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The Category of Partial Doi-Hopf Modules and Functors

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ABSTRACT - Let (H, A, C), (H', A', C') be two partial Doi-Hopf datums consisting of a Hopf algebra H, a partial right H-comodule algebra A and a partial right H-module coalgebra. Given $\alpha : H \to H'$, $\beta : A \to A'$ and $\gamma : C \to C'$, we define an induction functor between the category $\mathcal{M}(H)_A^C$ of all partial Doi-Hopf modules and the category $\mathcal{M}(H')_{A'}^{C'}$, and we prove that this functor has a right adjoint. Specially, we then give necessary and sufficient conditions for the functor $F : \mathcal{M}(H)_A^C \to \mathcal{M}(H)_A$ (exactly the category of right A-modules). This leads to a generalized notion of integrals. Moreover, from these results, we deduce a version of Maschke-type Theorems for partial Doi-Hopf modules. The applications of our results are considered.

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1. Introduction

Partial group actions were considered first by Exel [E1] in the context of operator algebras and they turned out to be a powerful tool in the study of C^*-algebras generated by partial isometries on a Hilbert space [E2]. A treatment from a purely algebraic point of view was given recently in

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[DEP], [DFP], [DZ] and [DE]. Partial Hopf actions were motivated by an attempt to generalize the notion of partial Galois extensions of commutative rings in [DFP] to a broader context. The definition of partial Hopf actions and co-actions were introduced by using the notions of partial entwining structures in [CJ].

The category $\mathcal{M}(H)_A^C$ of Doi-Hopf modules was introduced in [D], where H is a Hopf algebra, A a right H-comodule algebra and C a right H-module coalgebra. It is the category of the modules over the algebra A which are also comodules over the coalgebra C and satisfy certain compatibility condition involving H. The study of $\mathcal{M}(H)_A^C$ turned out to be very useful: it was shown in [D] that many categories such as the module and comodule categories over bialgebras, the Hopf modules category [S], Takeuchi's relative Hopf modules, graded modules, modules graded by G-sets, Long dimodules and the Yetter-Drinfeld category ([CMI], [RT], [Y]) are special cases of $\mathcal{M}(H)_A^C$.

As a general version of the category $\mathcal{M}(H)_A^C$ of Doi-Hopf modules, we shall consider a partial Doi-Hopf datum (H, A, C), and the category $\mathcal{M}(H)_A^C$ of so-called partial Doi-Hopf modules. The starting point of this paper is an attempt to discuss the results of [CR] in the partial case. This would have meant in particular giving generalizations of the induced and the coinduced functors. In the paper, we give a generalization of the induction functor and try to characterize the separability of the functor $F : \mathcal{M}(H)_A^C \to \mathcal{M}(H)_A$ which leads to the generalized integral of the partial Doi-Hopf datum (H, A, C). Also, a version of Maschke-type Theorems for partial Doi-Hopf modules is proved.

The paper is organized as follows.

In Section 2, we recall definitions and basic results related to Hopf partial action, and introduce the induction functor: given maps $\alpha : H \to H'$, $\beta : A \to A'$ and $\gamma : C \to C'$, we have a functor (called *induction functor*) $F = \bullet \otimes_A A' : \mathcal{M}(H)_A^C \to \mathcal{M}(H')_{A'}^{C'}$. This functor have a right adjoint G. In Section 3, we discuss the separability of the functor $F : \mathcal{M}(H)_A^C \to \mathcal{M}(H)_A$ that forgets the C-coaction,which leads to a generalized integral of a partial Doi-Hopf datum (H, A, C). The applications of our results are considered in Section 4.

2. Partial Module Coalgebras, Partial Comodule Algebras, Partial Doi-Hopf Modules

Throughtout this paper, k will be a field. Unless specified otherwise, all modules, algebras, coalgebras, bialgebras(or Hopf algebras), tensor products and homomorphisms are over k. ι denotes the identity mapping. H will be a Hopf algebra with an invertible antipode S and we will use Sweedler's sigma-notation extensively. For example, if $(C, \Delta_C, \varepsilon_H)$ is a coalgebra, then for all $c \in C$, we write

$$\Delta_C(c) = c_1 \otimes c_2 \in C \otimes C.$$

DEFINITION 2.1. Let H be a Hopf algebra. A k-algebra A is called a *partial right H-comodule algebra*, if there exists a k-linear map $\rho_A : A \to A \otimes H$, $\rho_A(a) = a_{[0]} \otimes a_{[1]}$ such that the following conditions satisfy:

$$\rho_A(ab) = \rho_A(a)\rho_A(b), \tag{2.1}$$

$$\rho_A(a_{[0]}) \otimes a_{[1]} = a_{[0]}1_{A[0]} \otimes a_{[1]1}1_{A[1]} \otimes a_{[1]2}, \tag{2.2}$$

$$\varepsilon(a_{[1]})(a_{[0]}) = a, \tag{2.3}$$

for all $a, b \in A$.

EXAMPLE 2.2. Let $e \in H$ be an idempotent such that $e \otimes e = \Delta(e)(e \otimes 1_H)$ and $\varepsilon(e) = 1$. Then we can define the following partial right H-coaction on $A = k$: $\rho(x) = x \otimes e \in k \otimes H$.

DEFINITION 2.3. Let H be a Hopf algebra. A k-coalgebra C is called a *partial right H-module coalgebra*, if there exists a k-linear map $\phi : C \otimes H \to C$, $\phi(c \otimes h) = c \cdot h$ such that the following conditions satisfy:

$$(c \cdot h) \cdot g = c \cdot hg, \tag{2.4}$$

$$(c \cdot h)_1 \cdot 1_H \otimes (c \cdot h)_2 = c_1 \cdot h_1 \otimes c_2 \cdot h_2, \tag{2.5}$$

$$\varepsilon(c \cdot h) = \varepsilon(c)\varepsilon(h), \tag{2.6}$$

for all $c \in C$ and $g, h \in H$.

EXAMPLE 2.4. Let $e \in H$ be a central idempotent such that $e \otimes e = \Delta(e)(e \otimes 1_H)$ and $\varepsilon(e) = 1$. Then we can define the partial right H-action on $C = H$: $g \cdot h = egh$.

