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# A criterium for monomiality

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ABSTRACT - Let $G$ be a solvable group. An odd degree rational valued character $\chi$ of $G$ is induced from a linear character of some subgroup of $G$. We extend this result to odd degree characters $\chi$ of $G$ that take values in certain cyclotomic extensions of $\mathbb{Q}$.

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## 1. Introduction

It follows from a well-known result of R. Gow [1] that an odd-degree rational-valued irreducible character of a solvable group is monomial. In this note, we slightly generalize Gow's result.

If $\chi$ is an irreducible complex character of a finite group $G$, let

$$n(\chi) = \gcd(n \mid \mathbb{Q}(\chi) \subseteq \mathbb{Q}_n)\,,$$

where $\mathbb{Q}(\chi)$ is the smallest field containing the values of $\chi$, and we denote by $\mathbb{Q}_n$ the $n$-th cyclotomic field.

THEOREM A. *Let $G$ be a solvable group. Let $\chi \in \mathrm{Irr}(G)$. If $\chi(1)$ is odd and $\gcd(\chi(1), n(\chi)) = 1$, then there exist a subgroup $U \subseteq G$ and a linear character $\lambda$ of $U$ such that $\lambda^G = \chi$. Moreover, if $\mu$ is a linear character of some subgroup $W \subseteq G$ and $\mu^G = \chi$, then there exists $g \in G$ such that $W = U^g$ and $\mu = \lambda^g$.*

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Theorem A is not true if $\chi(1)$ is even or if $G$ is not solvable ($SL_2(3)$ has a rational valued non-monomial character of degree 2, and $A_6$ has a rational valued non-monomial character of degree 5).

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## 2. Proof of Theorem A

We use the notation of [2]. First, we prove by induction on $|G|$ that $\chi$ is monomial.

STEP 1. We may assume $\chi$ is faithful and that there is no proper subgroup $H < G$ and $\psi \in \mathrm{Irr}(H)$ such that $\psi^G = \chi$ and $\mathbb{Q}(\psi) \subseteq \mathbb{Q}(\chi)$.

Let $K = \ker\chi$. If $K > 1$, all the hypotheses hold in $G/K$ so by induction we are done. For the second part $\psi(1)$ divides $\chi(1)$, and $n(\psi)$ divides $n(\chi)$, so $\gcd(\psi(1), n(\psi)) = 1$, and the inductive hypothesis applies.

STEP 2. $\mathbf{F}(G) = \prod_{p \nmid \chi(1)} \mathbf{O}_p G$.

Let $p$ be a prime. Suppose that $p$ divides $\chi(1)$. In particular $p$ is odd, and $p$ does not divide $n(\chi)$. Let $M$ be a normal $p$-subgroup of $G$. Since $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{n(\chi)}$ we have that $\mathbb{Q}_{|M|} \cap \mathbb{Q}(\chi) = \mathbb{Q}$. Hence $\chi_M$ is rational-valued. If $\xi \in \mathrm{Irr}(M)$, then $[\chi_M, \xi] = [\chi_M, \bar{\xi}]$. Since $\chi(1)$ is odd, there exists a real irreducible constituent $\xi$ of $\chi_M$. Since $|M|$ is odd, we have that $\xi = 1_M$, by Burnside's theorem. By Step 1, we know that $\chi$ is faithful and we conclude $M = 1$.

STEP 3. $F = \mathbf{F}(G)$ is abelian.

Let $M$ be a normal $p$-subgroup of $G$, where $p$ does not divide $\chi(1)$. It then follows that the irreducible constituents of $\chi_M$ are linear. Let $\lambda \in \mathrm{Irr}(M)$ be under $\chi$. We have that $M' \subseteq \ker \lambda^g = \ker \lambda^g$ for every $g \in G$. Then $M' \subseteq \mathrm{core}_G(\ker(\lambda)) \subseteq \ker(\chi) = 1$, so that $M$ is abelian. Hence $F$ is abelian by Step 2.

STEP 4. Let $N \triangleleft G$ and let $\theta \in \mathrm{Irr}(N)$ be under $\chi$. Let $g \in G$. Then $\theta^g = \theta^\sigma$ for some $\sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$. Also $\theta$ is faithful.

