REND. SEM. MAT. UNIV. PADOVA, Vol. 127 (2012)

# Another Look at Connections

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ABSTRACT - In this note we make use of some properties of vector fields on a manifold to give an alternate proof to [3] for the equivalence between connections and parallel transport on vector bundles over manifolds. Out of the proof will emerge a new approach to connections on a bundle as a consistent way to lift the dynamics of the manifold to the bundle.

This note is aimed at proving the equivalence between connections (covariant derivatives) and the geometric notion of parallel transport. A classical proof of this fact appears in [3]. A functorial approach to parallel transport and the equivalence with connections can be found in [7]. In the version presented here we give a global proof exploiting properties of flows of vector fields on compact manifolds. The idea is to think of a connection on a vector bundle as a compatible way of lifting vector fields $X$ on the base manifold to vector fields on the total space of the bundle, or equivalently to $X$-derivations acting on sections of the bundle. Even better, a connection can be described as a compatible way of lifting $\boldsymbol{R}$-actions (determined by the flows of vector fields) on the base manifold to $\boldsymbol{R}$-actions on the covering bundle.

Aside from its classical flavor, the equivalence of connections and parallel transport allows a description [5] of one dimensional topological field theories over a manifold (see Atiyah [1] for a definition of topological field theories in arbitrary dimensions). Two dimensional topological field theories over a manifold $M$ admit a similar description in terms of connections on vector bundles over $LM$ the free loop space of $M$, along with some Frobenius data between fibers encoding parallel transport along pairs of pants cf. [4]. A careful extension of the properties of flows of vector

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fields used here to the world of supermanifolds leads us to a characterization of supersymmetric one dimensional topological field theories over a manifold cf. [2].

## 1. Preliminaries on flows

In what follows, $M$ denotes a compact manifold.

PROPOSITION 1.1. *Let $X$ and $Y$ be vector fields on $M$, and let $\alpha, \beta : \boldsymbol{R} \times M \to M$ denote the flows determined by $X$, respectively $Y$. Then the flow $\gamma$ of the vector field $X + Y$ is given by*

$$\gamma_t(x) = \lim_{n\to\infty} \underbrace{(\alpha_{\frac{t}{n}}\beta_{\frac{t}{n}}) \circ \ldots \circ (\alpha_{\frac{t}{n}}\beta_{\frac{t}{n}})}_{n}(x).$$

REMARK. This is a version of the Trotter formula for flows of vector fields on manifolds.

The proof is not hard: if we simply consider the composition of the flows, we get infinitesimally the vector field $X + Y$, but not a group action. The above zig-zag composition still generates infinitesimally $X + Y$ and defines a group action as well. Details can be found in [2].

PROPOSITION 1.2. *Let $\alpha : \boldsymbol{R} \times M \to M$ be the flow of a vector field $X$ on the compact manifold $M$. If $f$ is a (positive) function on $M$ then the flow of $fX$ is given by*

$$\beta : \boldsymbol{R} \times M \to M : (t, x) \mapsto \alpha(s(t, x), x),$$

*where $s : \boldsymbol{R} \times M \to \boldsymbol{R}$ is the solution to*

$$\begin{cases} \dfrac{\partial s}{\partial t}(t, x) = f(\alpha(s(t, x), x)) \\ s(0, x) = 0, \text{ for all } x. \end{cases}$$

The proof is a routine check.

COROLLARY 1.3. *Let $X$ and $Y$ be vector fields on $M$. Then $X$ and $Y$ have the same (directed) trajectories if and only if $Y = fX$, for some positive function $f$ on $M$.*

COROLLARY 1.4. *If $Y = fX$, where $f$ is a positive function on $M$, and $c$ is an integral curve of $X$ then $c \circ \varphi$ is an integral curve of $Y$, for some (orientation-preserving) diffeomorphism $\varphi$ of $\boldsymbol{R}$.*

## 2. Main Result

After the above preliminaries we can state the main theorem. Consider an $n$-dimensional vector bundle $E$ over a compact manifold $M$. Denote by $\mathcal{P}(M)$ the pathspace of $M$, i.e. the space of all (piecewise) smooth paths $\gamma : I \to M$, where $I$ denotes an arbitrary interval. Then

THEOREM 2.1. *There is a natural 1-1 correspondence*

$$\left\{ \begin{array}{l} \text{Connections} \\ \text{on } E \text{ over } M \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Parallel transport maps} \\ \text{associated to } E \text{ over } M \end{array} \right\}.$$

Recall (compare with [3]) that a parallel transport map $\mathcal{P}$ on $E$ over $M$ is a (smooth) section of the pullback bundle

$$\begin{array}{ccc} p^*Hom(E,E) & \longrightarrow & Hom(E,E) \\ \downarrow & & \downarrow \\ \mathcal{P}(M) & \xrightarrow{p:=(i,e)} & M \times M, \end{array}$$

where $Hom(E,E)$ denotes the bundle whose fiber at $(x,y) \in M \times M$ is given by $Hom(E_x, E_y)$, and $i(\gamma)$, $e(\gamma)$ denote the starting respectively the ending point of the path $\gamma$. In other words, the map $\mathcal{P}$ associates to each path $\gamma$ in $M$ a linear map $\mathcal{P}(\gamma) : E_{i(\gamma)} \to E_{e(\gamma)}$. This correspondence is required to satisfy the following:

