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A Convergence Theorem for Immersions with L^2-Bounded Second Fundamental Form

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ABSTRACT - In this short note, we prove a convergence theorem for sequences of immersions from some closed surface Σ into some standard Euclidean space $\mathbb{R}^n$ with L^2-bounded second fundamental form, which is suitable for the variational analysis of the famous Willmore functional, where $n \geq 3$. More precisely, under some assumptions which are automatically verified (up to subsequence and an appropriate Möbius transformation of $\mathbb{R}^n$) by sequences of immersions from some closed surface Σ into some standard Euclidean space $\mathbb{R}^n$ arising from an appropriate stereographic projection of $\mathbb{S}^n$ into $\mathbb{R}^n$ of immersions from Σ into $\mathbb{S}^n$ and minimizing the L^2-norm of the second fundamental form with $n \geq 3$, we show that the varifolds limit of the image of the measures induced by the sequence of immersions is also an immersion with some minimizing properties.

1. Introduction and statement of the results

The question of convergence in a *suitable sense* of sequences of immersions with L^2-bounded second fundamental form and the regularity of the support of their varifolds limits are very important issues in the study of geometric variational problems arising from submanifold theory. A very well known example where this plays an important role is in the study of the celebrated Willmore functional who was (to the best of our knowledge) first considered in various works by Thomsen [11], and subsequently by Blaschke [1]. In 1965, Willmore [12] reintroduced, and studied it within the frame work of the conformal geometry of surfaces. The motivations of the

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study of Willmore functional stem from many areas of science. For example in molecular biology, it is known as the Helfrich Model [4], and is interpreted as surface energy for lipid bilayers. Another field where it plays an important role is solid mechanics. In fact, it arises as the limit energy for thin-plate theory, see [2]. Finally, in general relativity, the Willmore functional appears as the main term in the expression of the Hawking quasilocal mass, see [3].

In [7], J. Langer proved a convergence Theorem (after reparametrization) in the C^1-topology for sequences of immersions with L^p-bounded second fundamental form with $p > 2$. A crucial point in his argument is the fact that thanks to Sobolev embedding Theorem, such immersions are C^1 since $p > 2$. This latter point is not true for $p = 2$, as 2 is the borderline in the Sobolev embedding Theorem. Furthermore, as noticed by J. Langer [7] his result is no more true when $p = 2$. For these reasons, the issue of convergence theorem for sequence of immersions with L^2-bounded second fundamental form which is suitable for the variational analysis of the Willmore functional is not a trivial matter.

In this note, we prove a convergence theorem for a sequence of immersions from some closed surface Σ into some standard Euclidean space $\mathbb{R}^n$ ($n \geq 3$) with L^2-bounded second fundamental form, which is suitable for applying the direct method of Calculus of Variations to study the Willmore functional. In fact our convergence theorem (see Theorem 1.1 especially Corollary 1.3) has been used in the study of the Willmore boundary problem by R. Schätzle [8], see section 4, last paragraph of page 290.

In order to state our result in a clear way, we first fix some notation. In the following, we use $\mathbb{R}^n$, $n \geq 3$ to denote the standard n-dimensional Euclidean space, g_{euc} for its standard metric, $\mathbb{R}^{n+1}$ the standard $(n+1)$-dimensional Euclidean space and $\mathbb{S}^n$ its unit sphere. We use the notation e_{n+1} to denote the north pole of $\mathbb{S}^n$, namely $e_{n+1} := (0, \cdots, 1) \in \mathbb{R}^{n+1}$. Furthermore, we will use the following Möbius transformation $\Phi : \mathbb{R}^{n+1} \cup \{\infty\} \longrightarrow \mathbb{R}^{n+1} \cup \{\infty\}$ defined by the following formula

$$\Phi(x) = e_{n+1} + 2\,\frac{x - e_{n+1}}{|x - e_{n+1}|^2}. \tag{1}$$

The restriction of Φ to $\mathbb{S}^n$ will be called the stereographic projection of $\mathbb{S}^n$ to $\mathbb{R}^n \cup \{\infty\}$. For $x \in \mathbb{R}^n$, $r > 0$, $B_r(x)$ will denotes the Euclidean ball of $\mathbb{R}^n$ with center x and radius r. Moreover for $r > 0$, D_r will stand for the open disk of $\mathbb{R}^2$ of diameter r. We identify $\mathbb{R}^n$ with $\mathbb{R}^2 \times \mathbb{R}^{n-2}$, and denote by Π the projection of $\mathbb{R}^n$ onto the $\mathbb{R}^2$ factor. For k a nonnegative integer, $\mathcal{H}^k$ denotes the Hausdorff measure and $\mathcal{L}^k$ stands for the Lebesgue measure. Given Σ a closed surface, and $f : \Sigma \longrightarrow \mathbb{R}^n$ an immersion we set $g_f := f^*(g_{euc})$ the pull back of the standard metric g_{euc} by f. Furthermore, for an immersion $f : \Sigma \longrightarrow \mathbb{R}^n$, we use the notation $\vec{\boldsymbol{H}}_f$ to denote its mean curvature vector, A_f to denote its second fundamental form, μ_{g_f} the induced area measure on Σ by f, namely $d\mu_{g_f}$ is the volume form associated to g_f, and $\mu_f = f(\mu_{g_f})$ the varifold image of the immersion f, namely $\mu_f = f(\mu_{g_f}) := (x \longrightarrow \mathcal{H}^0(f^{-1}(x))\mathcal{H}^2 \lfloor f(\Sigma)$. We recall that $\mu_f = f(\mu_{g_f})$ is an integral 2-varifold on $\mathbb{R}^n$, see [9] for more informations. Given an integer rectifiable 2-varifold μ on $\mathbb{R}^n$, we use the notation A_μ to denote its weak second fundamental form and $spt\,\mu$ to denote its support, see [5] for more informations. For X a space, $A \subset X$, and μ a measure on X, we use the standard notation $\mu\lfloor A$ to mean the restriction of μ on A. We also use the notation $diam(A)$ to denote the diameter of A.

