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Realization Theorems for Valuated p^n-Socles

PATRICK W. KEEF

ABSTRACT - If n is a positive integer and p is a prime, then a valuated p^n-socle is said to be n-summable if it is isometric to a valuated direct sum of countable valuated groups. The functions from ω_1 to the cardinals that can appear as the Ulm function of an n-summable valuated p^n-socle are characterized, as are the n-summable valuated p^n-socles that can appear as the p^n-socle of some primary abelian group. The second statement generalizes a classical result of Honda from [9]. Assuming a particular consequence of the generalized continuum hypothesis, a complete description is given of the n-summable groups that are uniquely determined by their Ulm functions.

0. Terminology and introduction

Except where specifically noted, the term "group" will mean an abelian p-group, where p is a prime fixed for the duration of the paper. Our terminology and notation will be based upon [6]. A group is *Σ-cyclic* if it is isomorphic to a direct sum of cyclic groups. We also make use of concepts related to *valuated groups* and *valuated vector spaces* that can be found, for example, in [16] and [7], and that we briefly review: Let $\mathcal{O}$ be the class of ordinals and $\mathcal{O}_\infty = \mathcal{O} \cup \{\infty\}$, where we agree that $\alpha < \infty$ for all $\alpha \in \mathcal{O}_\infty$. A *valuation* on a group V is a function $|\ |_V : V \to \mathcal{O}_\infty$ such that for every $x, y \in V$, $|x \pm y|_V \geq \min\{|x|_V, |y|_V\}$ and $|px|_V > |x|_V$. As a result, for all $\alpha \in \mathcal{O}_\infty$, $V(\alpha) = \{x \in V : |x|_V \geq \alpha\}$ is a subgroup of V with $pV(\alpha) \subseteq V(\alpha+1)$. We say V is α-*bounded* if $V(\alpha) = \{0\}$; the *length* of V is the least α such that $V(\alpha) = V(\infty)$.

A homomorphism between two valuated groups is *valuated* if it does not decrease values and an *isometry* if it is bijective and preserves values. If $\{V_i\}_{i \in I}$, is a collection of valuated groups, then the usual direct sum, $V = \bigoplus_{i \in I} V_i$, has a natural valuation, where $V(\alpha) = \bigoplus_{i \in I} V_i(\alpha)$ for every $\alpha \in \mathcal{O}_\infty$.

(*) Indirizzo dell'A.: Department of Mathematics, Whitman College, Walla Walla, WA 99362, USA.
E-mail: keef@whitman.edu

If W is any subgroup of V, then restricting $|\ |_V$ to W turns W into a valuated group with $W(\alpha) = W \cap V(\alpha)$ for all $\alpha \in \mathcal{O}_\infty$. A valuated group W with $pW = \{0\}$ is called a *valuated vector space*; so each $W(\alpha)$ will be a subspace of W. We say a valuated vector space is *free* if it is isometric to a valuated direct sum of cyclic groups (of order p). If V is a valuated group, then its socle $V[p] = \{x \in V : px = 0\}$ is a valuated vector space, and V is *summable* if $V[p]$ is free. A group G is a valuated group using the height function (also denoted by $|\ |_G$) as its valuation; in this case $G(\alpha) = p^\alpha G$, and G is said to be *separable* if it is ω-bounded. If n is a fixed positive integer, it follows that the *p^n-socle* of G, written $G[p^n] = \{x \in G : p^n x = 0\}$, can be viewed as a valuated group.

In [4], an ∞-bounded valuated group V was defined to be a *valuated p^n-socle* if $p^n V = \{0\}$ and for every $x \in V[p^{n-1}]$ and every ordinal $\beta < |x|_V$, there is a $y \in V$ with $x = py$ and $\beta \leq |y|_V$. It easily follows that an ∞-bounded valuated vector space is a valuated p-socle. The p^n-socle of a reduced group G is always a valuated p^n-socle, and if $V = G[p^n]$, then we will say G is *supported* by V. A valuated p^n-socle is *realizable* if it is supported by some reduced group. [The parallel requirements that V be ∞-bounded and that G be reduced are convenient, but not strictly speaking necessary.]

Let $\mathcal{C}$ be the class of all cardinals. If V is a valuated group, then the *α-th Ulm invariant* of V is

$$f_V(\alpha) = r(V(\alpha)[p]/V(\alpha+1)[p]) \in \mathcal{C}.$$

We call $f_V : \mathcal{O} \to \mathcal{C}$ the *Ulm function* of V. Another definition of the Ulm function of a general valuated group is given in [10], and it is easy to verify that these agree for valuated p^n-socles. In general, when we say $f : \mathcal{O} \to \mathcal{C}$ is a function, we mean that its support is contained in some ordinal δ, and we identify f with its restriction $f : \delta \to \mathcal{C}$.

A valuated p^n-socle is said to be *n-summable* if it is isometric to the valuated direct sum of a collection of countable valuated groups (each of which will also be a valuated p^n-socle). It was shown in [4] that the theory of n-summable valuated p^n-socles parallels the theory of *dsc groups* (i.e., direct sums of countable groups - see Chapter XII of [6] for standard results on these groups). For example, the following is ([4], Theorem 2.7), which parallels ([6], Theorem 78.4).

THEOREM 0.1. *Suppose V and W are n-summable valuated p^n-socles. Then there is an isometry $V \cong W$ iff their Ulm functions agree, i.e., $f_V = f_W$.*

The parallel between n-summable valuated p^n-socles and dsc groups can be extended. A subgroup X of a valuated group V is *nice* if every coset $a+X$ has an element of maximal value. A *nice composition series* for V is an ascending chain of nice subgroups $\{X_i : i \leq \delta\}$ such that

(C1) $X_0 = \{0\}$, $X_\delta = V$;
(C2) for all $i<\delta$, $X_{i+1}/X_i \cong \mathbb{Z}_p$;
(C3) for all limit ordinals $\lambda \leq \delta$, $X_\lambda = \bigcup_{i<\lambda} X_i$.

A *nice system* for V is a collection $\mathcal{N}$ of nice subgroups of V such that

(S1) $\{0\} \in \mathcal{N}$;
(S2) $\mathcal{N}$ is closed under group sums;
(S3) If $S \subseteq V$ is countable, then $S \subseteq N$ for some countable $N \in \mathcal{N}$.

If V is a valuated p^n-socle, let λ_V be the unique ordinal such that $|V|_V = \{|x|_V : x \in V\} \subseteq \mathcal{O}_\infty$ is order-isomorphic to $\lambda_V \cup \{\infty\}$. The following is ([4], Theorem 2.1), which parallels ([6], Theorem 81.9).