A *partial Doi-Hopf datum* is a threetuple (H, A, C), where H is a Hopf algebra, A a partial right H-comodule algebra and C a partial right H-module coalgebra. Given a partial Doi-Hopf datum (H, A, C). A *partial*

Doi-Hopf module M is a right A-module and there exists a k-linear map $\rho : M \to M \otimes C$ such that

$$\rho_M^2(m) = m_{[0]} \cdot 1_{A[0][0]} \otimes m_{[1]1} \cdot 1_{A[0][1]} \otimes m_{[1]2} \cdot 1_{A[1]}, \tag{2.7}$$

where $\rho_M^2 = (\rho_M \circ \imath) \circ \rho_M$,

$$\rho(m \cdot a) = m_{[0]} \cdot a_{[0]} \otimes m_{[1]} \cdot a_{[1]}, \tag{2.8}$$

$$\varepsilon(m_{[1]}) m_{[0]} = m, \tag{2.9}$$

for all $m \in M$ and $a \in A$.

EXAMPLE 2.5. Let $e \in H$ be a central idempotent such that $e \otimes e = \Delta(e)(e \otimes 1_H)$ and $\varepsilon(e) = 1$. Then (H, k, H) is a partial Doi-Hopf datum.

$\mathcal{M}(H)_A^C$ will be the category of partial Doi-Hopf modules and A-linear, C-colinear maps. Now we can give the following result for the category of partial Doi-Hopf modules which is analogous of [Theorem 1.1, CR].

THEOREM 2.6. *Consider two partial Doi-Hopf datums* (H, A, C) *and* (H', A', C'), *and suppose that we have maps* $\alpha : H \to H'$, $\beta : A \to A'$ *and* $\gamma : C \to C'$ *which are respectively Hopf algebra, algebra and coalgebra maps satisfying*

$$\gamma(c \cdot h) = \gamma(c) \cdot \alpha(h), \tag{2.10}$$

$$\rho_A(\beta(a)) = \beta(a_{[0]}) \otimes \alpha(a_{[1]}), \tag{2.11}$$

for all $c \in C$, $h \in H$ *and* $a \in A$. *Then we have a functor* $F : \mathcal{M}(H)_A^C \to \mathcal{M}(H')_{A'}^{C'}$, *defined as follows:*

$$F(M) = M \otimes_A A',$$

where A' *is a left* A*-module via* β *and with structure maps defined by*

$$(m \otimes_A a') \cdot b' = m \otimes_A a'b', \tag{2.12}$$

$$\rho_{F(M)}(m \otimes_A a') = m_{[0]} \otimes_A a'_{[0]} \otimes \gamma(m_{[1]}) \cdot a'_{[1]}, \tag{2.13}$$

for all $a', b' \in A'$ *and* $m \in M$.

PROOF. Let us show that $M \otimes_A A'$ is an object of $\mathcal{M}(H')_{A'}^{C'}$. For this, we need to show that $M \otimes_A A'$ satisfies conditions (2.7)-(2.9). Notice that

$M \otimes_A A'$ satisfies (2.9) obviously. We restrict here to check that $M \otimes_A A'$ satisfies conditions (2.7) and (2.8). Take $m \in M$ and $a', b' \in A'$. Then

$$\begin{aligned}\rho_{F(M)}((m \otimes_A a') \cdot b') &= m_{[0]} \otimes_A a'_{[0]} b'_{[0]} \otimes \gamma(m_{[1]}) \cdot a'_{[1]} b'_{[1]} \\ &= (m_{[0]} \otimes_A a'_{[0]}) \cdot b'_{[0]} \otimes (\gamma(m_{[1]}) \cdot a'_{[1]}) \cdot b'_{[1]},\end{aligned}$$

i.e., (2.8) holds. For (2.7), for all $m \in M$ and $a' \in A'$, we have

$$\begin{aligned}&\rho^2_{F(M)}(m \otimes_A a') \\ &= m_{[0][0]} \otimes_A a'_{[0][0]} \otimes \gamma(m_{[0][1]}) \cdot a'_{[0][1]} \otimes \gamma(m_{[1]}) \cdot a'_{[1]} \\ &\overset{(2.7)}{=} m_{[0]} \cdot 1_{A[0]} \otimes_A a'_{[0][0]} \otimes \gamma(m_{[1]1} \cdot 1_{A[1]}) \cdot a'_{[0][1]} \otimes \gamma(m_{[1]2}) \cdot a'_{[1]} \\ &= m_{[0]} \otimes_A \beta(1_{A[0]}) a'_{[0][0]} \otimes \gamma(m_{[1]1}) \cdot \alpha(1_{A[1]}) a'_{[0][1]} \otimes \gamma(m_{[1]2}) \cdot a'_{[1]} \\ &\overset{(2.11)}{=} m_{[0]} \otimes_A 1_{A'[0]} a'_{[0][0]} \otimes \gamma(m_{[1]1}) \cdot 1_{A'[1]} a'_{[0][1]} \otimes \gamma(m_{[1]2}) \cdot a'_{[1]} \\ &\overset{(2.1)}{=} m_{[0]} \otimes_A a'_{[0][0]} \otimes \gamma(m_{[1]1}) \cdot a'_{[0][1]} \otimes \gamma(m_{[1]2}) \cdot a'_{[1]} \\ &= m_{[0]} \otimes_A a'_{[0]} 1_{A'[0][0]} \otimes \gamma(m_{[1]1}) \cdot a'_{[1]1} 1_{A'[0][1]} \otimes \gamma(m_{[1]2}) \cdot a'_{[1]2} 1_{A'[1]} \\ &\overset{(2.5)}{=} (m_{[0]} \otimes_A a'_{[0]}) \cdot 1_{A'[0][0]} \otimes (\gamma(m_{[1]}) \cdot a'_{[1]})_1 \cdot 1_{A'[0][1]} \otimes (\gamma(m_{[1]}) \cdot a'_{[1]})_2 \cdot 1_{A'[1]}.\end{aligned}$$

This is exactly what we have to show. □

THEOREM 2.7. *Under the assumptions of Theorem* 2.6, *we have a functor* $G : \mathcal{M}(H')^{C'}_{A'} \to \mathcal{M}(H)^C_A$ *which is right adjoint to* F. *G is defined by*

$$G(M') = \overline{M' \square_{C'} C} = \{m' \cdot \beta(1_{A[0]}) \otimes c \cdot 1_{A[1]}\},$$

where $m' \otimes c \in M' \otimes C$ *satisfies the following condition:*

$$m'_{[0]} \cdot \beta(1_{A[0][0]}) \otimes m'_{[1]} \cdot \alpha(1_{A[0][1]}) \otimes c \cdot 1_{A[1]} = m' \cdot \beta(1_{A[0][0]}) \otimes \gamma(c_1) \cdot \alpha(1_{A[0][1]}) \otimes c_2 \cdot 1_{A[1]}, \tag{2.14}$$

for all $M' \in \mathcal{M}(H')^{C'}_{A'}$, *and with structure maps*

$$\rho_{G(M')}(m' \cdot \beta(1_{A[0]}) \otimes c \cdot 1_{A[1]}) = m' \cdot \beta(1_{A[0][0]}) \otimes c_1 \cdot 1_{A[0][1]} \otimes c_2 \cdot 1_{A[1]}, \tag{2.15}$$

$$(m' \cdot \beta(1_{A[0]}) \otimes c \cdot 1_{A[1]}) \cdot a = m' \cdot \beta(a_{[0]}) \otimes c \cdot a_{[1]}, \tag{2.16}$$

for all $a \in A$.