Now having fixed this notation, we are ready to state our main result which reads as follows:

THEOREM 1.1. *Let Σ be a closed surface, and $f_m : \Sigma \longrightarrow \mathbb{R}^n$, $n \geq 3$ be a family of immersions. Assuming that*

1) $\int_\Sigma |A_m|^2 d\mu_{g_m} \leq E$, *$\mu_{g_m} \rightharpoonup \mu$ as varifolds, μ integer rectifiable 2-varifold, $spt\,\mu$ compact, $|A_m|^2\mu_{g_m} \rightharpoonup \nu$ as Radon measures with $g_m := g_{f_m}$, and $A_m := A_{f_m}$.*

2) *There exist $\bar{\varepsilon}_0 > 0$, $\rho_1 > 0$, such that if we denote by $\{x_k, k = 1, \cdots, K\} \subset spt\,\mu$, the finite number of bad points of $spt\,\mu$ verifying $\nu(\{x_k\}) \geq \bar{\varepsilon}_0^2$, then for every $x \in spt\,\mu \setminus \cup_{k=1}^K \overline{B_{\rho_1}(x_k)}$, we have there exists $r(x) > 0$ such that $\mu\lfloor B_{r(x)}(x) = \sum_{i=1}^{N_x} \mu^i(x)$ and this decomposition is unique up to relabelling and the $\mu^i(x)$'s are C^1-graphs in $B_{r(x)}(x)$.*

Then, (up to a subsequence) there exists m_0 depending only on E, n, Σ, $spt\,\mu$, ν, ρ_1, $\bar{\varepsilon}_0$, and the bad points $x_1, \cdots, x_K$, namely $m_0 = m_0(E, n, \Sigma, spt\,\mu, \nu, \rho_1, \bar{\varepsilon}_0, x_1, \cdots, x_K) > 0$, such that for every $m \geq m_0$, we have the existence of a closed surface Σ_m^∞, an immersion $F_m : \Sigma_m^\infty \longrightarrow spt\,\mu \subset \mathbb{R}^n$ such that

$$\mu\lfloor_{spt\,\mu \setminus \cup_{k=1}^K B_{\rho_1}(x_k)} \geq (x \longrightarrow \mathcal{H}^0(F_m^{-1}(x)))\mathcal{H}^2\lfloor F_m(\Sigma_{m,\rho_1}), \quad \text{and}$$

$$\int_{\Sigma_{m,\rho_1}} |A_{F_m}|^2 d\mu_{g_{F_m}} \leq \int_{spt\,\mu} |A_\mu|^2 d\mu,$$

where $\Sigma_{m,\rho_1} := f_m^{-1}(spt\,\mu \setminus \cup_{k=1}^K B_{\rho_1}(x_k))$. *Moreover, defining* $\Sigma^\infty_{m,\rho_1} := \Sigma^\infty_m \setminus F_m^{-1}(\cup_{k=1}^K B_{\rho_1}(x_k))$ *for* $m \geq m_0$, *we have that the surface* Σ^∞_m *verifies (still for* $m \geq m_0$*)*

$$\Sigma^\infty_{m,\rho_1} \cong \Sigma_{m,\rho_1}.$$

Remark 1.2. As already said in the abstract, the assumptions of Theorem 1.1 are automatically satisfied for a sequence of immersions (up to a subsequence and an appropriate Möbius transformation of $\mathbb{R}^n$) arising from an appropriate stereographic projection of $\mathbb{S}^n$ into $\mathbb{R}^n$ of immersions from Σ into $\mathbb{S}^n$ and minimizing the L^2-norm of the second fundamental form. In fact up to a subsequence, apart of $spt\,\mu$ compact, all the hypotheses in 1) follows from the minimizing property and the monotonicity formula as in (9) below. Moreover, up to an appropriate Möbius transformation of $\mathbb{R}^n$, the compactness of $spt\,\mu$ follows from a straightforward adaptation of the argument of Proposition 2.1 in [8] and the discussion right after until formula (2.7). On the other hand, the assumption 2) follows directly from the argument of Proposition 2.2 in [8].

