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A FEW REMARKS ON THE RESULTS OF ROSINSKI AND SUCHANECKI CONCERNING UNCONDITIONAL CONVERGENCE AND C-SEQUENCES

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In this exposé I would like to present some of the recent preliminary results obtained by my students J. Rosinski and Z. Suchanecki (the full version of their paper to be published elsewhere). The results deal with unconditional convergence of series in metric groups, unconditional almost sure convergence of group valued functions and the C-sequences i.e. the sequences in F-spaces that are summable after multiplication by all c_o-multipliers. In spite of their elementary character these results to some extent complement the work on the topic done, mainly in Paris, during last few years (cf. References).

§ I. UNCONDITIONAL CONVERGENCE IN METRIC GROUPS

Let (G,d) be an abelian, metric, complete group. We may assume (cf. [5]) that d is invariant under translations. Usage : $\|g\| = d(g,0)$, $g \in G$. We say that the series Σg_n, $g_n \in G$, is unconditionally convergent if it remains convergent after every permutation of its terms. The following "uniformization" lemma is a convenient tool.

Lemma 1 : The series Σg_n, $g_n \in G$, is unconditionnally convergent if and only if $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}\ \forall$ permutation (n_k) of $[N,\infty[$ and $\forall\ 1 \le M_1 \le M_2 < \infty$

$$\left\| \sum_{k=M_1}^{M_2} g_{n_k} \right\| < \varepsilon .$$

Proof : The implication $\Leftarrow$ is obvious. We prove $\Rightarrow$. Assume that $\exists\ \varepsilon > 0\ \forall\ N\ \exists$ permutation (n_k) of $[N,\infty[$ and $\exists\ M_1 \le M_2$ such that

$$\left\| \sum_{k=M_1}^{M_2} g_{n_k} \right\| > \varepsilon .$$

Now, if $N = 1$ we find a permutation $(n_k^{(1)})$ of $[1,\infty[$ and $M_1^{(1)} \le M_2^{(1)}$ such that

$$\left\| \sum_{k=M_1^{(1)}}^{M_2^{(1)}} g_{n_k^{(1)}} \right\| > \varepsilon .$$

Let $A_1 = \max\{n_k^{(1)} : M_1^{(1)} \le k \le M_2^{(1)}\}$. Take $N = A_1 + 1$ and find a permutation $(n_k^{(2)})$ of $[A_1 + 1,\infty[$ and $M_1^{(2)} \le M_2^{(2)}$ such that

$$\left\| \sum_{k=M_1^{(2)}}^{M_2^{(2)}} g_{n_k^{(1)}} \right\| > \varepsilon ,$$

and let $A_2 = \max\{n_k^{(1)} : M_1^{(2)} \leq k \leq M_2^{(2)}\}$. Proceeding as above one chooses accordingly $n_k^{(i)}$, $M_1^{(i)}$, $M_2^{(i)}$, $i = 1,2,\ldots$. Denoting by h_j those g_j which are not of the form $g_{n_k^{(i)}}$, $i = 1,2,\ldots$, $M_1^{(i)} \leq k \leq M_2^{(i)}$ we see that the series

$$\sum_{k=M_1^{(1)}}^{M_2^{(1)}} g_{n_k^{(1)}} + h_1 + \sum_{k=M_1^{(2)}}^{M_2^{(2)}} g_{n_k^{(2)}} + h_2 + \ldots$$

does not converge. A contradiction.

<u>Proposition 1</u> : <u>The series</u> Σg_n, $g_n \in G$, <u>is unconditionnally convergent if and only if for each bounded sequence of integers</u> $(i_n) \subset \mathbb{N}$ <u>the series</u> $\Sigma i_n g_n$ <u>is convergent</u>.

<u>Proof</u> : The implication $\Leftarrow$ with $i_n = \pm 1$ is essentially due to Orlicz (cf. [10]). To prove $\Longrightarrow$ assume Σg_n converges unconditionally. By Orlicz's theorem (cf. [10]) for arbitrary $\varepsilon_1, \varepsilon_2, \ldots = \pm 1$, the series $\Sigma \varepsilon_n g_n$ is unconditionnally convergent so that by lemma 1 $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}\ \forall$ permutation (n_k) of $[N,\infty[\ \ \forall\ M_1 \leq M_2$

$$\left\| \sum_{k=M_1}^{M_2} \varepsilon_{n_k} g_{n_k} \right\| < \varepsilon \quad .$$