PROOF. Let us first show that $G(M')$ is an object of $\mathcal{M}(H)^C_A$. It is routine to check that $G(M')$ is a right C-comodule. In order to prove that M is a right A-module, we need to show that $m' \cdot \beta(a_{[0]}) \otimes c \cdot a_{[1]} \in \overline{M' \square_{C'} C}$, for all

$a \in A$. Indeed,

$$
\begin{aligned}
&(m' \cdot \beta(a_{[0]}))_{[0]} \cdot \beta(1_{A[0][0]}) \otimes (m' \cdot \beta(a_{[0]}))_{[1]} \cdot \alpha(1_{A[0][1]}) \otimes c \cdot a_{[1]}1_{A[1]} \\
&\overset{(2.8)}{=} m'_{[0]} \cdot \beta(a_{[0]})_{[0]}\beta(1_{A[0][0]}) \otimes m'_{[1]} \cdot \beta(a_{[0]})_{[1]}\alpha(1_{A[0][1]}) \otimes c \cdot a_{[1]}1_{A[1]} \\
&\overset{(2.11)}{=} m'_{[0]} \cdot \beta(a_{[0][0]})\beta(1_{A[0][0]}) \otimes m'_{[1]} \cdot \alpha(a_{[0][1]})\alpha(1_{A[0][1]}) \otimes c \cdot a_{[1]}1_{A[1]} \\
&\overset{(2.1)}{=} m'_{[0]} \cdot \beta(1_{A[0][0]})\beta(a_{[0][0]}) \otimes m'_{[1]} \cdot \alpha(1_{A[0][1]})\alpha(a_{[0][1]}) \otimes c \cdot 1_{A[1]}a_{[1]} \\
&\overset{(2.14)}{=} m' \cdot \beta(1_{A[0][0]})\beta(a_{[0][0]}) \otimes \gamma(c_1) \cdot \alpha(1_{A[0][1]})\alpha(a_{[0][1]}) \otimes c_2 \cdot 1_{A[1]}a_{[1]} \\
&\overset{(2.1)}{=} m' \cdot \beta(a_{[0][0]})\beta(1_{A[0][0]}) \otimes \gamma(c_1) \cdot \alpha(a_{[0][1]})\alpha(1_{A[0][1]}) \otimes c_2 \cdot a_{[1]}1_{A[1]} \\
&\overset{(2.2,2.1)}{=} m' \cdot \beta(a_{[0]})\beta(1_{A[0][0]}) \otimes \gamma(c_1) \cdot \alpha(a_{[1]1})\alpha(1_{A[0][1]}) \otimes c_2 \cdot a_{[1]2}1_{A[1]} \\
&\overset{(2.10)}{=} m' \cdot \beta(a_{[0]})\beta(1_{A[0][0]}) \otimes \gamma(c_1 \cdot a_{[1]1}) \cdot \alpha(1_{A[0][1]}) \otimes c_2 \cdot a_{[1]2}1_{A[1]} \\
&\overset{(2.5)}{=} m' \cdot \beta(a_{[0]})\beta(1_{A[0][0]}) \otimes \gamma((c \cdot a_{[1]})_1) \cdot \alpha(1_{A[0][1]}) \otimes (c \cdot a_{[1]})_2 \cdot 1_{A[1]}.
\end{aligned}
$$

This is exactly what we have to show.

$G(M') \in \mathcal{M}(H)_A^C$ and the functorial properties are checked in a straightforward way. Let us finally show that G is a right adjoint to F. Take $M \in \mathcal{M}(H)_A^C$, we define $\eta_M : M \to GF(M) = (\overline{M \otimes_A A'})\Box_{C'} C$ as follows: for all $m \in M$,

$$\eta_M(m) = m_{[0]} \otimes_A \beta(1_{A[0]}) \otimes m_{[1]} \cdot 1_{A[1]}.$$

For all $a \in A$, we have

$$
\begin{aligned}
\eta_M(m \cdot a) &= (m \cdot a)_{[0]} \otimes_A \beta(1_{A[0]}) \otimes (m \cdot a)_{[1]} \cdot 1_{A[1]} \\
&\overset{(2.8)}{=} m_{[0]} \cdot a_{[0]} \otimes_A \beta(1_{A[0]}) \otimes m_{[1]} \cdot a_{[1]}1_{A[1]} \\
&= m_{[0]} \otimes_A \beta(a_{[0]})\beta(1_{A[0]}) \otimes m_{[1]} \cdot a_{[1]}1_{A[1]} \\
&\overset{(2.1)}{=} m_{[0]} \otimes_A \beta(1_{A[0]})\beta(a_{[0]}) \otimes m_{[1]} \cdot 1_{A[1]}a_{[1]} \\
&= (m_{[0]} \otimes_A \beta(1_{A[0]}) \otimes m_{[1]} \cdot 1_{A[1]}) \cdot a
\end{aligned}
$$

and

$$
\begin{aligned}
(\eta_M \otimes \iota) \circ \rho_M(m) &= m_{[0][0]} \otimes_A \beta(1_{A[0]}) \otimes m_{[0][1]} \cdot 1_{A[1]} \otimes m_{[1]} \\
&\overset{(2.7)}{=} m_{[0]} \cdot 1'_{A[0]} \otimes_A \beta(1_{A[0]}) \otimes m_{[1]1} \cdot 1'_{A[1]}1_{A[1]} \otimes m_{[1]2} \\
&= m_{[0]} \otimes_A \beta(1'_{A[0]}1_{A[0]}) \otimes m_{[1]1} \cdot 1'_{A[1]}1_{A[1]} \otimes m_{[1]2} \\
&\overset{(2.1)}{=} m_{[0]} \otimes_A \beta(1_{A[0]}) \otimes m_{[1]1} \cdot 1_{A[1]} \otimes m_{[1]2} \\
&= \rho_{GF(M)} \circ \eta_M(m).
\end{aligned}
$$

So $\eta_M \in \mathcal{M}(H)_A^C$.