Thus, we have the following corollary:

Corollary 1.3. *Let* Σ *be a closed surface, and* $f_m : \Sigma \longrightarrow \mathbb{R}^n$, $n \geq 3$ *be a family of immersions arising from the stereographic projection* $\Phi : \mathbb{S}^n \longrightarrow \mathbb{R}^n \cup \{\infty\}$ *of immersions from* Σ *into* $\mathbb{S}^n$ *where* Φ *is as in* (1). *Assuming that*

$$\int_\Sigma |A_{f_m}|^2 d\mu_{g_{f_m}} \longrightarrow \inf_{f:\,\Sigma \longrightarrow \mathbb{R}^n,\, f \text{ immersion}} \int_\Sigma |A_f|^2 d\mu_f \ \text{ as } \ m \longrightarrow +\infty,$$

then up to a subsequence and an appropriate Möbius transformation of $\mathbb{R}^n$, *we have that the same conclusions of Theorem* 1.1 *hold.*

Remark 1.4. We would like to point out that, if in addition the surface Σ in Theorem 1.1 is orientable then the corresponding limit surface Σ^∞_m is orientable as well. The same remark holds also for Corollary 1.3.

2. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1 from which Corollary 1.3 follows as already pointed out in Remark 1.2. We start by making two definitions following J. Langer [7].

DEFINITION 2.1. *Let Σ be a closed surface, q a point of Σ, $f:\Sigma\longrightarrow\mathbb{R}^n$ an immersion, $n\geq 3$, and $A:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ an Euclidean isometry which takes the origin to $f(q)$ and whose differential takes $e_n=(0,0,\cdots,0,1)$ to the inward normal to f at q. We define $U_{r,q}$ to be the q-component of $(\Pi\circ A^{-1}\circ f)^{-1}(D_r)$, namely the connected component of $(\Pi\circ A^{-1}\circ f)^{-1}(D_r)$ containing q.*

DEFINITION 2.2. *Let Σ be a closed surface, $f:\Sigma\longrightarrow\mathbb{R}^n$ an immersion, $n\geq 3$, and r,α positive real numbers. We say that $f:\Sigma\longrightarrow\mathbb{R}^n$ is an (r,α)-immersion if for every $q\in\Sigma$, $(A^{-1}\circ f)(U_{r,q})$ is the graph of a C^1-function $h:D_r\longrightarrow\mathbb{R}^{n-2}$ satisfying $\|\nabla h\|_{L^\infty}\leq\alpha$, where A is the Euclidean isometry in the definition of $U_{r,q}$.*

Now having fixed these definitions, we are ready to make the proof of Theorem 1.1. We divide the proof in 3 steps.

PROOF OF THEOREM 1.1

STEP 1. In this step, we show that, for every $x\in spt\,\mu\setminus\cup_{k=1}^{K}B_{\rho_1}(x_k)$, there exists $\frac{4}{3}r(x)>\rho_x>0$ (where $r(x)$ is as in Theorem 1.1) such that (up to a subsequence), we have that there exists closed pairwise disjoint disks $D_{m,i}(x)$ $i=1,\cdots,I_x$ and for every i, the sequence of measures $\mu_{m,i}(x):=\mathcal{H}^2\lfloor f_m(D_{m,i}(x))\cap B_{\frac{3}{4}\rho_x}(x)$ verifies the following properties

$$\mu_{m,i}(x)\rightharpoonup\mu_i(x)\quad\text{weakly as varifolds in } B_{\frac{3}{4}\rho_x}(x),\tag{2}$$

$$spt\,\mu_{m,i}(x)\to spt\,\mu_i(x)\quad\text{locally in Hausdorff distance in } B_{\frac{3}{4}\rho_x}(x),\tag{3}$$

and

$$\mu\lfloor B_{\frac{3}{4}\rho_x}(x)=\sum_{1=1}^{I_x}\mu_i(x).\tag{4}$$

In order to achieve (2)-(4), we first recall that the Willmore energy of f_m is defined by $\mathcal{W}(f_m):=\frac{1}{4}\int_\Sigma|\vec{\boldsymbol{H}}_m|^2d\mu_{g_m}$ and is (thanks to the Gauss-Bonnet theorem) related to $\int_\Sigma|A_m|^2d\mu_{g_m}$ by the following formula

$$\mathcal{W}(f_m)=\frac{1}{4}\int_\Sigma|A_m|^2d\mu_{g_m}+\pi\chi(\Sigma),\tag{5}$$

where $\vec{\boldsymbol{H}}_m$ is the mean curvature vector of f_m and $\chi(\Sigma)$ is the Euler-Poincaré characteristic of Σ. Thus, from the assumption of L^2-bounded second fundamental form, we infer that