Assume that (i_n) is bounded by i_o. By the above statement we get that $\forall\ \varepsilon > 0\ \exists\ N\ \forall\ M_1 \leq M_2$

$$\left\| \sum_{n=M_1}^{M_2} i_n g_n \right\| \leq 1 \left\| \sum_{n=M_1}^{M_2} \varepsilon_n g_n^{(1)} \right\| + 2 \left\| \sum_{n=M_1}^{M_2} \varepsilon_n g_n^{(2)} \right\| + \ldots + i_o \left\| \sum_{n=M_1}^{M_2} \varepsilon_n g_n^{(i_o)} \right\|$$

$$\leq \frac{i_o(i_o + 1)}{2}\, \varepsilon \quad ,$$

where $\varepsilon_n = \operatorname{sgn} i_n$ and $g_n^{(i)} = g_n$ if $i_n = i$ and $g_n^{(i)} = 0$ if $i_n \neq i$, $i = 1,\ldots,i_o$. That ends the proof.

<u>Theorem 1</u> : <u>If the series</u> Σg_n, $g_n \in G$, <u>is unconditionally convergent then there exists a sequence of integers</u> $(i_n) \subset \mathbb{N}$, $i_n \uparrow \infty$ <u>such that</u> $\Sigma i_n g_n$ <u>is unconditionally convergent</u>.

<u>Proof</u> : By Lemma 1 $\forall\ 1 \le r < \infty\ \exists\ N_r\ \forall$ permutation (n_k) of $[N_r, \infty[\ \forall\ M_1 \le M_2$ we have

$$\left\| \sum_{k=M_1}^{M_2} g_{n_k} \right\| < 3^{-r} .$$

We may assume that $N_1 < N_2 < \dots$. Put $i_n = r$ for $N_r \le n < N_{r+1}$. We shall show that the series $\sum i_n g_n$ satisfies the Cauchy condition for each permutation of its term. Assume that it is not the case i.e. that $\exists$ permutation (m_k) of $\mathbb{N}$ such that $\exists\ \varepsilon_o > 0\ \exists\ M_1 \le M_2 \le \dots$ such that

$$(*) \qquad \left\| \sum_{k=M_r}^{M_{r+1}-1} i_{m_k} g_{m_k} \right\| > \varepsilon_o , \quad r = 1,2,\dots .$$

Now, take $s \in \mathbb{N}$ so that $2^{-s} < \varepsilon_o$ and then take r_o such that for $k \ge M_{r_o}$, $m_k > N_s$. Then

$$\left\| \sum_{k=M_{r_o}}^{M_{r_o+1}-1} i_{m_k} g_{m_k} \right\| = \left\| \sum_{j=s}^{\infty} \sum_{\substack{k=M_{r_o} \\ N_j \le m_k < N_{j+1}}}^{M_{r_o+1}-1} i_{m_k} g_{m_k} \right\|$$

$$\le \sum_{j=s}^{\infty} j \left\| \sum_{\substack{k=M_{r_o} \\ N_j \le m_k < N_{j+1}}}^{M_{r_o+1}-1} g_{m_k} \right\| \le \sum_{j=s}^{\infty} j\, 3^{-j} < 2^{-s} < \varepsilon_o$$

what contradicts (*).

In the case of G being a linear space and of real (instead of integer) multipliers the picture is much different as there are examples [11] such spaces in which there exist unconditionnally convergent series $\sum x_n$ such that for some bounded multipliers $(\alpha_n) \subset \mathbb{R}$ the series $\sum \alpha_n x_n$ diverges. Some linear metric spaces in which unconditional convergence implies bounded multiplier convergence are found in [10], [13] and in references quoted therein. However, from the above theorem one gets immediately the following

Corollary 1 : *Let $f_n \in L^{\Phi}(\Omega,\mathfrak{F},\mu\,;E)$ where Φ is a (possibly bounded) Orlicz function, $\mu \geq 0$, $\mu(\Omega) < \infty$, and E is a Banach space. If the series Σf_n is unconditionally convergent in L^{Φ} then there exists a sequence $(i_n) \subset \mathbb{N}$, $i_n \uparrow \infty$ such that for $\forall\ (\alpha_n) \subset \mathbb{R}$, $|\alpha_n| \leq i_n$, the series $\Sigma \alpha_n f_n$ converges in L^{Φ} (unconditionally).*

Proof : Apply Theorem 1 with $G = L^{\Phi}(\Omega,\mathfrak{F},\mu\,;E)$ and $\|.\|$ being the usual Orlicz F-norm. Thus $\exists\ (i_n) \subset \mathbb{N}$, $i_n \uparrow \infty$, such that $\Sigma i_n f_n$ converges unconditionally in L^{Φ}. Put $\beta_n = \alpha_n / i_n$. Then $|\beta_n| \leq 1$ and by the main result of [13] $\Sigma \beta_n i_n f_n = \Sigma \alpha_n f_n$ converges in L^{Φ}.