Take $M' \in \mathcal{M}(H')_{A'}^{C'}$. Then we define $\delta_{M'} : FG(M') \to M'$, where

$$\delta_{M'}((m' \cdot \beta(1_{A[0]}) \otimes c \cdot 1_{A[1]}) \otimes_A a') = \varepsilon_C(c) m' \cdot a'.$$

Notice that $\delta_{M'}$ is A'-linear. That δ_N is C'-colinear is proved as follows:

$$\begin{aligned}
&(\delta_{M'} \otimes \imath) \circ (\rho_{FG(M')})(m' \cdot \beta(1_{A[0]}) \otimes c \cdot 1_{A[1]}) \otimes_A a') \\
&\quad = \delta_{M'}((m' \cdot \beta(1_{A[0][0]}) \otimes c_1 \cdot 1_{A[0][1]}) \otimes_A a'_{[0]}) \otimes \gamma(c_2 \cdot 1_{A[1]}) \cdot a'_{[1]} \\
&\quad = m' \cdot \beta(1_{A[0]}) a'_{[0]} \otimes \gamma(c \cdot 1_{A[1]}) \cdot a'_{[1]}.
\end{aligned}$$

Applying $\imath \otimes \imath \otimes \varepsilon_C$ to (2.14), this yields

$$\begin{aligned}
m' \cdot \beta(1_{A[0]}) \otimes \gamma(c \cdot 1_{A[1]}) &= \varepsilon(c) m'_{[0]} \cdot \beta(1_{A[0]}) \otimes m'_{[1]} \cdot \alpha(1_{A[1]}) \\
&= \varepsilon(c) m'_{[0]} \cdot \beta(1_{A[0]}) \otimes m'_{[1]} \cdot \alpha(1_{A[1]}).
\end{aligned}$$

Using the identity above, it follows that

$$\begin{aligned}
m' \cdot \beta(1_{A[0]}) a'_{[0]} \otimes \gamma(c \cdot 1_{A[1]}) \cdot a'_{[1]} &= \varepsilon(c) m'_{[0]} \cdot \beta(1_{A[0]}) a'_{[0]} \otimes m'_{[1]} \cdot \alpha(1_{A[1]}) a'_{[1]} \\
&\overset{(2.11)}{=} \varepsilon(c) m'_{[0]} \cdot 1_{A'[0]} a'_{[0]} \otimes m'_{[1]} \cdot 1_{A'[1]} a'_{[1]} \\
&\overset{(2.1)}{=} \varepsilon(c) m'_{[0]} \cdot a'_{[0]} \otimes m'_{[1]} \cdot a'_{[1]} \\
&= \rho_{M'} \circ \delta_{M'}((m' \cdot \beta(1_{A[0]}) \otimes c \cdot 1_{A[1]}) \otimes_A a').
\end{aligned}$$

This is what we need to show. We can check η and δ defined above are all natural transformations and they satisfy

$$G(\delta_{M'}) \circ \eta_{G(M')} = I, \quad \delta_{F(M)} \circ F(\eta_M) = I,$$

for all $M \in \mathcal{M}(H)_A^C$ and $M' \in \mathcal{M}(H')_{A'}^{C'}$.

The proof of Theorem is completed. □

REMARK 2.8. We consider (H, A, k) and the map $\imath_H$, $\imath_B$ and $\varepsilon_C : C \to k$. Now $\mathcal{M}(H)_A$(or $\mathcal{U}_A$), the category of right A-modules, and F is the functor which forgets the C-comodule structures. From Theorem 2.7, $G(M') = \{m' \cdot 1_{A[0]} \otimes c \cdot 1_{A[1]}\} = \overline{M' \otimes C}$ with structure maps

$$\rho_{G(M')}(m' \cdot_{A[0]} \otimes c \cdot 1_{A[1]}) = m' \cdot 1_{A[0][0]} \otimes c_1 \cdot 1_{A[0][1]} \otimes c_2 \cdot 1_{A[1]}, \tag{2.17}$$

$$(m' \cdot 1_{A[0]} \otimes c \cdot 1_{A[1]}) \cdot a = m' \cdot a_{[0]} \otimes c \cdot a_{[1]}, \tag{2.18}$$

for all $a \in A$ and $M' \in \mathcal{M}(H)_A$. The unit η of the adjunction (F, G) is given by $\eta^{M'} : M' \to G(M')$,

$$\eta^{M'}(m') = m'_{[0]} \cdot 1_{A[0]} \otimes m'_{[1]} \cdot 1_{A[1]}.$$

3. Integral of partial Doi-Hopf datum

DEFINITION 3.1. Let (H, A, C) be a partial Doi-Hopf datum. A k-linear map

$$\theta : C \otimes C \to A$$

is called a *normalized A-integral*, if θ satisfies the following conditions:

$$c_2 \cdot 1_{A[1]3} \otimes 1_{A[0]}\theta(d \cdot 1_{A[1]1} \otimes c_1 \cdot 1_{A[1]2}) \tag{3.1}$$

$$= c_2 \cdot 1_{A[1]} \otimes 1_{A[0][0][0]}\theta(d \cdot 1_{A[0][0][1]} \otimes c_1 \cdot 1_{A[0][1]})$$
$$= d_1 \cdot 1_{A[0][1]}\theta(d_2 \cdot 1_{A[1]} \otimes c)_{A[1]} \otimes 1_{A[0][0]}\theta(d_2 \cdot 1_{A[1]} \otimes c)_{[0]},$$

$$\theta(c_1 \otimes c_2) = 1_A\varepsilon(c), \tag{3.2}$$

$$a_{[0][0]}\theta(c \cdot a_{[0][1]} \otimes d \cdot a_{[1]}) = \theta(c \otimes d)a, \tag{3.3}$$

for all $a \in A$ and $c, d \in C$.