(6) $$\mathcal{W}(f_m) + \int_\Sigma |A_m|^2 d\mu_{g_m} \le \bar{E},$$

where $\bar{E}$ depends only on E and Σ. Next, we take $0 < \varepsilon_0 < \bar{\varepsilon}_0$ to be fixed later. Thus, by assumption, we have that for every $k = 1, \cdots, K$, there holds $\nu(\{x_k\}) > \varepsilon_0^2$. Hence, it is easy to see that, for every $x \in spt\,\mu \setminus \cup_{k=1}^K B_{\rho_1}(x_k)$, there exists $4r(x) > 3\rho_x > 0$ such that

$$\nu(\overline{B_{\rho_x}(x)}) < \varepsilon_0^2.$$

Now, since by assumption we have $|A_m|^2 \mu_{g_m} \rightharpoonup \nu$ as Radon measures, then for m large enough, we get

$$\int_{B_{\rho_x}(x)} |A_m|^2 d\mu_{g_m} \le \nu(\overline{B_{\rho_x}(x)}) < \varepsilon_0^2.$$

Hence, applying the graphical decomposition Lemma of L. Simon, see Lemma 2.1 in [10], we have that for every $x \in spt\,\mu \setminus \cup_{k=1}^K B_{\rho_1}(x_k)$, and for m large enough, $f_m^{-1}(\overline{B_{\rho_x}(x)}) = \sum_{i=1}^{I_x^m} D_{m,i}(x)$ where $D_{m,i}(x)$, $i = 1, \cdots, I_x^m$, $I_x^m \le CE$ are closed pairwise disjoint sets. In addition, the following properties hold: there exist affine 2-planes $L_{m,i}(x) \subset \mathbb{R}^n$, smooth functions $u_{m,i}^x : \overline{\Omega_{m,i}(x)} \subset L_{m,i}(x) \longrightarrow \overline{\Omega_{m,i}(x)}^\perp$, where $\Omega_{m,i}(x) = \Omega_{m,i}^0(x) \setminus \cup_{k=1}^{K_{x,i}^m} d_{m,i,k}(x)$, $\Omega_{m,i}^0(x)$ are simply connected and open, $d_{m,i,k}(x)$ are closed pairwise disjoint disks, with

$$\rho^{-1}|u_{m,i}^x| + |\nabla u_{m,i}^x| \le C(E,n)\varepsilon_0^{\frac{1}{4n+6}},$$

and closed pairwise disjoint pimples $P_{m,i,1}(x), \cdots, P_{m,i,J_{x,i}^m}(x) \subset D_{m,i}(x)$ which are disks such that

(7) $$f_m(D_{m,i}(x) \setminus \cup_{j=1}^{J_{x,i}^m} P_{m,i,j}(x)) = graph(u_{m,i}^x) \cap \overline{B_{\frac{\rho_x}{2}}(x)},$$

and

(8) $$\sum_{i=1}^{I_x^m} \sum_{j=1}^{J_{x,i}^m} diam(f_m(P_{m,i,j}(x))) \le C(E,n)\varepsilon_0^{\frac{1}{2}}\rho_x.$$

On the other hand, using (A.16) in [6], we obtain

(9) $$r^{-2}\mu_m(B_r(x)) \le CW(f_m) \quad \forall r > 0,$$

where $\mu_m := \mu_{f_m} = f_m(\mu_{g_m}) = (x \longrightarrow \mathcal{H}^0(f_m^{-1}(x))\mathcal{H}^2 \lfloor f_m(\Sigma)$ and $g_m = g_{f_m} = f_m^*(g_{euc})$. Thus, we derive

$$r^{-2}\mu_m(B_r(x)) \leq C(E, n, \Sigma), \qquad \forall r > 0, \tag{10}$$

and

$$\sum_{i=1}^{I_x^m} \sum_{j=1}^{J_{x,i}^m} \mu_m(f_m(P_{m,i,j}(x))) \leq C(E, n, \Sigma)\varepsilon_0 \rho_x^2. \tag{11}$$

Now, since

$$\begin{aligned} \mathcal{H}^2(graph(u_{m,i}^x) \cap \overline{B_{\frac{\rho_x}{2}}(x)}) &\leq \sqrt{1 + |\nabla u_{m,i}^x|_{L^\infty}^2}\ \mathcal{L}^2(\Pi L_{m,i}(x)(graph(u_{m,i}^x) \cap \overline{B_{\frac{\rho_x}{2}}(x)}) \\ &\leq \sqrt{1 + C(E,n)\varepsilon_0^{\frac{1}{2n+3}}}\ \omega_2 \rho_x^2, \end{aligned} \tag{12}$$

then, using (8) combined with the diameter bound of L. Simon [10] and (6) (which give the area control of $f_m(P_{m,i,j}(x))$ the image by f_m of the pimples $P_{m,i,j}(x)$), we obtain

$$\mu_{g_m}(D_{m,i}(x)) \leq (1 + C(E,n)\varepsilon_0^{\frac{1}{2n+3}})\ \omega_2 \rho_x^2 \quad \forall\, m, i. \tag{13}$$