§ II. UNCONDITIONAL ALMOST SURE CONVERGENCE FOR GROUP-VALUED FUNCTIONS

Consider a sequence (f_n) of measurable functions on the finite measure space $(\Omega,\mathfrak{F},\mu)$ with values in $(G,\|.\|)$. We say that the series Σf_n is convergent unconditionally almost everywhere if after every permutation of its terms the series converges μ-a.e.

Notation : $\|f\|_F = \int_\Omega [\|f(\omega)\|/(1+\|f(\omega)\|)]\mu(d\omega)$. As in Section I, the following uniformization lemma will be instrumental.

Lemma 2 : *The series Σf_n is unconditionally a.e. convergent if and only if $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}\ \forall$ permutation (n_k) of $[N,\infty[$ $\forall$ bounded integer valued measurable functions $M_1(\omega) \leq M_2(\omega)$ we have that*

$$\Big\| \sum_{k=M_1}^{M_2} f_{n_k} \Big\|_F < \varepsilon .$$

Proof : $\Rightarrow$. Firstly, let us notice that

$$(**) \qquad \|f\|_F > \varepsilon \text{ implies that } \mu\{\omega : \|f(\omega)\| > \tfrac{\varepsilon}{2}\} > \tfrac{\varepsilon}{2} .$$

Assume to the contrary that $\exists\ \varepsilon > 0\ \forall\ N\ \exists$ permutation (n_k) of $[N,\infty[$ $\exists$ bounded measurable functions $M_1(\omega) \leq M_2(\omega)$ such that

$$\Big\| \sum_{k=M_1}^{M_2} f_{n_k} \Big\|_F > \varepsilon .$$

If $N=1$ we find a permutation $(n_k^{(1)})$ of $[1,\infty[$ and $M_1^{(1)}(\omega) \le M_2^{(1)}(\omega)$ such that, by (**)

$$\frac{\varepsilon}{2} < \mu\{\omega : \|\sum_{k=1}^{M_2^{(1)}(\omega)} f_{n_k^{(1)}}(\omega)\| + \|\sum_{k=1}^{M_1^{(1)}(\omega)-1} f_{n_k^{(1)}}(\omega)\| > \frac{\varepsilon}{2}\}$$

$$\le \mu\{\omega : 2 \max_{1\le k\le M^{(1)}} \|\sum_{j=1}^{k} f_{n_j^{(1)}}(\omega)\| > \frac{\varepsilon}{2}\}$$

where $M^{(1)} = \max_{\omega} M_2^{(1)}(\omega)$. Then put $A_1 = \max\{n_k^{(1)} : k \le M^{(1)}\}$ and find a permutation $(n_k^{(2)})$ of $[A_1+1,\infty[$ and an integer $M^{(2)}$ such that

$$\frac{\varepsilon}{2} < \mu\{\omega : 2 \max_{1\le k\le M^{(2)}} \|\sum_{j=1}^{k} f_{n_j^{(2)}}(\omega)\| > \frac{\varepsilon}{2}\} \quad .$$

Proceeding by induction we find permutations $(n_k^{(i)})$ and integers $M^{(i)}$, $i = 1,2,\ldots$ such that the series

$$\sum_{k=1}^{M^{(1)}} f_{n_k^{(1)}} + \varphi_1 + \sum_{k=1}^{M^{(2)}} f_{n_k^{(2)}} + \varphi_2 + \cdots$$

(where φ_j are those f_j that are missing in the sums constructed above) does not satisfy the Cauchy condition for a.e. convergence. A contradiction.

The implication $\Leftarrow$ follows immediately from the lemma of [10] which says that if $\Sigma f_n(\omega)$ diverges on the set of positive measure then $\exists\ \varepsilon > 0\ \exists\ F$, $\mu(F) > 0\ \exists\ N_1 < N_2 < \ldots$ such that

$$\max_{N_i\le k\le N_{i+1}} \|\sum_{j=N_i}^{k} f_j(\omega)\| > \varepsilon \ , \quad \forall\ \omega \in F,\ i = 1,2,\ldots \ .$$

Exactly as in Section I (Proposition 1), using Lemma 2 and Theorem 2 of [10] one can prove (but only one implication can be proved this way !)