Let $T = I_G(\theta)$ be the stabilizer of $\theta$ in $G$, and write $T^*$ for the semi-inertia subgroup of $\theta$, this is $T^* = \{g \in G | \theta^g = \theta^\sigma \text{ for some } \sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})\}$. By part ($b$) of Lemma 2.2 of [3], if $\psi \in \mathrm{Irr}(T|\theta)$ is the Clifford correspondent for $\chi$, then $\eta = \psi^{T^*} \in \mathrm{Irr}(T^*)$ induces $\chi$ and $\mathbb{Q}(\eta) = \mathbb{Q}(\chi)$. By Step 1, we have that $T^* = G$. So that every $G$-conjugate of $\theta$ is actually a Galois conjugate. Thus $\ker\theta^g = \ker\theta$ for every $g \in G$. It follows that $\ker(\theta) \triangleleft G$ and $\ker(\theta)$ is contained in $\ker(\chi)$ by Clifford's theorem. So $\theta$ is faithful by Step 1.

Step 5. If $\lambda \in \mathrm{Irr}(F)$ is under $\chi$, then $\lambda^G = \chi$.

Let $\lambda \in \mathrm{Irr}(F)$ be under $\chi$. If $y \in G$ is such that $\lambda^y = \lambda$, then we have $[x, y] \in \ker(\lambda)$ for every $x \in F$. By step 4, $\lambda$ is faithful, so the element $y$ centralizes $F$. Since $F$ is self-centralizing, necessarily $y \in F$. We have proved $I_G(\lambda) = F$. This implies $\lambda^G$ is irreducible and thus $\lambda^G = \chi$. This finishes the proof that $\chi$ is monomial.

Now, we work by induction on $|G|$ to show that if $U$ and $V$ are subgroups of $G$ and $\lambda \in \mathrm{Irr}(U)$ and $\mu \in \mathrm{Irr}(V)$ are linear such that $\lambda^G = \chi = \mu^G$, then the pairs $(U, \lambda)$ and $(V, \mu)$ are $G$-conjugate. Since $K = \ker(\chi) \subseteq \mathrm{core}_G(\ker(\lambda)) \cap \mathrm{core}_G(\ker(\mu))$ we may assume that $\chi$ is faithful, for if $K > 1$ then we can work in $G/K$. If $p$ is a prime not dividing $\chi(1)$, then $\mathbf{O}_p(G)$ is contained in both $U$ and $V$, because $|G : U| = \chi(1) = |G : V|$. By Step 2 (which only required that $\chi$ is faithful), we have that

$$\mathbf{F}(G) = \prod_{p \nmid \chi(1)} \mathbf{O}_p(G) \subseteq U \cap V\,.$$

Now $\lambda_F$ and $\mu_F$ are both under $\chi$. So that $\mu_F = (\lambda_F)^g$ for some $g \in G$ by Clifford's theorem. We may assume that $\mu_F = \lambda_F = \nu$, by replacing the pair $(U, \lambda)$ by some $G$-conjugate. Thus $U$ and $V$ are contained in $T = I_G(\nu)$ and also in $T^*$, the semi-interia subgroup of $\nu$. Since $\lambda^G$ and $\mu^G$ are irreducible, also $\lambda^T$ and $\mu^T$ are irreducible. By uniqueness of the Clifford correspondent, we deduce that $\lambda^T = \mu^T$. In particular $\lambda^{T^*} = \mu^{T^*} = \psi \in \mathrm{Irr}(T^*|\nu)$. We know that $\mathbb{Q}(\psi) = \mathbb{Q}(\chi)$, again using Lemma 2.2 of [3]. If $T^* < G$, then the result follows by induction. Hence, we may assume $T^* = G$. In particular, arguing as in the first part of the proof, we conclude that $\nu^G = \chi$. This implies that $U = F = V$ and the theorem is proven.

Recently we have given a related criterium for monomiality in which the odd degree hypothesis is replaced by certain oddness related to Sylow normalizers (see [4]). This result and our Theorem A seem to be independent. Under the hypothesis of Theorem A, it can also be proved that in fact $\chi$ is **supermonomial**, that is, that every character inducing $\boldsymbol{\chi}$ is monomial.

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