Next, using the monotonicity formula (A.6) in [6], we infer that

$$\begin{aligned} \sigma^2 \mu_{g_m}(D_{m,i}(x) \cap f_m^{-1}(B_\sigma(x))) &\leq \frac{9}{16}\,(1+\delta)\rho_x^2 \mu_{g_m}(D_{m,i}(x) \cap f_m^{-1}(B_{\frac{3}{4}\rho_x}(x))) + \\ &\qquad + C(1+\delta^{-1}) \int_{B_{\rho_x}(x)} |\vec{\boldsymbol{H}}_m|^2 d\mu_m, \\ &\leq \frac{9}{16}(1+\delta)\rho_x^{-2}\mu_{g_m}(D_{m,i}(x)) + C(1+\delta^{-1})\varepsilon_0^2, \end{aligned}$$

for σ such that $B_\sigma(x) \subset B_{\frac{3\rho_x}{4}}(x)$, and for every $\delta > 0$. Now, we take $\tau < \frac{1}{2}$ and choose δ such that $\frac{9}{16}(1+\delta) < 1 + \frac{\tau}{2}$. Next, we come to the choice of ε_0 that we choose depending only on E, n and τ such that $0 < \varepsilon_0 < \bar{\varepsilon}_0$ and (thanks to (13))

$$\frac{\mu_{g_m}(D_{m,i}(x) \cap f_m^{-1}(B_\sigma(x)))}{\omega_2 \sigma^2} < 1 + \tau.$$

In particular

$$f_m : D_{m,i}(x) \cap f_m^{-1}(B_{\frac{3\rho_x}{4}}(x)) \longrightarrow B_{\frac{3\rho_x}{4}}(x) \quad \text{is an embedding.} \tag{14}$$

Now, setting,

$$\mu_{m,i}(x) := \mathcal{H}^2 \lfloor f_m(D_{m,i}(x)) \cap B_{\frac{3\rho_x}{4}}(x) = \mu_m \lfloor f_m(D_{m,i}(x)) \cap B_{\frac{3\rho_x}{4}}(x) = \\ = f_m(\mu_{g_m}) \lfloor f_m(D_{m,i}(x)) \cap B_{\frac{3\rho_x}{4}}(x),$$

and applying again the monotonicity formula as in (9), and using the fact that $I_x^m \leq CE$, we can suppose that up to a subsequence $I_x^m = I_x$, and for every i

$$\mu_{m,i}(x) \rightharpoonup \mu_i(x) \quad \text{weakly as varifolds in } B_{\frac{3\rho_x}{4}}(x),$$

$$spt\,\mu_{m,i}(x) \to spt\,\mu_i(x) \quad \text{locally in Hausdorff distance in } B_{\frac{3\rho_x}{4}}(x),$$

and

$$\mu \lfloor B_{\frac{3\rho_x}{4}}(x) = \sum_{1=1}^{I_x} \mu_i(x),$$

as desired.

REMARK 2.3. We would like to point out that ρ_x depends not only on x but also on $E, n, spt\,\mu, v, \bar{\varepsilon}_0, \rho_1$ and $\{x_1, \cdots, x_K\}$. However, for the sake of clarity in the exposition, we have chosen to emphasize only the dependence to x .

STEP 2. In this second step, we show the existence of $\tilde{m}_0$ positive depending only on $E, n, \Sigma, spt\,\mu, v, \bar{\varepsilon}_0, \rho_1$ and the bad points x_k, $k = 1, \cdots, K$, namely $\tilde{m}_0 = \tilde{m}_0(E, n, \Sigma, spt\,\mu, v, \bar{\varepsilon}_0, \rho_1, x_1, \cdots, x_K) > 0$ such that for every $m \geq \tilde{m}_0$, we have the existence of an immersion $\bar{F}_m : \Sigma_{m,\rho_1} \longrightarrow spt\,\mu \subset \mathbb{R}^n$ with the following property

$$\mu_{F_m} = G_m(x)\mathcal{H}^2 \lfloor \bar{F}_m(\Sigma_{m,\rho_1}) \leq \mu, \tag{15}$$

where

$$G_m(x) = \mathcal{H}^0(\bar{F}_m^{-1}(x)), \quad x \in \Sigma_{m,\rho_1}.$$

For this end, we first use the fact that $spt\,\mu$ is compact to infer that there exists $x^1, \cdots, x^N \in spt\,\mu \setminus \cup_{k=1}^K B_{\rho_1}(x_k)$ such that $spt\,\mu \setminus \cup_{k=1}^K B_{\rho_1}(x_k) \subset \cup_{k=1}^N B_{\rho_{x^k}}(x^k)$ with N depending only on $spt\,\mu$, ρ_1 and the bad points x_k, $k = 1, \cdots, K$. Now, to continue, we recall that from the arguments of the proof of Step 1, we have that for every $k = 1, \cdots, N$, there exists $m_k^0 = m(x^k, E, n, \bar{\varepsilon}_0, \rho_{x^k})$ depending only on x^k, E, n, ρ_{x^k}, and $\bar{\varepsilon}_0$ such that for every $m \geq m_k^0$, there holds:

$f_m^{-1}(\overline{B_{\rho_{x^k}}(x^k)})$ decomposes into finitely many closed pairwise disjoint sets $D_{m,i}(x^k)$, and that for every $m \geq m_k^0$ and i,

the set $D_{m,i}(x^k)$ contains closed pairwise disjoint disks $P_{m,i,j}(x^k)$,

verifying

$$f_m(D_{m,i}(x^k) \setminus \cup_{j=1}^{J^m_{x^k,i}} P_{m,i,j}(x^k)) \text{ is a graph of a smooth function,}$$

and for every $m \geq m_k^0$, i and j,

the image by f_m of the pimples $P_{m,i,j}(x^k)$ (namely $f_m(P_{m,i,j}(x^k))$) has very small diameter.