<u>Proposition 2</u> : <u>If the series</u> Σf_n <u>is unconditionally almost everywhere convergent then for each bounded sequence</u> $(i_n) \subset \mathbb{N}$ <u>the series</u> $\Sigma i_n f_n$ <u>is convergent a.e.</u>

Now, one can prove, following the lines of the proof of Theorem 1 (or of [6]), but utilizing Lemma 2 instead of Lemma 1, the following

Theorem 2 : If the series Σf_n is convergent unconditionally a.e. then one can find a sequence of integers (i_n), $i_n \uparrow \infty$ such that $\Sigma i_n f_n$ converges uncondionally a.e.

Using the above Theorem 2 and the Theorem 2 from [10] one gets immediately Theorem 3 of [10] as a

Corollary 2 : If $(G, \|.\|)$ is a Banch space and if the series Σf_n is uncondionally a.e. convergent then $\exists\, (i_n)$, $i_n \uparrow \infty$ such that $\forall\, (\alpha_n) \subset \mathbb{R}$, $|\alpha_n| < i_n$ the series $\Sigma \alpha_n f_n$ converges unconditionally a.e.

§ III. C-SEQUENCES AND C-SPACES

Recall that a sequence (x_n) of an F-space X is said to be a C-sequence if $\forall\, (\alpha_n) \in c_o$ the series $\Sigma \alpha_n x_n$ converges. An F-space X is said to be a C-space if for any C-sequence $(x_n) \subset X$ the series Σx_n converges. L. Schwartz [12] has proved that if in an F-space X every C-sequence converges to zero then X is a C-space, and Bessaga-Pełczynski [1] have shown that if X is not a C-space then it contains a subspace isomorphic to c_o. Some examples of C-spaces are in [12] and [14] and in [2] one can find alternative characterizations of C-spaces.

In what follows $(E, \|.\|)$ will be a real Banach space and we shall say that E is of cotype p $(2 \le p < \infty)$ if the convergence a.s. of the series $\Sigma \varepsilon_n x_n$ (ε_n-Rademachers) implies that $\Sigma \|x_n\|^p < \infty$. Equivalently E is of cotype-p iff $\exists\, C > 0\ \forall\, n \in \mathbb{N}\ \forall\, x_1, \ldots, x_n \in E$

$$(***) \qquad \sum_{i=1}^{n} \|x_i\|^p \le C\, E \,\Big\| \sum_{i=1}^{n} \varepsilon_i\, x_i \Big\|^p$$

(cf. e.g. [9]). We shall also have need of the following "uniformization" Lemma the proof thereof is a straightforward adaptation of the proof of Lemma 1 from [10].

Lemma 3 : If the series Σf_n, $f_n \in L^o(T,\mathfrak{F},\mu\,;E)$ is unbounded on a set of positive measure μ then $\exists\, F \in \mathfrak{F}$, $\mu(F) > 0$ and $\exists\, N_1 < N_2 < \ldots$ such that

$$\max_{N_i \le k < N_{i+1}} \Big\| \sum_{j=N_i}^{k} f_j(t) \Big\| > i \quad , \quad \forall\, t \in F,\ i = 1,2,\ldots \quad .$$

The theorem proved below is, in the case $E = \mathbb{R}$, due to Kolmogorov and Khinchine [7] (cf. also Kwapien [8]) but the proof given below differs essentially from those of [7] and [8].

Theorem 3 : Let E be a Banch space. Then the following two conditions are equivalent

(a) for arbitrary finite measure space $(T,\mathfrak{F},\mu)$ and for arbitrary C-sequence $(f_n) \subset L^o(T,\mathfrak{F},\mu\,;E)$ the series $\Sigma \|f_n(t)\|^p$ is μ-a.e. convergent

(b) E is of cotype p.

Proof : (a) $\Rightarrow$ (b). Let $T = [0,1]$, $\mathfrak{F}$-Borel σ-algebra, μ-Lebesgue measure. Take $(x_n) \subset E$ such that $\Sigma r_n(t)\, x_n$ is μ-a.e. convergent, r_n being the usual Rademacher functions. By the contraction principle (Kahane [4]) for any $(\alpha_n) \in c_o$, $\Sigma \alpha_n r_n(t) x_n$ converges μ-a.e. what implies that $(r_n x_n)$ is a C-sequence in $L^o(T,\mathfrak{F},\mu\,;E)$. Hence the series $\Sigma \|r_n(t)x_n\|^p = \Sigma \|x_n\|^p$ converges so that E is of cotype p.