(1) $\mathcal{P}(\gamma_x) = 1_{E_x}$, where $\gamma_x$ is a constant map at $x \in M$.
(2) (Invariance under reparametrization) $\mathcal{P}(\gamma \circ \alpha) = \mathcal{P}(\gamma)$, where $\alpha$ is an orientation-preserving diffeomorphism of intervals.
(3) (Compatibility under juxtaposition) $\mathcal{P}(\gamma_2 \star \gamma_1) = \mathcal{P}(\gamma_2) \circ \mathcal{P}(\gamma_1)$, where $\gamma_2 \star \gamma_1$ is the juxtaposition of $\gamma_1$ and $\gamma_2$.

PROOF. It is well known how a connection $\nabla$ gives rise to parallel transport: given a path $c$ in $M$, pull-back the connection along $c$ and solve a differential equation $(c^*\nabla)s = 0$. The solutions $s$ to this differential equation are the parallel sections along $c$ and will define an isomorphism between the fibers at the endpoints of the path; see for example [6] for details.

We now need to show how a parallel transport map $\mathcal{P}$ associated to the bundle $E$ over $M$ gives rise to a connection on $E$ over $M$. For this, let $X$ be a vector field on $M$ and $\alpha : \boldsymbol{R} \to \mathrm{Diff}(M)$ the flow of $X$, where $\mathrm{Diff}(M)$ stands for the group of diffeomorphisms of $M$. Denote by $P = GL(E)$ the frame bundle of $E$, a principal bundle on $M$ with structure group $G = GL(n)$, where $n$ is the rank of the bundle $E$, and let $\pi : P \to M$ be the projection map. Let $\mathrm{Diff}^G(P)$ denote the group of $G$-equivariant diffeomorphisms of $P$, i.e. diffeomorphisms of $P$ that preserve the $G$-action on $P$.

Parallel transport allows us to lift the flow $\alpha$ of $X$ to a group homomorphism $\tilde{\alpha} : \boldsymbol{R} \to \mathrm{Diff}^G(P)$ which is the flow of a vector field $\tilde{X}$ on $P$ that is $G$-invariant. Moreover, the vector field $\tilde{X}$ is an $X$-derivation, i.e.

$$\tilde{X} : \mathcal{C}^\infty(P) \to \mathcal{C}^\infty(P), \qquad \tilde{X}(fg) = X(f)g + f\tilde{X}(g),$$

for $f \in \mathcal{C}^\infty(M)$ and $g \in \mathcal{C}^\infty(P)$ (note that $\mathcal{C}^\infty(M)$ defines an obvious action on $\mathcal{C}^\infty(P)$ as well as on $Vect(P)$, the space of vector fields on $P$, via $\pi^*$). This holds since $\tilde{X}$ is $G$-invariant and therefore $\tilde{X}(\pi^*(f)) = \pi^*(X(f))$.

Further, $\tilde{X}$ extends to an $X$-derivation on $\mathcal{C}^\infty(P; V)$, the space of $V$-valued functions on $P$, where $V$ is another notation for $\boldsymbol{R}^n$. Again, since $\tilde{X}$ is $G$-invariant, it preserves $\mathcal{C}^\infty(P; V)^G$ the space of $G$-equivariant $V$-valued functions on $P$, which canonically identifies with $\Gamma(E)$, the space of sections on $E$. We have defined a correspondence

$$Vect(M) \ni X \longmapsto \{\tilde{X} : \Gamma(E) \to \Gamma(E)\}.$$

This defines a connection if the following two conditions hold:

$$\widetilde{X+Y} = \tilde{X} + \tilde{Y} \tag{1}$$

$$\widetilde{fX} = f\tilde{X}, \tag{2}$$

for $X, Y$ vector fields on $M$, and $f$ a smooth function on $M$. Let us first show property (1). Consider $X, Y$ vector fields on $M$, with flows $\alpha$, respectively $\beta$, and let $\gamma$ denote the flow of $X + Y$. Then

$$\begin{aligned}
\widetilde{X+Y} &= \frac{d}{dt}\Big|_{t=0} \tilde{\gamma}_t \\
&= \frac{d}{dt}\Big|_{t=0} \lim_{n\to\infty} (\widetilde{\alpha_{t/n}\beta_{t/n}})^{(n)} \\
&= \frac{d}{dt}\Big|_{t=0} \lim_{n\to\infty} (\tilde{\alpha}_{t/n}\tilde{\beta}_{t/n})^{(n)} \\
&= \tilde{X} + \tilde{Y}.
\end{aligned}$$