Now, setting $\bar{m}_0 := \max_{k=1}^N m_k^0$, and recalling that N depends only on $spt\,\mu$, ρ_1 and the bad points x_k, $k = 1, \cdots, K$ and for every $k = 1, \cdots, N$, ρ_{x^k} depends only on x^k, E, n, v, $spt\,\mu$, $\bar{\varepsilon}_0$, ρ_1 and the bad points $x_1, \cdots, x_K$ (see Remark 2.3), we have that $\bar{m}_0$ depends only on E, n, $spt\,\mu$, v, $\bar{\varepsilon}_0$, ρ_1, and $x_1, \cdots, x_K$, namely $\bar{m}_0 = \bar{m}_0(E, n, spt\,\mu, v, \bar{\rho}_1, \varepsilon_0, x_1, \cdots, x_K)$. Next, we have that by replacing the image of the pimples $P_{m,i,j}(x^k)$, namely $f_m(P_{m,i,j}(x^k))$ (for $k = 1, \cdots, N$ and $m \geq \bar{m}_0$) by extensions of the graphs, we obtain new immersions $\hat{f}_m : \Sigma_{m,\rho_1} \longrightarrow \mathbb{R}^n$ still for $m \geq \bar{m}_0$, where we recall that $\Sigma_{m,\rho_1} := f_m^{-1}(spt\,\mu \setminus \cup_{k=1}^K B_{\rho_1}(x_k))$. We point out that $\hat{f}_m$ does not depend on the good points x^k, $k = 1, \cdots, N$, since for each x^k, we have replaced the corresponding image (by f_m) of the pimples and the result gives $\hat{f}_m$. On the other hand, we have that the extensions can be done so that the $\hat{f}_m$'s obtained are $\left(\dfrac{3\rho}{254}, C(E,n)\varepsilon_0^{\frac{1}{4n+6}}\right)$-immersions with $\rho < \min_{k=1}^N \rho_{x^k}$(for the definition of these type of immersions see Definition 2.2), and up to taking $\bar{m}_0$ bigger, we have that for $m \geq \bar{m}_0$ the following holds:

(16) $T_{f_m(q)}(f_m(\Sigma_{m,\rho_1}))$ is very close to $T_{\hat{f}_m(q)}(\hat{f}_m(\Sigma_{m,\rho_1}))$, for every $q \in \Sigma_{m,\rho_1}$,

where for every $q \in \Sigma_{m,\rho_1}$, $T_{f_m(q)}(f_m(\Sigma_{m,\rho_1}))$ and $T_{\hat{f}_m(q)}(\hat{f}_m(\Sigma_{m,\rho_1}))$ denote respectively the tangent space of $f_m(\Sigma_{m,\rho_1}))$ at $f_m(q)$ and of $\hat{f}_m(\Sigma_{m,\rho_1})$ at $\hat{f}_m(q)$. In addition, we have that thanks to (7), the extensions can be done so that after replacing the image by f_m of the pimple $P_{m,i,j}(x^k)$, namely $f_m(P_{m,i,j}(x^k))$, we stay in the bigger balls $B_{\frac{3\rho_{x^k}}{4}}(x^k)$, and this for every $k = 1, \cdots, N$. Now, using (16) and the existence of nearest point projection, we infer that there exists a projection

$$\Pi_m : f_m(\Sigma_{m,\rho_1}) \longrightarrow \hat{f}_m(\Sigma_{m,\rho_1}),$$

still for $m \geq \bar{m}_0$. Next, given any point $z \in \hat{f}_m(\Sigma_{m,\rho_1})$, we have there exists i