THEOREM 0.2. *Suppose V is a valuated p^n-socle and $\lambda_V \leq \omega_1$. Then the following are equivalent:*

(a) *V is n-summable;*
(b) *V has a nice system;*
(c) *V has a nice composition series.*

If $\lambda_V \leq \omega_1$ and $\phi : |V|_V \to \lambda_V \cup \{\infty\}$ is an order-preserving bijection and $|x|'_V = \phi(|x|_V)$ for every $x \in V$, then V is also a valuated p^n-socle using $|\ |'_V$ and virtually anything that is true using one valuation (e.g., that V is n-summable) is also true using the other. It therefore makes sense to restrict our attention to those n-summable valuated p^n-socles that are ω_1-bounded. This leads to two of the main questions that are addressed in this paper:

(1) If $f : \omega_1 \to \mathcal{C}$ is a function, when does $f = f_V$ for some n-summable valuated p^n-socle V?

(2) If V is an ω_1-bounded n-summable valuated p^n-socle, when is V realizable?

Complete answers are given to both questions. For obvious reasons, a function satisfying (1) will be called *n-summable*. Section 1 is a discussion of n-summable functions, which are characterized in Theorem 1.10. This combinatorial condition, which we describe later, can be viewed as a generalization of the classical notion of an *admissible* function (see [6], Theorem 83.6).

Section 2 is a consideration of question (2). In Theorem 2.11 it is shown that V is realizable iff for every countable limit ordinal λ and every $\alpha < \lambda$ we have

$$\sum_{\lambda+n-1\leq\beta<\lambda+\omega} f_V(\beta) \leq \Big(\sum_{\alpha<\beta<\lambda} f_V(\beta)\Big)^{\aleph_0}.$$

Naturally, a group G is *n-summable* if $G[p^n]$ is n-summable as a valuated p^n-socle. These groups are considered in [5], [13], [14] and [15]. An n-summable group will always be summable (since a countable valuated vector space is free), and so $p^{\omega_1}G = \{0\}$ (see, for example, [6], Theorem 84.3). Therefore, any n-summable valuated p^n-socle that is realizable must be ω_1-bounded; this is another reason for restricting to the ω_1-bounded case. It is known that a given valuated vector space is often supported by different groups that are not isomorphic. This suggests a third question.

(3) If V is an n-summable valuated p^n-socle, when is V uniquely realizable, in the sense that any two groups supported by V are isomorphic?

Since n-summable valuated p^n-socles are classified by their Ulm functions, the last question can be restated as follows:

(3′) Describe the n-summable groups G that are uniquely determined by their Ulm functions; that is, those that have the property that if G' is another n-summable group with $f_G = f_{G'}$, then G is isomorphic to G'.

Assuming a natural statement regarding cardinal arithmetic that is a consequence of the generalized continuum hypothesis, this question is answered in Theorem 3.4. With this cardinality assumption, it is shown that V is uniquely realizable iff $V(\omega + n - 1)$ is countable; and in this case, every group G supported by V will be a dsc group. In particular, when $n = 1$, Corollary 3.5 states that these groups agree with those described in ([2], Theorem 2.6).

Theorem 2.11 (i.e., the solution to the above question (2)) is a generalization of the classical "Existence Theorem for Principal p-Groups" from [9]; in fact, for $n = 1$, it reduces to precisely this result. However, there are several important differences. First, it is fairly clear that if $n = 1$, then *any* function $f : \omega_1 \to \mathcal{C}$ will satisfy (1); i.e., any such function is the Ulm function of some free valuated vector space. On the other hand, if $n > 1$, then f will be the Ulm function of an n-summable valuated p^n-socle iff it satisfies Theorem 1.10. Second, the proofs of our main results are considerably less complicated; i.e., a few pages in length, as opposed to the over twenty pages needed to prove the Existence Theorem in [9]. And third, we can apply our techniques to answer question (3) whenever our condition on cardinal arithmetic holds; the latter question was not even considered in [9].

1. Realizing Ulm Functions

In this section we give an explicit description of those functions from ω_1 to the cardinals that are n-summable, in that they can appear as the Ulm function of an n-summable valuated p^n-socle. By considering (valuated) direct sums, it follows that the sum of a collection of n-summable functions is n-summable.

If α is an ordinal, then $\alpha = q_\omega(\alpha) + r_\omega(\alpha)$, where $q_\omega(\alpha)$ is 0 or a limit and $r_\omega(\alpha) < \omega$. We say α is an *n-limit* if $q_\omega(\alpha) > 0$ and $r_\omega(\alpha) < n - 1$, and otherwise, we say α is *n-isolated*. An n-limit α is an *n, ω-limit* if $q_\omega(\alpha)$ has countable cofinality; clearly, if $\alpha < \omega_1$, then it is an n, ω-limit iff it is an n-limit. Note that all ordinals are 1-isolated and α is 2-isolated iff it is isolated in the usual sense of the term.

For a function $f : \mathcal{O} \to \mathcal{C}$, we denote the support of f by $\mathrm{supp}(f)$, and we further let $\mathrm{supp}_I(f) = \{\beta \in \mathrm{supp}(f) : \beta \text{ is } n\text{-isolated}\}$ and $\mathrm{supp}_L(f) = \{\beta \in \mathrm{supp}(f) : \beta \text{ is an } n\text{-limit}\}$. We begin with an elementary observation.

LEMMA 1.1. *If V is a valuated p^n-socle, $f = f_V : \mathcal{O} \to \mathcal{C}$ and $\beta \in \mathrm{supp}_L(f)$, then $q_\omega(\beta)$ is a limit point of* $\mathrm{supp}_I(f)$.

PROOF. If δ is n-isolated and $\delta < \lambda \overset{\text{def}}{=} q_\omega(\beta)$, then let α be the smallest ordinal such that $\delta < \alpha \in \mathrm{supp}(f)$; so $\alpha \le \beta$. If α is an n-limit, then $\alpha = \mu + k$ where $\mu = q_\omega(\alpha) > \delta$ and $k = r_\omega(\alpha) < n - 1$. Let $x \in V(\alpha)[p] - V(\alpha+1)[p]$ and find y such that $|y|_V = \mu$ and $p^k y = x$. Since $k + 1 < n$ and $p^{k+1} y = px = 0$, there is a $z \in V(\delta + 1)$ such that $pz = y$. If $\alpha' = |z|_V > \delta$, then $\alpha' < \mu \le \alpha$, and there is a $z' \in V(\alpha' + 1)$ such that $pz' = y$. Therefore, $z - z' \in V(\alpha')[p] - V(\alpha' + 1)[p]$, and so $f(\alpha') \ne 0$. However, this contradicts the choice of α. So $\alpha \in \mathrm{supp}_I(f)$ and $\delta < \alpha < \lambda$, proving the result. □

We will say $f : \mathcal{O} \to \mathcal{C}$ is *n-isolated* if every $\alpha \in \mathrm{supp}(f)$ is n-isolated; in particular, f will always be 1-isolated. The next elementary observation is a description of the Ulm functions of the class of strongly n-summable valuated p^n-socles (see [4], Corollary 1.7).

