma 1. For each $n \in N$ we have

$$\sup_{\substack{(\sigma,\tau) \\ \sigma,\tau \in \mathbf{S}(n)}} \int (X_\tau - X_\sigma)dP = \sup_{\substack{(\sigma,\tau) \\ \sigma,\tau \in \mathbf{S} \\ \tau \geq \sigma \geq n}} \int |E^{\mathcal{F}_\sigma}(X_\tau) - X_\sigma| dP$$

Proof: Easy: Note that for $\sigma,\tau \in \mathbf{S}$, the set $A = \{\sigma \leq \tau\}$ belongs to both $\mathcal{F}_\sigma$ and $\mathcal{F}_\tau$ [respectively, for $\sigma,\tau \in \mathbf{S}$ with $\tau \geq \sigma$, the set $B = \{X_\sigma \leq E^{\mathcal{F}_\sigma}(X_\tau)\}$ belongs to $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$] and then use the "localization" property b) of $\mathbf{S}$.

Lemma 2. Let Y and Z be elements of $\mathcal{MC}[(X_n)_{n \in N};\mathbf{S}]$. Then $Y \vee Z$ and $Y \wedge Z$ also belong to $\mathcal{MC}[(X_n)_{n \in N};\mathbf{S}]$.

Proof: Elementary (use again the "localization" property of $\mathbf{S}$).

Using Lemmas 1 and 2 we may easily derive the following corollary of Theorem 2 which extends the "Generalized Fatou Inequality" of Chacon ([5]; see also [2] and [4]):

Theorem 3 (Generalized Fatou Inequality). Suppose that there is a sequence $(\tau(n))_{n \in N}$ with $\tau(n) \in \mathbf{S}(n)$ such that $(X_{\tau(n)})_{n \in N}$ is L^1-bounded. Let Y and Z be integrable elements of $\mathcal{MC}[(X_n)_{n \in N};\mathbf{S}]$. Then we have for each $n \in N$:

$$\text{(I)} \qquad \int (Y - Z)\,dP \leq \sup_{\substack{(\sigma,\tau) \\ \sigma,\tau \in \mathbf{S}(n)}} \int (X_\tau - X_\sigma)\,dP;$$

or alternatively,

$$\text{(I')} \qquad \int |Y - Z|\,dP \leq \sup_{\substack{(\sigma,\tau) \\ \sigma,\tau \in \mathbf{S} \\ \tau \geq \sigma \geq n}} \int |E^{\mathcal{F}_\sigma}(X_\tau) - X_\sigma|\,dP.$$

Remarks. 1) For other related results, such as the "amart convergence theorem" see for instance [4] (see also [1],[6],[3]).

2) Further applications of the above techniques will be given in a forthcoming paper.

Acknowledgment. I am indebted to J. L. Doob for comments that considerably improved the terminology of this paper.

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