THEOREM 3.2. *For any partial Doi-Hopf datum (H, A, C), the following assertions are equivalent,*

(1) *η in Remark 2.8 is a split natural monomorphism.*
(2) *The forgetful functor F is separable.*
(3) *There exists a normalized A-integral $\theta : C \otimes C \to A$.*

PROOF. (1)$\Longleftrightarrow$(2) follows by Rafael Theorem ([R]).
(3)$\Longrightarrow$(1). For any partial Doi-Hopf module M, we define

$$\nu^M : \overline{M \otimes C} \to M, \;\; \nu^M(m \cdot 1_{A[0]} \otimes c \cdot 1_{A[1]}) = m_{[0]}\theta(m_{[1]} \otimes c),$$

for all $m \in M$ and $c \in C$.

Now, we shall check that $\nu^M \in \mathcal{M}(H)_A^C$. In fact, for all $m \in M$, $c \in C$ and $a \in A$,

$$\begin{aligned}
\nu^M((m \cdot 1_{A[0]} \otimes c \cdot 1_{A[1]}) \cdot a) &= \nu^M(m \cdot a_{[0]} \otimes c \cdot a_{[1]}) \\
&= (m \cdot a_{[0]})_{[0]}\theta((m \cdot a_{[0]})_{[1]} \otimes c \cdot a_{[1]}) \\
&\overset{(2.8)}{=} m_{[0]} \cdot a_{[0][0]}\theta(m_{[1]} \cdot a_{[0][1]} \otimes c \cdot a_{[1]}) \\
&\overset{(3.3)}{=} m_{[0]}\theta(m_{[1]} \otimes c)a \\
&= \nu^M(m \cdot 1_{A[0]} \otimes c \cdot 1_{A[1]}) \cdot a.
\end{aligned}$$

Hence it is a morphism of A-module. Next, we shall check that ν is a morphism of C-comodule. It is sufficient to check that

$$\rho_M \circ \nu^M = (\nu^M \otimes \imath) \circ \rho_{G(M)}$$

holds. For all $m \in M$ and $c \in C$, we have

$$\begin{aligned}
&\rho_M \circ \nu^M(m \cdot 1_{A[0]} \otimes c \cdot 1_{A[1]}) \\
&= \rho_M(m_{[0]} \cdot \theta(m_{[1]} \otimes c)) \\
&= (m_{[0]} \cdot \theta(m_{[1]} \otimes c))_{[0]} \otimes (m_{[0]} \cdot \theta(m_{[1]} \otimes c))_{[1]} \\
&\overset{(2.8)}{=} m_{[0][0]} \cdot \theta(m_{[1]} \otimes c)_{[0]} \otimes m_{[0][1]} \cdot \theta(m_{[1]} \otimes c)_{[1]} \\
&\overset{(2.7)}{=} m_{[0]} \cdot 1_{[0][0]}\theta(m_{[1]2} \cdot 1_{[1]} \otimes c)_{[0]} \otimes m_{[1]1} \cdot 1_{[0][1]}\theta(m_{[1]2} \cdot 1_{[1]} \otimes c)_{[1]}
\end{aligned}$$

and

$$\begin{aligned}
&(\nu^M \otimes \imath) \circ \rho_{G(M)}(m \cdot 1_{A[0]} \otimes c \cdot 1_{A[1]}) \\
&= (\nu^M \otimes \imath)(m \cdot 1_{A[0][0]} \otimes c_1 \cdot 1_{A[0][1]} \otimes c_2 \cdot 1_{A[1]}) \\
&= \nu^M(m \cdot 1_{A[0][0]} \otimes c_1 \cdot 1_{A[0][1]}) \otimes c_2 \cdot 1_{A[1]} \\
&= (m \cdot 1_{A[0][0]})_{[0]}\theta((m \cdot 1_{A[0][0]})_{[1]} \otimes c_1 \cdot 1_{A[0][1]}) \otimes c_2 \cdot 1_{A[1]} \\
&= m_{[0]} \cdot 1'_{A[0]}1_{A[0][0]}\theta(m_{[1]} \cdot 1_{A[0][1]} \otimes c_1 \cdot 1_{A[1]}) \otimes c_2 \cdot 1'_{A[1]} \\
&= m_{[0]} \cdot 1_{A[0]}\theta(m_{[1]} \otimes c_1) \otimes c_2 \cdot 1_{A[1]}.
\end{aligned}$$

Using (3.1), we can get the desired result. For all $m \in M$, Since

$$\begin{aligned}
\nu^M \circ \eta_M(m) &= \nu^M(m_{[0]} \otimes m_{[1]}) \\
&= m_{[0][0]}\theta(m_{[0][1]} \otimes m_{[1]}) \\
&\overset{(2.7)}{=} m_{[0]} \cdot 1_{A[0][0]}\theta(m_{[1]1} \cdot 1_{[0][1]} \otimes m_{[1]2} \cdot 1_{A[1]}) \\
&= m_{[0]} \cdot \theta(m_{[1]1} \otimes m_{[1]2}) \\
&\overset{(3.2)}{=} m_{[0]}\varepsilon(m_{[1]}) = m.
\end{aligned}$$

So it follows that ν splits η. It is evidently natural.

(1)$\Longrightarrow$(3). We consider the following partial Doi-Hopf module $G(A)$. Evaluating at this object, the retraction ν of the unit η yields a morphism

$$\nu^{G(A)} : \overline{A \otimes C \otimes C} \to \overline{A \otimes C},$$

where $\overline{A \otimes C \otimes C} = \, < a1_{[0][0]} \otimes c \cdot 1_{A[0][1]} \otimes d1_{A[1]}, a \in A, c, d \in C >$. From $\nu \circ \eta = I$, we have

$$\nu^{G(A)}(a1_{A[0][0]} \otimes c_1 \cdot 1_{A[0][1]} \otimes c_2 \cdot 1_{A[1]}) = a \otimes c.$$

It can be used to construct θ as follows:

$$\theta : C \otimes C \to A,$$
$$\theta(c \otimes d) = (\imath \otimes \varepsilon_C) \circ \nu^{G(A)}(1_{A[0][0]} \otimes c \cdot 1_{A[0][1]} \otimes d \cdot 1_{A[1]}).$$

For all $c \in C$, since

$$\begin{aligned}\theta(c_1 \otimes c_2) &= (\imath \otimes \varepsilon_C) \circ \nu^{G(A)}(1_{A[0][0]} \otimes c_1 \cdot 1_{A[0][1]} \otimes c_2 \cdot 1_{A[1]}) \\ &= (id_A \otimes \varepsilon_C)(1_A \otimes c) = 1_A \varepsilon_C(c).\end{aligned}$$

Hence (3.2) holds. It can be seen to obey (3.3) by naturality and the A-module map property of ν.