LEMMA 1.2. *If $f : \mathcal{O} \to \mathcal{C}$ is n-isolated, then it is n-summable.*

PROOF. For an n-isolated ordinal β, let $k = \min\{n, \beta + 1\}$ and $V_\beta = \langle x_\beta \rangle$ be a cyclic valuated p^n-socle of order p^k with $|x_\beta|_{V_\beta} = 0$, when $\beta < n - 1$, and otherwise, $|x_\beta|_{V_\beta} = \beta - (n - 1)$. We then let V be the valuated direct sum of $f(\beta)$ copies of V_β for all β. It can easily be checked that $f = f_V$, so that f is n-summable, as required. □

In particular, any function $f : \mathcal{O} \to \mathcal{C}$ is 1-summable. We will say $f : \mathcal{O} \to \mathcal{C}$ is an *n, ω-limit* if it is the characteristic function of a set of the form $\{\gamma_i\}_{i<\omega} \cup \{\beta\}$, where β is an n, ω-limit ordinal, and γ_i for $i < \omega$ is a strictly ascending sequence of n-isolated ordinals with limit $q_\omega(\beta)$. Unsurprisingly, a countable valuated p^n-socle V is an *n, ω-limit* if f_V is an n, ω-limit function. The following result gives a concrete picture of such objects.

LEMMA 1.3. *If $f : \mathcal{O} \to \mathcal{C}$ is an n, ω-limit function, then f is n-summable.*

PROOF. Suppose, as above, f is the characteristic function of $\{\gamma_i\}_{i<\omega} \cup \{\beta\}$; set $\lambda = q_\omega(\beta), k = r_\omega(\beta)$. There is clearly no loss of generality in assuming $\gamma_0 \geq n - 1$. Let W be generated by y and $\{x_i\}_{i<\omega}$ subject to the relations $p^{k+1}y = 0$, and for $i < \omega$, $p^n x_i = y$. It is straightforward to check that a valuation on W can be defined by the formulas: $|p^\ell y|_W = \lambda + \ell$ for $0 \leq \ell \leq k$, $|p^\ell x_i|_W = \gamma_i - (n-1) + \ell$ for $0 \leq \ell \leq n-1$, and if $z = jy + \ell_1 x_{i_1} + \cdots + \ell_m x_{i_m}$, where $p^n \nmid \ell_1, \ldots, \ell_i$, then

$$|z|_W = \min\{|jy|_W, |\ell_1 x_{i_1}|_W, \ldots, |\ell_m x_{i_m}|_W\}.$$

It can also be checked that if $z \in pW = \langle px_i : i < \omega \rangle$ and $\alpha < |z|_W$, then there is a $z' \in W$ such that $pz' = z$ and $\alpha \leq |z'|_W$.

From this, it follows that $V = W[p^n]$ is a countable valuated p^n-socle. It also follows that there is a valuated decomposition

$$V[p] = \langle p^k y \rangle \oplus \left(\bigoplus_{i<\omega} \langle p^{n-1}(x_i - x_{i+1}) \rangle \right).$$

This implies that $f_V = f$, as required. □

If $f : \mathcal{O} \to \mathcal{C}$ is a function, and $\alpha \leq \beta \leq \infty$, then let $\int_\alpha^\beta f = \sum_{\alpha \leq \gamma < \beta} f(\gamma)$. We say f is *countable* if $\int_0^\infty f \leq \aleph_0$.

THEOREM 1.4. *Suppose $f : \mathcal{O} \to \mathcal{C}$ is a countable function. Then the following are equivalent:*

(a) *f is n-summable;*

(b) *f is the sum of a collection of n, ω-limit functions and an n-isolated function;*

(c) *If $\beta \in \mathrm{supp}_L(f)$, then $q_\omega(\beta)$ is a limit point of $\mathrm{supp}_I(f)$.*

PROOF. First, assuming (a), then (c) follows from Lemma 1.1.

Next, assuming (b), then (a) follows from Lemmas 1.2 and 1.3.

To complete the proof, suppose (c) holds for f; we will then verify (b). Let

$$I = \{(\beta, \gamma) : \beta \in \mathrm{supp}_L(f) \text{ and } \gamma < f(\beta)\},$$

$\kappa = |I| \leq \aleph_0$ and $\{(\beta_j, \gamma_j)\}_{j<\kappa}$ be a listing of I. For each $j<\kappa$ there is a strictly increasing sequence, $\{\alpha_{j,\ell}\}_{\ell<\omega} = C_j \subseteq \mathrm{supp}_I(f)$, with limit $q_\omega(\beta_j)$; we can also obviously pick these so that if $q_\omega(\beta_j) = q_\omega(\beta_{j'})$, then $C_j = C_{j'}$.

Let $\{\mathcal{N}_j\}_{j<\kappa}$ be disjoint infinite subsets of ω. For each $j<\kappa$, we inductively define $T_j \subseteq C_j$ by the equation

$$T_j = \{\alpha_{j,\ell} : \ell \in \mathcal{N}_j\} - (T_0 \cup \cdots \cup T_{j-1}).$$

Clearly, these sets are disjoint.

Claim: T_j is infinite. The first term is infinite, so it will suffice to show that if $j'<j$, then $\{\alpha_{j,\ell} : \ell \in \mathcal{N}_j\} \cap T_{j'}$ is finite. If $q_\omega(\beta_j) = q_\omega(\beta_{j'})$, then $C_j = C_{j'}$ and clearly

$$\{\alpha_{j,\ell} : \ell \in \mathcal{N}_j\} \cap T_{j'} \subseteq \{\alpha_{j,\ell} : \ell \in \mathcal{N}_j\} \cap \{\alpha_{j',\ell} : \ell \in \mathcal{N}_{j'}\} = \emptyset.$$

On the other hand, if $q_\omega(\beta_j) \neq q_\omega(\beta_{j'})$, then $\{\alpha_{j,\ell} : \ell \in \mathcal{N}_j\} \cap T_{j'} \subseteq C_j \cap C_{j'}$ is finite, since C_j and $C_{j'}$ have different suprema.

For every $(\beta_j, \gamma_j) \in I$, let f_j be the characteristic function of $\{\beta_j\} \cup T_j$; so f_j is an n, ω-limit function. In addition, if β is an n-limit ordinal, then

$$\Big(\sum_{j<\kappa} f_j\Big)(\beta) = |\{(\beta_j, \gamma_j) \in I : \beta = \beta_j\}| = f(\beta).$$

And if β is n-isolated, then $\big(\sum_{j<\kappa} f_j\big)(\beta)$ equals 1 if $\beta \in \bigcup_{j<\kappa} T_j \subseteq \mathrm{supp}_I(f)$, and otherwise, it equals 0. It follows that there is an n-isolated function g such that $f = \big(\sum_{j<\kappa} f_j\big) + g$, thereby establishing the result. □

Since a function $f : \mathcal{O} \to \mathcal{C}$ is n-summable iff it is the sum of a collection of countable n-summable functions, Theorem 1.4 immediately implies the next result.