and $x \in \{x^1, \cdots x^N\}$ such that $z \in f_m(D_{m,i}(x)) \cap B_{\frac{3\rho_x}{4}}(x)$. Now, using again (16) and the fact that $spt\,\mu_{m,i}(x) \to spt\,\mu_i(x)$ locally in Hausdorff distance in $B_{\frac{3\rho_x}{4}}(x)$, see (3) in Step 1, we have that there exists $\tilde{m}_0 > \bar{m}_0$ depending only on E, n, Σ, v, $spt\,\mu$, $\bar{\varepsilon}_0$, ρ_1 and the bad points $x, \cdots, x_K$ (namely $\tilde{m}_0 = \tilde{m}_0(E, n, \Sigma, v, spt\,\mu, \rho_1, \bar{\varepsilon}_0, x_1, \cdots, x_K)$) such that, for $m \geq \tilde{m}_0$, we have that by following a smooth transversal field to $\hat{f}_m$, we can project z to a point $Q_m(z)$ belonging to $spt\,\mu_i(x)$. We claim that this process gives a well-defined map $Q_m : \hat{f}_m(\Sigma_{m,\rho_1}) \longrightarrow spt\,\mu$ for $m \geq \tilde{m}_0$. Indeed, let us suppose that $m \geq \tilde{m}_0$ and $z \in f_m(D_{m,i}(y_1))) \cap f_m(D_{m,j}(y_2)) \cap B_{\frac{3\rho_{y_1}}{4}}(y_1) \cap B_{\frac{3\rho_{y_2}}{4}}(y_2)$ for two different points y_1 and $y_2 \in \{x^1, \cdots x^N\}$ and show that by this process we obtain a unique value for $Q_m(z)$. In order to do that, we argue as follows. First of all, using (14), we infer that $z \in f_m(D_{m,i}(y_1)) \cap D_{m,j}(y_2)) \cap B_{\frac{3\rho_{y_1}}{4}}(y_1) \cap B_{\frac{3\rho_{y_2}}{4}}(y_2)$. Thus, we have that there exists $\theta < \frac{3\rho_z}{8}$ and l such that

$$(17) \qquad D_{m,i}(z) \cap D_{m,j}(z) \cap D_{m,l}(z) \neq \emptyset \quad \text{and} \quad B_\theta(z) \subset B_{\frac{3\rho_{y_1}}{4}}(y_1) \cap B_{\frac{3\rho_{y_2}}{4}}(y_2).$$

In fact, since $z \in f_m(D_{m,i}(y_1) \cap D_{m,j}(y_2))$, then there exists a point q such that

$$(18) \qquad q \in D_{m,i}(y_1) \cap D_{m,j}(y_2) \quad \text{and} \quad z = f_m(q).$$

On the other hand, we have also that $z \in B_{\frac{3\rho_{y_1}}{16}}(y_1) \cap B_{\frac{3\rho_{y_2}}{4}}(y_2)$. Now, from $z \in B_{\frac{3\rho_{y_1}}{4}}(y_1) \cap B_{\frac{3\rho_{y_2}}{4}}(y_2)$, we infer the existence of $\theta < \frac{3\rho_z}{8}$ such that $B_\theta(z) \subset B_{\frac{3\rho_{y_1}}{8}}(y_1) \cap B_{\frac{3\rho_{y_2}}{8}}(y_2)$. Next, since $z = f_m(q)$ and $z \in \overline{B_{\frac{3\rho_z}{4}}(z)}$, then $q \in f_m^{-1}(\overline{B_{\frac{3\rho_z}{4}}(z)})$. Now, using Step 1, we have the following decomposition for $f_m^{-1}(\overline{B_{\rho_z}(z)})$

$$(19) \qquad f_m^{-1}(\overline{B_{\rho_z}(z)}) = \sum_{i=1}^{I_z} D_{m,i}(z).$$

Thus, from (19), we infer that there exists l such that $q \in D_{m,l}(z)$. Hence, we have proved the existence of l and θ with the desired properties. To continue the proof of the claim we first define

$$\mu_{m,i,j,l}(y_1, y_2, z) := \mathcal{H}^2 \lfloor f_m(D_{m,i}(y_1) \cap D_{m,j}(y_2) \cap D_{m,l}(z)) \cap B_{\frac{3\rho_{y_1}}{4}}(y_1) \cap \\ \cap B_{\frac{3\rho_{y_2}}{4}}(y_2) \cap B_{\frac{3\rho_z}{4}}(z).$$

Now, from Step 1 and the fact that $\theta < \frac{3\rho_z}{8}$, we get

$$\mu_{m,i,j,l}(y_1, y_2, z) \rightharpoonup \mu_l(z) \quad \text{weakly as varifolds in } B_\theta(z).$$

On the other hand, using again Step 1 and the fact that $B_\theta(z) \subset B_{\frac{3\rho_{y_1}}{4}}(y_1) \cap B_{3\frac{\rho_{y_2}}{4}}(y_2)$, we infer that

$$\mu_{m,i,j,l}(y_1, y_2, z) \rightharpoonup \mu_i(y_1) \quad \text{weakly as varifolds in } B_\theta(z),$$
$$\text{and}$$
$$\mu_{m,i,j,l}(y_1, y_2, z) \rightharpoonup \mu_j(y_2) \quad \text{weakly as varifolds in } B_\theta(z).$$