The verification of (3.1) is more involved. For any C-comodule M, we consider the partial Doi-Hopf module $M \otimes A$. The A-actions and C-coaction are defined as follows:

$$\begin{cases} (m \otimes a) \cdot b = m \otimes ab, \\ \rho^{M \otimes A}(m \otimes a) = m_{[0]} \otimes a_{[0]} \otimes m_{[1]} \cdot a_{[1]}, \end{cases}$$

for all $m \in M, a, b \in A$. For C-comodule C, there is a partial Doi-Hopf module $C \otimes A$ and the map

$$\xi : C \otimes A \to \overline{A \otimes C}, \ \xi(c \otimes a) = a_{[0]} \otimes c \cdot a_{[1]}$$

induces a morphism of partial Doi-Hopf modules $C \otimes A \to \overline{A \otimes C}$. Thus by naturality of ν, we have the following commutative diagram

$$\begin{array}{ccc} GF(C \otimes A) & \xrightarrow{\nu^{C \otimes A}} & C \otimes A \\ GF(\xi) \downarrow & & \downarrow \xi \\ GF(\overline{A \otimes C}) & \xrightarrow{\nu^{\overline{A \otimes C}}} & A \otimes C \end{array}$$

Explicitly, for all $a \in A, c, d \in C$, we have

(3.4) $\xi \circ \nu^{C \otimes A}(c \otimes a 1_{A[0]} \otimes d \cdot 1_{A[1]}) = (\nu^{\overline{A \otimes C}})(a_{[0]} 1_{A[0][0]} \otimes c \cdot a_{[1]} 1_{A[0][1]} \otimes d 1_{A[1]}).$

We consider next the following partial Doi-Hopf module $C \otimes C$ with partial C-coaction given by comultiplication in the second factor. Then

$$\chi = \Delta \otimes \imath : C \otimes A \to C \otimes C \otimes A, \ \chi(c \otimes a) = c_1 \otimes c_2 \otimes a$$

induces a morphism of partial Doi-Hopf modules $C \otimes A \to C \otimes C \otimes A$. Thus by naturality of ν, the following diagram

$$\begin{array}{ccc} GF(C\otimes A) & \xrightarrow{\nu^{C\otimes A}} & C\otimes A \\ GF(\chi)\downarrow & & \downarrow\chi \\ GF(C\otimes C\otimes A) & \xrightarrow{\nu^{C\otimes C\otimes A}} & C\otimes C\otimes A \end{array}$$

The commutative diagram above is equivalent to

$$(3.5)\quad \chi\circ\nu^{C\otimes A}(c\otimes a1_{A[0]}\otimes d\cdot 1_{A[1]})=\nu^{C\otimes C\otimes A}(c_1\otimes c_2\otimes a1_{A[0]}\otimes d\cdot 1_{A[1]}),$$

for all $c,d\in C$ and $a\in A$. Finally, for any $c\in C$, the map

$$f_c: C\otimes A\to C\otimes C\otimes A,\quad d\otimes a\mapsto c\otimes d\otimes a$$

induces a morphism of partial Doi-Hopf modules $C\otimes A\to C\otimes C\otimes A$. Hence by naturality of ν, we have

$$(3.6)\quad c\otimes\nu^{C\otimes A}(e\otimes a1_{A[0]}\otimes d\cdot 1_{A[1]})=\nu^{C\otimes C\otimes A}(c\otimes e\otimes a1_{A[0]}\otimes d\cdot 1_{A[1]}),$$

for all $c,e,d\in C$ and $a\in A$. From (3.5) and (3.6),

$$(3.7)\quad \chi\circ\nu^{C\otimes A}(c\otimes a1_{A[0]}\otimes d\cdot 1_{A[1]})=c_1\otimes\nu^{C\otimes A}(c_2\otimes a1_{A[0]}\otimes d\cdot 1_{A[1]}),$$

for all $c,d\in C$ and $a\in A$.

From $\nu^{G(A)}$ being C-colinear, it follows that

$$\begin{aligned}&\rho^{G(A)}\circ\nu^{G(A)}(1_{A[0][0]}\otimes c\cdot 1_{A[0][1]}\otimes d\cdot 1_{A[1]})\\ &\qquad=\nu^{G(A)}(1_{A[0][0][0]}\otimes c\cdot 1_{A[0][0][1]}\otimes d_1\cdot 1_{A[0][1]})\otimes d_2\cdot 1_{A[1]},\end{aligned}$$

for all $c,d\in C$.

For all $c,d\in C$, since

$$\begin{aligned}&c_2\cdot 1_{A[1]}\otimes 1_{A[0][0][0]}\theta(d\cdot 1_{A[0][0][1]}\otimes c_1\cdot 1_{A[0][1]})\\ &=c_2\cdot 1_{A[1]3}\otimes 1_{A[0]}\theta(d\cdot 1_{A[1]1}\otimes c_1\cdot 1_{A[1]2})\\ &=\tau_{A,C}\circ(\iota\otimes\varepsilon_C\otimes\iota)(\nu^{G(A)}(1_{A[0]}1'_{A[0][0]}\otimes d\cdot 1_{A[1]1}1'_{[0][1]}\otimes c_1\cdot 1_{A[1]2}1'_{A[1]})\otimes c_2\cdot 1_{A[1]3})\\ &=\tau_{A,C}\circ(\iota\otimes\varepsilon_C\otimes\iota)(\nu^{G(A)}(1_{A[0][0][0]}\otimes d\cdot 1_{A[0][0][1]}\otimes c_1\cdot 1_{A[0][1]})\otimes c_2\cdot 1_{A[1]})\\ &=\tau_{A,C}\circ(\iota\otimes\varepsilon_C\otimes\iota)\rho_{G(A)}\circ\nu^{G(A)}(1_{A[0][0]}\otimes d\cdot 1_{A[0][1]}\otimes c\cdot 1_{A[1]})\\ &=\tau_{A,C}\circ\nu^{G(A)}(1_{A[0][0]}\otimes d\cdot 1_{A[0][1]}\otimes c\cdot 1_{A[1]}),\end{aligned}$$

where the second equals is followed from ν being a left A-module. In fact, for all $a \in A$, the map

$$f_a : \overline{A \otimes C} \to \overline{A \otimes C},\ f_a(b1_{A[0]} \otimes c \cdot 1_{A[1]}) = ab1_{A[0]} \otimes c \cdot 1_{A[1]}$$