Corollary 1.5. *A function $f : \mathcal{O} \to \mathcal{C}$ is n-summable iff it is the sum of a collection of n, ω-limit functions and an n-isolated function.*

Corollary 1.6. *If V is an n-summable valuated p^n-socle, then it is isometric to a valuated direct sum $\oplus_{i \in I} V_i$, where each V_i is either cyclic or an n, ω-limit.*

Proof. By Corollary 1.5, f_V is the sum of a collection of n, ω-limit functions and an n-isolated function, and each term is the Ulm function of a valuated p^n-socle that is an n, ω-limit or a valuated direct sum of cyclics. So the result follows from Theorem 0.1. □

Corollary 1.7. *If α is an ordinal, $\lambda = q_\omega(\alpha)$ and $k = r_\omega(\alpha)$, then α is the length of some n-summable valuated p^n-socle V iff $0 < k < n$ implies that λ has countable cofinality. In fact, if λ has countable cofinality or $k \geq n$, then we can choose V to be countable.*

Proof. If λ has cofinality κ, then let $\{\beta_i\}_{i<\kappa}$ be a strictly increasing set of n-isolated ordinals with limit λ. First, if $k = 0$, let $K = \{\beta_i\}_{i<\kappa}$. Next, if $0 < k < n$, then $\kappa = \omega$, $\alpha - 1$ is an n-limit and we let $K = \{\beta_i\}_{i<\omega} \cup \{\alpha - 1\}$. Lastly, if $k \geq n$, then $\alpha - 1 = \lambda + k - 1$ is n-isolated and we let $K = \{\alpha - 1\}$. In any of these cases, let f be the characteristic function of K. It follows from Theorem 1.5 that $f = f_V$ for some n-summable valuated p^n-socle V. It is easy to check that V will have length α, and that if λ has countable cofinality or $k \geq n$, then V will be countable.

Conversely, suppose λ is a limit ordinal of uncountable cofinality and $0 < k < n$. If the valuated p^n-socle V is either cyclic or an n, ω-limit, then $f_V(\alpha - 1) = 0$. It follows that there cannot be an n-summable valuated p^n-socle of length α. □

We now restrict our attention to functions $f : \omega_1 \to \mathcal{C}$. If λ is a limit ordinal, then let

$$f'(\lambda) = \int_{\lambda}^{\lambda+n-1} f \quad \text{and} \quad \bar{f}(\lambda) = \inf\left\{ \int_{\alpha}^{\lambda} f : \alpha < \lambda \right\}.$$

Observe that $\bar{f}(\lambda)$ is either 0 or an infinite cardinal. We say f is *n-thin* if there is a closed and unbounded subset $C \subseteq \omega_1$ consisting of limit ordinals λ such that $f'(\lambda) = 0$. Further, we say f is *n-admissible* if

(1.A) for every limit $\lambda < \omega_1$ we have $f'(\lambda) \leq \overline{f}(\lambda)$; and
(1.B) either f is n-thin, or $\{\alpha < \omega_1 : f(\alpha) \geq \aleph_1\}$ is unbounded in ω_1.

Any function $f : \omega_1 \to \mathcal{C}$ is 1-isolated and hence 1-summable. In addition, if $n = 1$, then $f'(\lambda) = 0$, so f is 1-thin and 1-admissible.

We now point out that for $n > 1$, the study of n-summable functions can be reduced to the study of 2-summable functions, which will simplify our discussion. If $n > 1$ and α is isolated, then let $f'(\alpha) = f(\alpha + n - 2)$. If $n = 2$, then $f' = f$.

LEMMA 1.8. *If $f : \omega_1 \to \mathcal{C}$ and $n > 1$, then*

(a) *f is n-isolated iff f' is 2-isolated;*
(b) *f is an n, ω-limit iff f' is a $2, \omega$-limit;*
(c) *f is n-admissible iff f' is 2-admissible;*
(d) *f is n-summable iff f' is 2-summable.*

PROOF. (a), (b) and (c) follow immediately from the definitions, and (d) follows from (a), (b) and Corollary 1.5. □

LEMMA 1.9. *Suppose $f : \omega_1 \to \mathcal{C}$ is 2-admissible and $\lambda < \omega_1$ is a limit.*

(a) *If $\overline{f}(\lambda)$ is uncountable, $\alpha < \lambda$ and $\gamma < \overline{f}(\lambda)$, then there is an isolated ordinal α' such that $f(\alpha')$ is uncountable, $\alpha < \alpha' < \lambda$ and $\gamma < f(\alpha')$.*
(b) *If $f(\lambda) \neq 0$, then λ is a limit point of* $\mathrm{supp}_I(f)$.

PROOF. We verify (a), the proof of (b) being even more straightforward.

Since $\overline{f}(\lambda) \leq \int_{\alpha+1}^{\lambda} f$, we have $\gamma' \stackrel{\text{def}}{=} \max\{\gamma, \omega\} < f(\alpha')$ for some α' with $\alpha < \alpha' < \lambda$. Choose α' to be the smallest ordinal satisfying these conditions. We need to show that α' is isolated, so assume that it is actually a limit.

By (1.A), $\gamma' < f(\alpha') \leq \overline{f}(\alpha')$, so $\overline{f}(\alpha')$ is uncountable and $\gamma < \overline{f}(\alpha')$. Arguing as in the last paragraph with λ replaced by α', we have $\gamma' < f(\alpha'')$ for some α'' with $\alpha < \alpha'' < \alpha'$, contradicting the minimality of α'. □

We now come to the main point of this section, the characterization of n-summable functions defined on ω_1.

THEOREM 1.10. *For a function $f : \omega_1 \to \mathcal{C}$, the following are equivalent:*

(a) *f is n-summable;*
(b) *f is the sum of a collection of n, ω-limit functions and an n-isolated function;*
(c) *f is n-admissible.*

Proof. If $n = 1$ then any such function satisfies any of these conditions. So we may assume $n > 1$. Replacing f by f', by Lemma 1.8, we may assume $n = 2$.

By Corollary 1.5, (a) and (b) are equivalent, so we need to show they are equivalent to (c). Suppose first that (b) holds, and let $f = \sum_{i\in I} f_i + g$, where each f_i is a $2,\omega$-limit and g is 2-isolated. It is easy to check that (1.A) holds for g and each f_i, which readily implies that it holds for f, as well.

Regarding (1.B), observe first that if $\{\alpha < \omega_1 : f(\alpha) \geq \aleph_1\}$ is unbounded in ω_1, then we are clearly done; so we may assume it is bounded, say by $\mu < \omega_1$. If (1.B) fails, then the set $S = \mathrm{supp}_L(f) \cap (\mu, \omega_1)$ is stationary in ω_1. For every $\lambda \in S$ there is an $i_\lambda \in I$ such that $f_{i_\lambda}(\lambda) \neq 0$. Since each f_i is a $2,\omega$-limit function, it follows that the assignment $\lambda \mapsto i_\lambda$ is injective. For every $\lambda \in S$ we can then find an $\alpha_\lambda \in (\mu, \lambda)$ such that $f_{i_\lambda}(\alpha_\lambda) \neq 0$. Note that $\lambda \mapsto \alpha_\lambda$ will be a regressive function, so by Fodor's Lemma (see, for example, [12], Theorem 8.7), there is an $\alpha \in (\mu, \omega_1)$ such that $\alpha_\lambda = \alpha$ for all λ in an uncountable subset R of S. This implies that

$$f(\alpha) \geq \sum_{i\in I} f_i(\alpha) \geq \sum_{\lambda\in R} f_{i_\lambda}(\alpha) = |R| = \omega_1.$$

This contradicts the fact that $f(\alpha)$ is countable for all $\alpha > \mu$. Therefore, we have shown that (b) implies (c).