The third equality follows from the compatibility of lifting paths via parallel transport with concatenation of paths. For the second property (2), let $f$ be a (positive) function on $M$ and $X$ a vector field on $M$, with flow $\alpha$. By the Proposition 1.2 above, the flow $\alpha^f$ of the vector field $fX$ is given by the composition

$$\boldsymbol{R} \times M \xrightarrow{1\times\Delta} \boldsymbol{R} \times M \times M \xrightarrow{s\times 1} \boldsymbol{R} \times M \xrightarrow{\alpha} M,$$

for some $s : \boldsymbol{R} \times M \to \boldsymbol{R}$. The flow $\widetilde{\alpha^f}$ of $\widetilde{fX}$ is then given by the composition

$$\boldsymbol{R} \times P \xrightarrow{1\times\Delta} \boldsymbol{R} \times P \times P \xrightarrow{\tilde{s}\times 1} \boldsymbol{R} \times P \xrightarrow{\tilde{\alpha}} P,$$

where $\tilde{s} : \boldsymbol{R} \times P \to \boldsymbol{R}$ is defined by the composition

$$\boldsymbol{R} \times P \xrightarrow{1\times\pi} \boldsymbol{R} \times M \xrightarrow{s} \boldsymbol{R}.$$

But the above composition $\tilde{\alpha}(\tilde{s} \times 1)(1 \times \Delta)$ is nothing else than the flow of $f\tilde{X}$. The vector fields $\widetilde{fX}$ and $f\tilde{X}$ have the same flow, so they must be the same. The connection in the direction of the vector field $X$ is now determined by defining

$$\nabla_X := \tilde{X} : \Gamma(E) \to \Gamma(E).$$

It is clear that the family of derivations $\{\nabla_X\}$ parametrized by the space *Vect*($M$) of vector fields on the manifold $M$ defines a connection $\nabla$ on the bundle $E$ over $M$.

We are left to check that the two constructions are inverse to each other. The map "$\longrightarrow$" which associates to a connection its parallel transport map is injective, as it is well known that the parallel transport of a connection *recovers* the connection.

The proof is complete if we can show that

$$\longrightarrow \circ \longleftarrow \; = id.$$

Indeed, start with a parallel transport map $\mathcal{P}$ associated to the bundle $E$ over $M$. Denote by $\rho$ the standard representation of $G$ on $V = \boldsymbol{R}^n$. Then $E$ can be canonically identified with the associated bundle $P \times_\rho V$, whose elements are classes $[p, v] \in P \times V/\sim$ (we say that $(p, v)$ and $(p', v')$ are in the same class if $p' = pg$ and $v' = g^{-1}v$, for some $g \in G$). Recall the isomorphism

$$\mathcal{C}^\infty(P; V)^G \longrightarrow \Gamma(E) : \qquad f \longmapsto \{s(x) = [p, f(p)],\ \text{where } \pi(p) = x\}.$$

The parallel transport then defines as above a connection $\nabla$ given by

$$\nabla_X[p, f(p)] = [p, \tilde{X}(f)(p)],$$

in the direction of a vector field $X$ on $M$ ($\tilde{X}$ denotes the lift of $X$ to the bundle $P$ via parallel transport as well as the extension as a derivation to $V$-valued functions on $P$). Now, if $c : \boldsymbol{R} \to M$ is a curve in $M$ and $f \in C^\infty(c^*P; V)^G$, we also have

$$(c^*\nabla)_{\partial_t}[p, f(p)] = [p, v(f)(p)],$$

where $v$ denotes the lift of $\partial_t$, the standard vector field on $\boldsymbol{R}$, to $c^*P$. Therefore, a section $s = [p, f(p)]$ along $c$ is parallel with respect to the connection $\nabla$ if and only if $f$ is constant in the direction of the vector field $v$, which happens if and only if $s$ is parallel along $c$ with respect to the parallel transport $\mathcal{P}$. This finishes the proof of the theorem. □

Let $q : \mathrm{Diff}^G(P) \to \mathrm{Diff}(M)$ denote the obvious descending map. In the proof of Theorem 2.1 we only used parallel transport along flows of vector fields. This allows us to redefine a connection (on the principal $G$-bundle $P$) as a lift

$$\left\{ \begin{array}{c} \text{Homomorphisms} \\ \boldsymbol{R} \to \mathrm{Diff}(M) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Homomorphims} \\ \boldsymbol{R} \to \mathrm{Diff}^G(P) \end{array} \right\}$$

$$\alpha \longmapsto \tilde{\alpha}$$

that preserves the zig-zag composition of Proposition 1.1 and the action of the space of (positive) functions on $M$ given by Proposition 1.2. Here, "lift" means that a homomorphism $\alpha$ must map to a homomorphism $\tilde{\alpha}$ that descends to $\alpha$, i.e. such that $q\tilde{\alpha} = \alpha$. The lift should also preserve the trivial homomorphisms.

It is an interesting problem to see how the *flatness* condition for a connection can be expressed in terms of lifting the dynamics of a manifold as above. It should be a lift that preserves the flow formula for the Lie bracket, but we are not aware of such a formula.

*Acknowledgments.* This note is based on an idea of Stephan Stolz that connections allow liftings of group actions. We warmly acknowledge the financial support provided by a BitDefender scholarship while visiting the Institute of Mathematics of the Romanian Academy "Simion Stoilow".

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Manoscritto pervenuto in redazione il 22 febbraio 2011.