So, from the uniqueness of varifolds limit, we have that $\mu_l(z) = \mu_i(y_1) = \mu_j(y_2)$ in $B_\theta(z)$. Thus, by the uniqueness up to relabelling of the decomposition of μ, and the fact that $\rho_{y_i} < \frac{4}{3} r(y_i)$ for $i = 1, 2$, we have that $\mu_i(y_1) = \mu_j(y_2)$ in $B_{\frac{3\rho_{y_1}}{4}}(y_1) \cap B_{\frac{3\rho_{y_2}}{4}}(y_2)$. Hence, the projection Q_m is well-defined for $m \geq \tilde{m}_0$. Thus, we have that there exists $\tilde{m}_0$ depending only on E, n, Σ, $spt\,\mu$, v, $\bar{\varepsilon}_0$, ρ_1, and $x_1, \cdots, x_K$ such that for every $m \geq \tilde{m}_0$, the following composition is very well defined

$$\bar{F}_m = Q_m \circ \Pi_m \circ f_m.$$

Now, from the hypothesis that the μ^i's are C^1-graphs and the uniqueness of the decomposition, we infer that $\bar{F}_m : \Sigma_{m,\rho_1} \longrightarrow spt\,\mu \subset \mathbb{R}^n$ is an immersion for every $m \geq \tilde{m}_0$. Furthermore, setting

$$G_m(x) = \mathcal{H}^0(\bar{F}_m^{-1}(x)), \quad x \in \Sigma_{m,\rho_1},$$

for $m \geq \tilde{m}_0$ and using the definition of $\bar{F}_m$, we have

$$\mu_{\bar{F}_m} = G_m(x)\mathcal{H}^2 \lfloor \bar{F}_m(\Sigma_{m,\rho_1}) \leq \mu,$$

for $m \geq \tilde{m}_0$ as desired.

STEP 3. In this step, we finish the proof of Theorem 1.1. To do so, we first use Step 2 and fill the regions corresponding to the bad points, namely the set $\cup_{k=1}^K B_{\rho_1}(x_k)$, to obtain (for $m \geq \tilde{m}_0$) a closed surface Σ_m^∞ and immersion $F_m : \Sigma_m^\infty \longrightarrow spt\,\mu \subset \mathbb{R}^n$ such that setting

$$\Sigma_{m,\rho_1}^\infty := \Sigma_m^\infty \setminus F_m^{-1}(\cup_{k=1}^K B_{\rho_1}(x_k)), \tag{20}$$

and recalling that

$$\Sigma_{m,\rho_1} := f_m^{-1}(spt\,\mu \setminus \cup_{k=1}^K B_{\rho_1}(x_k)),$$

we obtain

$$F_m = \bar{F}_m \quad \text{in} \quad \Sigma_{m,\rho_1},$$
$$\text{and} \tag{21}$$
$$\Sigma_{m,\rho_1}^\infty \cong \Sigma_{m,\rho_1},$$

still for $m \geq \tilde{m}_0$. On the other hand, from the definition of $\bar{F}_m$ (see Step 2) and (21), it is easy to see that

$$\int_{\Sigma_{m,\rho_1}} |A_{F_m}|^2 d\mu_{g_{F_m}} \leq \int_{\Sigma_{m,\rho_1}} |A_{f_m}|^2 d\mu_{g_m} + \varepsilon_m, \tag{22}$$

with $\varepsilon_m \longrightarrow 0$ as $m \longrightarrow +\infty$. Now, using assumption 1), we have also that

$$\int_{\Sigma_{m,\rho_1}} |A_{f_m}|^2 d\mu_{g_m} \longrightarrow \int_{spt\,\mu \setminus \cup_{k=1}^{K} B_{\rho_1}(x_k)} |A_\mu|^2 d\mu \ \text{ as } \ m \longrightarrow +\infty. \tag{23}$$

Next, since x_k, $k = 1, \cdots, K$ are the bad point of $spt\,\mu$ verifying $\nu(\{x_k\}) \geq \bar{\varepsilon}_0^2$ for $k = 1, \cdots, K$, then we have

$$\int_{spt\,\mu \setminus \cup_{k=1}^{K} B_{\rho_1}(x_k)} |A_\mu|^2 d\mu < \int_{spt\,\mu} |A_\mu|^2 d\mu. \tag{24}$$

Thus, from (15), (21)-(24), we infer that there exists m_0 depending only on E, n, Σ, ν, $spt\,\mu$, $\bar{\varepsilon}_0$, ρ_1 and the bad points $x, \cdots, x_K$, namely $m_0 = m_0(E, n, \Sigma, spt\,\mu, \nu, \bar{\varepsilon}_0, \rho_1, x_1, \cdots, x_K)$, such that $m_0 > \tilde{m}_0$ and for every $m \geq m_0$ there holds:

$$\mu \lfloor spt\,\mu \setminus \cup_{k=1}^{K} B_{\rho_1}(x_k) \geq (x \longrightarrow \mathcal{H}^0(F_m^{-1}(x))\mathcal{H}^2 \lfloor F_m(\Sigma_{m,\rho_1}) \quad \text{and}$$

$$\int_{\Sigma_{m,\rho_1}} |A_{F_m}|^2 d\mu_{g_{F_m}} \leq \int_{spt\,\mu} |A_\mu|^2 d\mu.$$

Hence, since $m_0 > \tilde{m}_0$, then (21) completes the proof of the Theorem. □

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