Conversely, supposing (c) holds, we verify that (b) follows. Let U be the closure of $\{\alpha < \omega_1 : f(\alpha) \geq \aleph_1\}$ in the order topology. Define $f_u, f_c : \omega_1 \to \mathcal{C}$ by the conditions $\mathrm{supp}(f_u) \subseteq U$, $\mathrm{supp}(f_c) \subseteq \omega_1 - U$ and $f = f_u + f_c$. Clearly, $f_c(\alpha) \leq \aleph_0$ for all $\alpha < \omega_1$, and if α is isolated, then $f_u(\alpha)$ will be 0 or uncountable. We will be done if we can verify the following:

Claim 1. (b) holds for f_u.

Claim 2. (b) holds for f_c.

Starting with Claim 1, let

$$I = \{(\beta, \gamma) : \beta \in \mathrm{supp}_L(f_u) \text{ and } \gamma < f_u(\beta) = f(\beta)\}.$$

If $i = (\beta, \gamma) \in I$, then clearly $\overline{f}(\beta)$ is uncountable. Therefore, by Lemma 1.9(a), there is a strictly increasing sequence $K_i = \{\alpha_{i,j}\}_{j<\omega} \subseteq \mathrm{supp}_I(f_u)$ with limit β such that $\gamma < f(\alpha) = f_u(\alpha)$ for all $\alpha \in K_i$. Let $f_i : \omega_1 \to \kappa$ be the characteristic function of $\{\beta\} \cup K_i$; so f_i is an n-limit.

We need to establish two facts:

(1) for all $\beta \in \mathrm{supp}_L(f_u)$, we have $\sum_{i\in I} f_i(\beta) = f_u(\beta)$; and

(2) for all $\alpha \in \mathrm{supp}_I(f_u)$, we have $\sum_{i\in I} f_i(\alpha) \leq f_u(\alpha)$.

For (1), if $\beta \in \mathrm{supp}_L(f_u)$, then $f_i(\beta) = 1$ iff $i = (\beta, \gamma)$, where $\gamma < f_u(\beta)$. This happens for precisely $f_u(\beta)$ elements $i \in I$, so that (1) follows. Regarding (2), if $\alpha \in \mathrm{supp}_I(f_u)$, then $f_u(\alpha) \geq \aleph_1$. Further, if $i = (\beta, \gamma) \in I$ and $f_i(\alpha) = 1$, then $\beta > \alpha$ and $\gamma < f_u(\alpha)$. Since this can happen for at most $\aleph_1 \cdot f_u(\alpha) = f_u(\alpha)$ elements $i \in I$, we can conclude that (2) follows. So we have verified Claim 1.

Turning to Claim 2, we first define a closed and unbounded subset $C \subseteq \omega_1$ as follows: If U is unbounded, then we let $C = U$. On the other hand, if U is bounded by $\mu < \omega_1$, then by (1.B) we can find such a subset $D \subseteq (\mu, \omega_1)$ consisting of limit ordinals λ such that $f(\lambda) = 0$. In this case, we let $C = U \cup D$. Since $\mathrm{supp}(f_c) \subseteq \omega_1 - U$ and $f(\lambda) = 0$ for all $\lambda \in D$, it follows that $\mathrm{supp}(f_c) \subseteq \omega_1 - C$ and that f_c agrees with f on $\omega_1 - C$.

Since C is closed, $\omega_1 - C$ is the disjoint union of sets of the form $(\delta_i, \varepsilon_i)$ for $i \in I$, where $\varepsilon_i < \omega_1$. Let $f_c = \sum_{i \in I} f_i$ where $\mathrm{supp}(f_i) \subseteq (\delta_i, \varepsilon_i)$. Since f_i is countable, Claim 2, and hence the entire result, will follow once we show f_i satisfies Theorem 1.4(c) (where $n = 2$).

Let $\lambda < \omega_1$ be a limit ordinal such that $f_i(\lambda) \neq 0$; we need to show that λ is a limit point of $\mathrm{supp}_I(f_i)$. Note that $\delta_i < \lambda < \varepsilon_i$ and by Lemma 1.9(b), λ is a limit point of $\mathrm{supp}_I(f)$. Since f, f_c and f_i agree on $(\delta_i, \varepsilon_i)$, it follows that λ is a limit point of $\mathrm{supp}_I(f_i)$, as required. □

2. Realizing Valuated p^n-Socles

In this section we discuss which ω_1-bounded valuated p^n-socles V appear as the p^n-socle of some group. In fact, we will start by being a bit more general. We say a valuated group V is *group-like* if for all ordinals α, we have $(pV)(\alpha + 1) = p(V(\alpha))$; in other words, if $x \in pV$ and $\alpha < |x|_V$, then there is a $y \in V$ such that $\alpha \leq |y|_V$ and $py = x$. Note that if $p^n V = \{0\}$, then $pV \subseteq V[p^{n-1}]$, which readily implies that any valuated p^n-socle is group-like. We say a valuated group V is *supported* by a group G if V is an essential subgroup of G so that for all $x \in V$, $|x|_V = |x|_G$. We will say V is *realizable* if it is supported by some group G. For valuated p^n-socles, this agrees with our previous terminology.

Recall from [16] that if V is a valuated group, there is a group $H(V)$ containing V as a nice subgroup such that $H(V)/V$ is totally projective. In this construction we may clearly assume that V and $H(V)$ have the same length.

PROPOSITION 2.1. *Suppose V is a valuated group. Then V is realizable iff it is group-like and there is a balanced subgroup $Z \subseteq H(V)$ such that there is a valuated decomposition $H(V)[p] = Z[p] \oplus V[p]$.*

PROOF. Suppose first that V is supported by the group G. To show that V is group-like, let $x \in (pV)(\alpha + 1)$. Then $x = pz$ for some $z \in V$. In addition, there is a $y \in p^\alpha G$ such that $py = x$. Note that $p(y - z) = 0$ so that $y - z \in G[p] \subseteq V$. Therefore, $y = z + (y - z) \in V \cap p^\alpha G = V(\alpha)$ and $py = x$, as required.

Next, since V is a nice subgroup of $H(V)$ and $H(V)/V$ is totally projective, by ([6], Corollary 81.4) the identity map $V \to V$ extends to a homomorphism $\pi : H(V) \to G$. It is easy to see that the range of π is a pure subgroup of G containing V. And since V is essential in G, we can conclude that π is surjective. Let Z be its kernel.

Since for all α we have $(p^\alpha G)[p] = V(\alpha)[p] \subseteq \pi((p^\alpha H(V))[p]) \subseteq (p^\alpha G)[p]$, it follows that Z is balanced in $H(V)$. The identity map $V[p](= G[p]) \to V[p](\subseteq H(V))$ being an isometry leads to the valuated decomposition $H(V)[p] = Z[p] \oplus V[p]$, giving one implication.