is a morphism in the category $\mathcal{U}_A^C$. Hence by naturality of ν, we have that ν is a left A-module. Since

$$\begin{aligned}
& d_1 \cdot 1_{A[0][1]}\theta(d_2 \cdot 1_{A[1]} \otimes c)_{[1]} \otimes 1_{A[0][0]}\theta(d_2 \cdot 1_{A[1]} \otimes c)_{[0]} \\
= {} & d_1 \cdot (1_{A[0]}\theta(d_2 \cdot 1_{A[1]} \otimes c))_{[1]} \otimes (1_{A[0]}\theta(d_2 \cdot 1_{A[1]} \otimes c))_{[0]} \\
= {} & d_1 \cdot (1_{A[0]}(\imath \otimes \varepsilon_C) \circ \nu^{G(A)}(1'_{A[0][0]} \otimes d_2 \cdot 1_{A[1]}1'_{A[0][1]} \otimes c \cdot 1'_{A[1]}))_{[1]} \\
& \otimes (1_{A[0]}(\imath \otimes \varepsilon_C) \circ \nu^{G(A)}(1'_{A[0][0]} \otimes d_2 \cdot 1_{A[1]}1'_{A[0][1]} \otimes c \cdot 1'_{A[1]}))_{[0]} \\
= {} & d_1 \cdot ((\imath \otimes \varepsilon_C) \circ \nu^{G(A)}(1_{A[0]}1'_{A[0][0]} \otimes d_2 \cdot 1_{A[1]}1'_{A[0][1]} \otimes c \cdot 1'_{A[1]}))_{[1]} \\
& \otimes ((\imath \otimes \varepsilon_C) \circ \nu^{G(A)}(1_{A[0]}1'_{A[0][0]} \otimes d_2 \cdot 1_{A[1]}1'_{A[0][1]} \otimes c \cdot 1'_{A[1]}))_{[0]} \\
\overset{(3.4)}{=} {} & d_1 \cdot ((\imath \otimes \varepsilon_C) \circ \xi \circ \nu^{C\otimes A}(d_2 \otimes \cdot 1_{A[1]}1_{A[0]} \otimes c \cdot 1_{A[1]}))_{[1]} \\
& \otimes ((\imath \otimes \varepsilon_C) \circ \xi \circ \nu^{C\otimes A}(d_2 \otimes \cdot 1_{A[0]} \otimes c \cdot 1_{A[1]}))_{[0]}
\end{aligned}$$

Let $\nu^{C\otimes A}(d \otimes a1_{A[0]} \otimes c \cdot 1_{A[1]}) = c_i \otimes a_i$. Then

$$\begin{aligned}
& d_1 \cdot ((\imath \otimes \varepsilon_C) \circ \xi \circ \nu^{C\otimes A}(d_2 \otimes \cdot 1_{A[1]}1_{A[0]} \otimes c \cdot 1_{A[1]}))_{[1]} \\
& \qquad \otimes ((\imath \otimes \varepsilon_C) \circ \xi \circ \nu^{C\otimes A}(d_2 \otimes \cdot 1_{A[0]} \otimes c \cdot 1_{A[1]}))_{[0]} \\
\overset{(3.7)}{=} {} & c_{i1} \cdot ((\imath \otimes \varepsilon_C) \circ \xi(c_{i2} \otimes a_i))_{[1]} \otimes ((\imath \otimes \varepsilon_C) \circ \xi(c_{i2} \otimes a_i))_{[0]} \\
= {} & c_{i1} \cdot a_{i[1]} \otimes a_{i[0]} = \sum \tau_{A,C} \circ \xi(c_i \otimes a_i) \\
= {} & \sum \tau_{A,C} \circ \xi \circ \nu^{C\otimes A}(d \otimes a1_{A[0]} \otimes c \cdot 1_{A[1]}).
\end{aligned}$$

Hence we can get (3.1) by using (3.4). □

4. Applications

4.1 – *Maschke-type Theorems for partial Doi-Hopf modules*

Since separable functors reflect well the semisimplicity of the objects of a categogy, by Theroem 3.2, we will prove the Maschke-type theorems for partial Doi-Hopf modules.

COROLLARY 4.1. *Let (H, A, C) be a partial Doi-Hopf datum, and $M, N \in \mathcal{U}_A^C$. Suppose that there exists a total integral $\theta : C \otimes C \to A$. Then a monomorphism (resp. epimorphism) $f : M \to N$ splits in $\mathcal{U}_A^C$, if the monomorphism (resp. epimorphism) f splits as an A-module morphism.*

4.2 – *Partial relative modules*

Let H be a Hopf algebra and A a partial right H-comodule algebra. Then the threetuple (H, A, H) is a partial Doi-Hopf datum. The category $\mathcal{M}(H)_A^H$ is called a partial (H, A)-Hopf module category and denoted by $\mathcal{U}_A^H$.

COROLLARY 4.2. *Let H be a Hopf algebra and A a partial right H-comodule algebra. Then the following statements are equivalent:*

(1) *The forgetful functor $F : \mathcal{U}_A^H \to \mathcal{U}_A$ is separable,*
(2) *There exists a normalized A-integral $\theta : H \otimes H \to A$.*

We will now introduce the partial total integral for the partial right H-comodule algebra, and investigate the difference between the partial total integral and the total integral in sense of Doi.

PROPOSITION 4.3. *Let H be a Hopf algebra and A a partial right H-comodule algebra. If $\theta : H \otimes H \to k$ is a normalized A-integral for (H, A, H), the k-linear map*

$$\varphi : H \to A,\ \ \varphi(h) = \theta(1_H \otimes h),$$

for all $h \in H$, satisfies the relations:

$$\varphi(h)_{[0]} \otimes \varphi(h)_{[1]} = \varphi(h_1)1_{A[0]} \otimes h_2 1_{A[1]}, \tag{4.1}$$

$$\varphi(1_H) = 1_A. \tag{4.2}$$

PROOF. Notice first that $\varphi(1_H) = \theta(1_H \otimes 1_H) = \varepsilon_H(1_H)1_A = 1_A$. Since

$$\begin{aligned}
&h_2 1_{A[1]} \otimes \theta(g \otimes h_1)1_{A[0]}\\
&= h_2 1_{A[1]} \otimes 1_{A[0][0][0]}\theta(g 1_{A[0][0][1]} \otimes h_1 1_{A[0][1]})\\
&= g_1 1_{A[0][1]}\theta(g_2 1_{A[1]} \otimes h)_{[1]} \otimes 1_{A[0][0]}\theta(g_2 1_{A[1]} \otimes h)_{[0]}\\
&= g_1(1_{A[0]}\theta(g_2 1_{A[1]} \otimes h))_{[1]} \otimes (1_{A[0]}\theta(g_2 1_{A[1]} \otimes h))_{[0]}\\
&= g_1(\theta(g_2 \otimes h))_{[1]} \otimes (\theta(g_2 \otimes h))_{[0]}
\end{aligned}$$

It follows by taking $g = 1_H$ that

$$h_2 1_{A[1]} \otimes \theta(1_H \otimes h_1)1_{A[0]} = \theta(1_H \otimes h)_{[1]} \otimes \theta(1_H \otimes h)_{[0]}.$$

So (4.1) holds.