(b) $\Rightarrow$ (a). Assume $\mu(T) = 1$. Let $\varepsilon_1, \varepsilon_2, \ldots$ be some Rademacher's r.v.s. on certain probability space (Ω,P). We first show that for arbitrary C-sequence $(f_n) \subset L^o(T,\mathfrak{F},\mu\,;E)$ the series $\Sigma\, \varepsilon_n(\omega)\, f_n(t)$ is $P \times \mu$-a.e. bounded on $\Omega \times T$. Indeed, assume it is not. Then by Lemma 3 $\exists\, \delta > 0$ and $N_1 < N_2 < \ldots$ such that

$$(P \times \mu)\Big\{ \max_{N_i \le k < N_{i+1}} \Big\| \sum_{j=N_i}^{k} \varepsilon_j(\omega) f_j(t) \Big\| > i \Big\} \ge \delta \quad .$$

By the Fubini theorem, $\forall\, i = 1,2, \ldots$

$$\delta \le \int_T P\Big\{ \max_{N_i \le k < N_{i+1}} \Big\| \sum_{j=N_i}^{k} \varepsilon_j(\omega) f_j(t) \Big\| > i \Big\}\, \mu(dt)$$

$$\le 2 \int_T P\Big\{ \Big\| \sum_{j=N_i}^{N_{i+1}-1} \varepsilon_j(\omega) f_j(t) \Big\| > i \Big\}\, \mu(dt)$$

$$= 2 (P \times \mu)\Big\{ \Big\| \sum_{j=N_i}^{N_{i+1}-1} \varepsilon_j(\omega) f_j(t) \Big\| > i \Big\} \quad .$$

the last inequality being motivated by the fact that for fixed t $\varepsilon_j(\omega)f_j(t)$, $j = 1,2, \ldots$ form a sequence of independent and symmetric random vectors [3]. Therefore $\forall\, i\ \exists\ \omega_i$ such that

$$\mu\{\|\sum_{j=N_i}^{N_{i+1}-j} \varepsilon_j(\omega_i)f_j(t)\| > i\} \geq \frac{\delta}{2} \quad .$$

Taking $\alpha_j = \varepsilon_j(\omega_i)/\sqrt{i}$ for $N_i \leq j < N_{i+1}$, $(\alpha_j) \in c_o$, and we get that

$$\mu\{\|\sum_{j=N_i}^{N_{i+1}-1} \alpha_j f_j\| > \sqrt{i}\} \geq \frac{\delta}{2}$$

what contradicts the assumption that (f_n) is a C-sequence in $L^o(T,\mathfrak{J},\mu\,;E)$.

Now, by the Fubini theorem for μ almost all $t \in T$

$$M(\omega) = \sup_n \|\sum_{j=1}^{n} \varepsilon_i(\omega)f_i(t)\| < \infty$$

with probability $P = 1$. Then by Th. 2.4 of [4] $M^p \in L^1$ for each $p \geq 1$. Because E is of cotype p then by (***) we get that

$$\sum_{j=1}^{n} \|f_j(t)\|^p \leq C \int_\Omega \|\sum_{j=1}^{n} \varepsilon_j(\omega)f_j(t)\|^p dP \leq C \int_\Omega M^p(\omega)\,dP$$

so that $\sum\|f_n(t)\|^p < \infty$ for μ-a.a. $t \in T$. Q.E.D.

Using the above Theorem 3 one can get (as e.g. in L. Schwartz [12])

<u>Corollary 3</u> : <u>If</u> E <u>is of cotype</u> p <u>for some</u> $2 \leq p < \infty$ <u>then</u> $\forall\, q$, $0 \leq q < \infty$, $L^q(T,\mathfrak{J},\mu\,;E)$ <u>is a</u> C-<u>space</u>.

Same results for Orlicz spaces of E valued functions analogous to [14]. Gilles Pisier made a remark to the effect that in view of the, rather deep, results of J. Hoffmann-Jørgensen [15] and S. Kwapień [16] one can obtain the more general result saying that E is a C-space iff $\forall\, q$, $0 \leq q < \infty$, $L^q(E)$ is a C-space. For $1 \leq q < \infty$ this actually is proven in [15] and [16].

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