Conversely, suppose V is group-like and we are given $Z \subseteq H(V)$ as indicated. Let $G = H(V)/Z$ and $\pi : H(V) \to G$ be the natural epimorphism.

It is clear that π maps V onto an essential subgroup of G. We will be done if we can show that for all $x \in V$, we have $|x|_V = |\pi(x)|_G$. We certainly have $|x|_V = |x|_{H(V)} \leq |\pi(x)|_G$. We establish the reverse inequality by induction on the order of x.

Suppose first that $x \in V[p]$ and $\alpha = |\pi(x)|_G$. Since Z is a balanced subgroup of $H(V)$, there is a $y \in (p^\alpha H(V))[p]$ such that $\pi(y) = x$. We must have $y = x + z$, where $z \in Z[p]$, so that $\alpha \leq |y|_H \leq |x|_V$, as required.

Suppose now that this holds for all elements of $V[p^{k-1}]$, and $x \in V[p^k]$, where $\alpha = |\pi(x)|_G$. Since $px \in V[p^{k-1}]$, by induction $\alpha + 1 \leq |\pi(px)|_G = |px|_V$. Since V is group-like, there is a $y \in V(\alpha)$ such that $py = px$. Note that $x - y \in V[p]$ and $|\pi(x - y)|_G \geq \alpha$, so that $|x - y|_V \geq \alpha$. However, this implies that $|x|_V = |(x - y) + y|_V \geq \alpha$, as required. □

PROPOSITION 2.2. *If V is an ω-bounded valuated group, then V is realizable iff it is group-like.*

PROOF. By Proposition 2.1, if V is realizable, then it is group-like. Therefore, assume V is ω-bounded and group-like. Since V is ω-bounded, $H(V)$ will be separable. And since V is nice in $H(V)$, the quotient $H(V)/V$ will be a separable totally projective group, i.e., it is Σ-cyclic. This implies that there is a valuated decomposition $H(V)[p] = F \oplus V[p]$, where F is a free, ω-bounded valuated vector space (see, for example, [11], Lemma 1). It follows that there is a pure (and hence isotype) Σ-cyclic subgroup $Z \subseteq H(V)$ such that $Z[p] = F$. Since F is closed in $H(V)[p]$ in the induced p-adic topology, it follows that Z is closed in $H(V)$. This means that $G \stackrel{\text{def}}{=} H(V)/Z$ is separable. Therefore, Z is nice, and hence balanced, in $H(V)$. So by Proposition 2.1, V is realizable. □

Since any valuated p^n-socle is group-like, the next statement follows directly from Proposition 2.2.

COROLLARY 2.3. *If V is an ω-bounded valuated p^n-socle, then it is realizable.*

The main purpose of this section is to describe precisely when a given n-summable valuated p^n-socle is realizable. As discussed in the introduction, it is also natural to ask when a group G supported by an n-summable valuated p^n-socle V is *determined* by V, i.e., if G' is any other group supported by V, then $G \cong G'$. If such a G is determined by V, we will say V is *uniquely* realizable.

For example, if V is an ω-bounded n-summable valuated p^n-socle, then by Corollary 2.3, V is realizable. In fact, the summability of $V[p]$ implies that any group supported by V will be Σ-cyclic. In other words, ω-bounded n-summable valuated p^n-socles are uniquely realizable. In the next section, we consider when this remains the case for groups of greater length. For now, we present a simple example.

EXAMPLE 2.4. Suppose V is a valuated p^n-socle which is n-summable and $\omega 2 = \omega + \omega$-bounded. If $f = f_V$ and $\overline{f}(\omega) \geq \int_{\omega+n-1}^{\omega 2} f \geq \aleph_1$, then V is realizable, but not uniquely realizable. In fact, there are groups A and A' supported by V such that $A/p^\omega A$ is Σ-cyclic and $A'/p^\omega A'$ is not.

PROOF. Since f is n-admissible, $\overline{f}(\omega) \geq f'(\lambda)$, which readily implies that f is an admissible function; so there is a dsc group A with $f = f_A$. It follows that $A[p^n]$ is n-summable, so by Theorem 0.1, there is an isometry $A[p^n] \cong V$; in other words, V is realizable.

On the other hand, let $M = \{m < \omega : f(m) \geq \aleph_0\}$, and let $k \geq n-1$ be an integer such that $f(\omega + k) \geq \aleph_1$. Let $B = \bigoplus_{m \in M} \mathbb{Z}_{p^{m+k+2}}$. Next, let X be a pure subgroup of the torsion completion $\overline{B}$ containing B such that X/B has rank $\aleph_1$. It can be verified that $Y = X/B[p^{k+1}]$ is n-summable and

$$f_Y(\alpha) = \begin{cases} 1, & \text{if } \alpha \in M; \\ \aleph_1, & \text{if } \alpha = \omega + k; \\ 0, & \text{otherwise.} \end{cases}$$

If we let $A' = A \oplus Y$, then A' will also be n-summable. We have constructed A' so that $f_{A'} = f_A = f$. This implies that $A'[p^n]$ is also isometric to V.

Clearly, since A is a dsc group, $A/p^\omega A$ is Σ-cyclic. On the other hand, $A'/p^\omega A'$ has a summand isomorphic to

$$Y/p^\omega Y = (X/B[p^{k+1}])/(X[p^{k+1}]/B[p^{k+1}]) \cong X/X[p^{k+1}] \cong p^{k+1}X,$$

which is not Σ-cyclic. It follows that $A'/p^\omega A'$ is also not Σ-cyclic, so that V is not uniquely realizable. □

If λ is a limit ordinal and V is a valuated group, then the *λ-topology* on V uses $V(\beta)$ for $\beta < \lambda$ as a neighborhood base of 0. Naturally, a subgroup W of V is said to be *λ-dense* if it is dense in this topology, i.e., for all $\beta < \lambda$ we have $V = V(\beta) + W$. We let $L_\lambda V$ be the completion of V in the λ-topology, i.e., the inverse limit of $V/V(\beta)$ over $\beta < \lambda$. As an exception to our overall restriction to primary groups, $L_\lambda V$ may have elements of infinite order. However, if V is a valuated p^n-socle, then $p^n L_\lambda V = \{0\}$. There is a natural map $V \to L_\lambda V$ whose kernel is $V(\lambda)$, leading to an inclusion $V/V(\lambda) \subseteq L_\lambda V$. We then set $P_\lambda V = (L_\lambda V)[p]/(V/V(\lambda))[p]$ and $Q_\lambda V = L_\lambda V/(V/V(\lambda))$. If G is a group and λ has countable cofinality, $G/p^\lambda G$ will be isotype and λ-dense in $L_\lambda G$, and $Q_\lambda G$ will be divisible.