DEFINITION 4.4. Let H be a Hopf algebra and A a right partial H-comodule algebra. A k-linear map $\varphi : H \to A$ is call a partial total integral for (H, A), if φ satisfies the conditions (4.1) and (4.2).

REMARK 4.5. If $1_{A[0]} \otimes 1_{A[1]} = 1_A \otimes 1_H$, then the right partial H-comodule algebra A is just the ordinary right H-comodule algebra, and the partial total integral is the same with the total integral in sense of Doi in (D2).

Let $\varphi : H \to A$ be the total integral for the partial right H-coalgebra A, and define

$$\theta : H \otimes H \to A,\ \theta(h \otimes g) = 1_{A[0]}\varphi(gS^{-1}(1_{A[1]}h)),$$

for all $g, h \in H$.

THEOREM 4.6. *Let A be a partial right H-comodule algebra and $\varphi : H \to A$ be a partial total integral. If*

$$g\varphi(h)_{[1]} \otimes \varphi(h)_{[0]} = \varphi(h)_{[1]}g \otimes \varphi(h)_{[0]},$$

$$\varphi(h) \in Z(A)\ (\textit{the center of } A),\ 1_{A[0]}\varphi(S^{-1}(1_{[1]})) = 1_A,$$

Then θ is a normalized A-integral.

PROOF. For all $a \in A$ and $g, h \in H$, one has

$$\begin{aligned} a_{[0][0]}\theta(ga_{[0][1]} \otimes ha_{[1]}) &= a_{[0][0]}1_{A[0]}\varphi(ha_{[1]}S^{-1}(1_{A[1]}ga_{[0][1]})) \\ &= a_{[0]}1_{A[0]}\varphi(ha_{[1]2}S^{-1}(ga_{[1]1}1_{A[1]})) \\ &= 1_{A[0]}\varphi(hS^{-1}(g1_{A[1]}))a \\ &= \theta(g \otimes h) \end{aligned}$$

and

$$\begin{aligned} &g_1(\theta(g_2 \otimes h))_{[1]} \otimes (\theta(g_2 \otimes h))_{[0]} \\ &= g_1 1_{A[0][1]}\varphi(hS^{-1}(1_{A[1]}g_2))_{[1]} \otimes 1_{A[0][0]}\varphi(hS^{-1}(1_{A[1]}g_2))_{[0]} \\ &= \varphi(hS^{-1}(1_{A[1]}g_2))_{[1]}g_1 1_{A[0][1]} \otimes 1_{A[0][0]}\varphi(hS^{-1}(1_{A[1]}g_2))_{[0]} \\ &= h_2S^{-1}(1_{A[1]}g_2)_2 1'_{A[1]}g_1 1_{A[0][1]} \otimes 1_{A[0][0]}\varphi(h_1S^{-1}(1_{A[1]}g_2)_1)1'_{A[0]} \\ &= h_2S^{-1}(1_{A[1]1})S^{-1}(g_2)g_1 1_{A[0][1]} \otimes 1_{A[0][0]}\varphi(h_1S^{-1}(1_{A[1]2}g_3)) \\ &= h_2S^{-1}(1_{A[1]2})1_{A[1]1}1'_{A[1]} \otimes 1_{A[0]}1'_{A[0]}\varphi(h_1S^{-1}(1_{A[1]3}g)) \\ &= h_2 1'_{A[1]} \otimes 1_{A[0]}1'_{A[0]}\varphi(h_1S^{-1}(1_{A[1]}g)) \\ &= h_2 1_{A[1]} \otimes \theta(g \otimes h_1)1_{A[0]}. \end{aligned}$$

$$\begin{aligned} \theta(h_1 \otimes h_2) &= 1_{A[0]}\varphi(h_2S^{-1}(1_{A[1]}h_1)) \\ &= \varepsilon_H(h)1_{A[0]}\varphi(S^{-1}(1_{A[1]})) = \varepsilon_H(h)1_A \end{aligned}$$

So θ is a normalized A-integral. □

4.3 – *Partial Doi-Hopf Datum* (H, k, H)

COROLLARY 4.7. *Under the assumptions of Example* 2.5. *Then the following statements are equivalent:*

(1) *The forgetful functor* $F : \mathcal{M}(H)^H \to \mathcal{M}_k$*(the category of all vector spaces) is separable.*

(2) *k-linear map* $\theta : H \otimes H \to k$ *such that the following conditions are satisfied:*

$$\theta(h_1 \otimes h_2) = \varepsilon_H(h), \tag{4.3}$$

$$\theta(eh \otimes eg) = \theta(h \otimes g), \tag{4.4}$$

$$eh_2\theta(g \otimes h_1) = eg_1\theta(eg_2 \otimes h). \tag{4.5}$$

Take $e = 1$. Then the partial Doi-Hopf datum (H, k, H) is just the Doi-Hopf datum, the category $\mathcal{M}(H)^H$ is the category $\mathcal{M}^H$ of H-Hopf module. Suppose that φ is the right integral of H^*, the map $\theta : H \otimes H \to k$ is defined by

$$\theta(h \otimes g) = \varphi(gS^{-1}(h)).$$

By the properties of the right integral φ, we can check that θ satisfies (4.3)-(4.4).

COROLLARY 4.8. *Let H be a finite dimensional cosemisimple Hopf algebra. The forgetful functor* $F : \mathcal{M}^H \to \mathcal{M}_k$ *is separable.*

PROOF. Since H is a finite dimensional cosemisimple Hopf algebra, it follows that there exists a right integral $\varphi \in H^*$ such that $\varphi(1_H) = 1$. The desired total integral θ can be constructed by using φ. □

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