We now review some additional concepts from [4]. If α is an ordinal and V is a valuated p^n-socle, then a subgroup W of V is *α-high* if it is maximal with respect to $W \cap V(\alpha) = \{0\}$. An α-high subgroup W of V is *n-isotype* in V (i.e., it is a valuated p^n-socle under the induced valuation), and if α is a limit, W is α-dense in V. If α is n-isolated, then there is a valuated decomposition $V = W \oplus U$, which we refer to as a *standard α-decomposition* of V. If, in addition, $\alpha \geq n-1$, then $U \subseteq V(\alpha - n + 1)$. We now connect these definitions.

LEMMA 2.5. *Suppose V is a valuated p^n-socle, $\lambda < \omega_1$ is a limit ordinal and W is λ-high in V. Suppose further that G is a group supported by W, $\pi : L_\lambda G \to Q_\lambda G$ is the canonical epimorphism and $\mu : V \to L_\lambda G$ is the natural extension of the homomorphism $W \subseteq G \to L_\lambda G$ (so that $V(\lambda)$ is the kernel of μ). Then there is a natural isomorphism*

$$P_\lambda V \cong (Q_\lambda G)[p]/\pi(\mu(V))[p].$$

PROOF. Note that $(L_\lambda G)[p^n]$ will be complete in the (induced) λ-topology and $W \subseteq V$ will be λ-dense in it; so we can identify $(L_\lambda G)[p^n]$ with $L_\lambda W \cong L_\lambda V$. Therefore,

$$\begin{aligned} P_\lambda(V) &= (L_\lambda V)[p]/(V/V(\lambda))[p] \\ &= (L_\lambda G)[p]/\mu(V)[p] \\ &\cong [(L_\lambda G)[p]/G[p]]/[\mu(V)[p]/G[p]] \\ &\cong (Q_\lambda G)[p]/\pi(\mu(V))[p], \end{aligned}$$

completing the proof. □

We now describe a natural way to construct a group supported by a valuated p^n-socle. The following is the main inductive step.

LEMMA 2.6. *Suppose V is a valuated p^n-socle, $\lambda < \lambda' < \omega_1$ are limit ordinals and $W_\lambda \subseteq W_{\lambda'}$ are, respectively, λ and λ'-high in V. Let G_λ be a group supported by W_λ and A be a group supported by $W_{\lambda'}(\lambda)$ (where, technically, we shift the valuation in the latter group by λ). Then there is a group $G_{\lambda'}$ containing G_λ supported by $W_{\lambda'}$ such that $p^\lambda G_{\lambda'} = A$ iff*

$$(*) \qquad \int_{\lambda+n-1}^{\lambda+\omega} f \leq r(P_\lambda V).$$

PROOF. Let $A = J \oplus K$, where J is a maximal p^{n-1}-bounded summand of A. There is a standard $\lambda + n - 1$-decomposition $W_{\lambda'} = W_{\lambda+n-1} \oplus Y$, where $W_{\lambda+n-1}$ is $\lambda + n - 1$-high in V containing W_λ, $W_{\lambda+n-1}(\lambda) = J$ and $Y = K[p^n]$. Let D be a divisible hull of $W_{\lambda+n-1}/W_\lambda$ and E be a divisible hull of K. There is an embedding $J = W_{\lambda+n-1}(\lambda) \cong J' \stackrel{\text{def}}{=} [W_{\lambda+n-1}(\lambda) + W_\lambda]/W_\lambda \subseteq D$.

By a standard construction (cf., [8], Theorem 106), the existence of $G_{\lambda'}$ is equivalent to the existence of a commutative pull-back diagram

$$\begin{array}{ccccccccc} & & & & A & = & A & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & G_\lambda & \to & G_{\lambda'} & \to & D \oplus E & \to & 0 \\ & & \| & & \downarrow & & \downarrow \phi & & \\ 0 & \to & G_\lambda & \to & L_\lambda G_\lambda & \stackrel{\pi}{\to} & Q_\lambda G_\lambda & \to & 0 \end{array}$$

Therefore, given G_λ, the existence of $G_{\lambda'}$ is equivalent to the existence of a homomorphism $\phi : D \oplus E \to Q_\lambda G_\lambda$ with kernel $J' \oplus K \cong A$.

The natural homomorphism $\mu : V \to L_\lambda G_\lambda$, whose kernel is $V(\lambda)$, induces a homomorphism $\gamma_0 : W_{\lambda+n-1}/W_\lambda \to Q_\lambda G_\lambda$, whose kernel is J'. The divisibility of $Q_\lambda G_\lambda$ implies that γ_0 extends to a homomorphism $\gamma_1 : D \to Q_\lambda G_\lambda$. Since $W_{\lambda+n-1}/W_\lambda$ is (algebraically) the direct sum of a collection of copies of $\mathbb{Z}_{p^n}$ and $p^{n-1}J' = \{0\}$, it follows that $(W_{\lambda+n-1}/W_\lambda)/J'$ is essential in D/J'. This implies that the kernel of γ_1 is also J'. In addition, by Lemma 2.5, there is an isomorphism

$$P_\lambda(V) \cong (Q_\lambda G)[p]/\pi(\mu(V))[p] = (Q_\lambda G)[p]/\gamma_1(D)[p].$$

A standard computation shows that the rank of E/K agrees with the rank of a basic subgroup of K, which is $\int_{\lambda+n-1}^{\lambda+\omega} f$. Therefore, there is a homomorphism $\sigma : E \to Q_\lambda G_\lambda$ such that $\phi = \gamma_1 + \sigma : D \oplus E \to Q_\lambda G_\lambda$ has kernel $J' \oplus K \cong A$ iff ($*$) holds. □

We will say the valuated group V has an *ω_1-high tower* if it is the smoothly ascending union of a chain of subgroups indexed by the countable limit ordinals, $\{W_\lambda\}_{\lambda<\omega_1}$, such that for each λ, W_λ is λ-high in V. For example, if V is α-bounded where $\alpha < \omega \cdot \omega$, then V clearly has an ω_1-high tower. If V is a valuated p^n-socle or a group, then V has an ω_1-high tower iff the socle $V[p]$ has an ω_1-high tower. In particular, if V is a group or valuated p^n-socle that is summable, then it has an ω_1-high tower.

THEOREM 2.7. *Suppose the valuated p^n-socle V has ω_1-high tower and $f = f_V$. Then V is realizable iff for every countable limit ordinal λ, we have*

$$\int_{\lambda+n-1}^{\lambda+\omega} f \le r(P_\lambda(V)).$$

PROOF. Let W_λ be as above. For each limit ordinal $\lambda < \omega_1$, we inductively construct an ascending chain of groups G_λ supported by W_λ. If λ is a limit of limit ordinals, we can construct G_λ by taking a union. Suppose then that we have constructed W_λ and we want to construct $W_{\lambda+\omega}$. By Corollary 2.3, we can find a group A supported by $W_{\lambda+\omega}(\lambda)$. So by Lemma 2.6, we can extend G_λ to get the required $G_{\lambda+\omega}$ iff our cardinality condition is satisfied. □

We will need a pair of observations about the above construction, so assume the notation in the proof of Lemma 2.6. If $\int_{\lambda+n-1}^{\lambda+\omega} f = r(P_\lambda(V))$ is infinite, then we could construct two different extensions $\phi, \phi' : D \oplus E \to Q_\lambda G_\lambda$ of γ_1, both with kernel $J' \oplus K \cong A$, such that ϕ maps onto the torsion subgroup of $Q_\lambda G_\lambda$, and ϕ' does not. If $G_{\lambda'}$ and $G'_{\lambda'}$ are the groups constructed with these two homomorphisms, we could conclude that $Q_\lambda G_{\lambda'}$ is torsion-free and $Q_\lambda G'_{\lambda'}$ is not (in other words, $G_\lambda/p^\lambda G_\lambda$ is *λ-torsion complete* and $G_{\lambda'}/p^\lambda G_{\lambda'}$ is not). This establishes the first part of the next result.

COROLLARY 2.8. *Suppose the n-summable valuated p^n-socle V is realizable and $f = f_V$. If either*

(1) *for some limit ordinal $\lambda < \omega_1$, $\int_{\lambda+n-1}^{\lambda+\omega} f = r(P_\lambda(V))$ is infinite, or*

(2) $\overline{f}(\omega) \geq \int_{\omega+n-1}^{\omega 2} f \geq \aleph_1$,

then there are non-isomorphic groups G and G' supported by V.

PROOF. Regarding (2), let $W_{\omega 2}$ be an $\omega 2$-high subgroup of V. As in Example 2.4, we can construct non-isomorphic groups A and A' of length $\omega 2$ supported by $W_{\omega 2}$; in particular, $A/p^\omega A$ and $A'/p^\omega A'$ will not be isomorphic. If we extend these to groups G and G' supported by V, then since $G/p^\omega G \cong A/p^\omega A$ and $G'/p^\omega G' \cong A'/p^\omega A'$, we can conclude that G and G' are not isomorphic. □

We have the following immediate consequence of Theorem 2.7, which is an extension of Corollary 2.3.

COROLLARY 2.9. *Any $\omega + n - 1$-bounded valuated p^n-socle V is realizable.*

We now specialize this to the case of valuated p^n-socles that are n-summable. The next result allows us to compute the required invariants.

LEMMA 2.10. *If V is an n-summable valuated p^n-socle, $f \stackrel{\text{def}}{=} f_V$ and λ is a countable limit ordinal, then $r(P_\lambda V) = (\overline{f}(\lambda))^{\aleph_0}$.*

PROOF. Let $\xi = \overline{f}(\lambda)$ and $\beta_0 < \lambda$ be chosen so that $\int_{\beta_0}^{\lambda} f = \int_{\beta}^{\lambda} f = \xi$ for all $\beta_0 \leq \beta < \lambda$. We may clearly assume that β_0 is n-isolated, so there is a standard β_0-decomposition $V = W \oplus U$, where W is β_0-high in V. It follows that $P_\lambda V$ is naturally isomorphic to $P_\lambda U$. Replacing V by U, we may assume $\int_{0}^{\lambda} f = \xi$.

Since V is a valuated direct sum of countable groups, the same will be true of $V_\lambda \stackrel{\text{def}}{=} V/V(\lambda)$, and $B \stackrel{\text{def}}{=} V_\lambda[p]$. Since a countable valuated vector space is free, we can conclude that B is free. Clearly, $f_B(\beta) = f(\beta)$ for all $\beta < \lambda$, and we can identify $(L_\lambda V)[p]$ with $L_\lambda B$, so that $P_\lambda V \cong P_\lambda B$.

If α_i is a strictly increasing sequence that converges to λ, then we can write $B = \bigoplus_{i<\omega} B_i$, where for each $i<\omega$ we have $B_i(\alpha_i) = B_i$ and $B_i(\alpha_{i+1}) = \{0\}$. It follows that $L_\lambda B$ is isometric to $\overline{B} = \prod_{i<\omega} B_i$, so that $P_\lambda V$ is isomorphic to $\overline{B}/B$. Since $\overline{B}/B$ has rank $\xi^{\aleph_0}$, the result follows. □

This brings us to the main goal of this section, i.e., the generalization of the Existence Theorem for Principal p-Groups from [9] promised in the introduction. It is an immediate consequence of Theorem 2.7, Lemma 2.10 and the fact that an ω_1-bounded n-summable valuated p^n-socle has an ω_1-high tower.

THEOREM 2.11. *Suppose V is an ω_1-bounded n-summable valuated p^n-socle and $f = f_V$. Then V is realizable iff for every countable limit ordinal λ we have*

$$\int_{\lambda+n-1}^{\lambda+\omega} f \leq \overline{f}(\lambda)^{\aleph_0}.$$

For example, if $f(\alpha) = 1$ for $\alpha \leq \omega$, $f(\omega+1) = \kappa$ and $f(\alpha) = 0$ for $\alpha > \omega + 1$, then f is 2-admissible, so there is a valuated p^2-socle V with $f = f_V$. However, this V is realizable iff $\kappa \leq 2^{\aleph_0}$.

3. Unique Realization

We say a function $f : \omega_1 \to \mathcal{C}$ is ω_1-*countable*, if $f(\alpha)$ is countable for all $\alpha < \omega_1$. The following is similar to ([4], Corollary 1.7).

PROPOSITION 3.1. *Suppose V is a valuated p^n-socle and $f \stackrel{\text{def}}{=} f_V$ is ω_1-countable. Then V is n-summable iff it is summable and f is n-thin.*

PROOF. Suppose first that V is n-summable, so that f is n-admissible. It follows immediately that V is summable. And it follows from (1.B) that f is n-thin.

Conversely, suppose V is summable and f is n-thin. Let $\alpha_0 = 0$ and for $0 < i < \omega_1$, let α_i be a strictly increasing enumeration of a CUB subset of countable limit ordinals such that $f'(\alpha_i) = 0$ for all $0 < i < \omega_1$. After collecting terms, there is a valuated decomposition $V[p] = \bigoplus_{i<\omega_1} S_i$, where $S_0 \subseteq V[p]$ is α_1-high and for $i > 0$, $S_i \subseteq V(\alpha_i + n - 1)[p]$ is α_{i+1}-high. Clearly, each S_i is countable.

Let W_i be an α_{i+1}-high subgroup of $V(\alpha_i)$ such that $W_i[p] = S_i$. It follows that W_i is countable and n-isotype in V. By ([4], Corollary 1.6), there is a valuated decomposition $V = \bigoplus_{i<\omega_1} W_i$, completing the argument. □

COROLLARY 3.2. *Suppose G is a group such that f_G is ω_1-countable. Then G is n-summable iff it is summable and f_G is n-thin. In this case, $G/p^\alpha G$ will be countable for all $\alpha < \omega_1$.*

PROOF. The first conclusion follows by applying Proposition 3.1 to $G[p^n]$. To verify the last sentence, if H is $p^{\alpha+\omega}$-high in G, then there is an isomorphism $H/p^\alpha H \cong G/p^\alpha G$. Since $H[p]$ is countable, so is H. Therefore, $G/p^\alpha G$ is also countable. □

For example, if $f(\alpha) = 1$ for all $\alpha < \omega_1$, and V is a free valuated vector space with $f = f_V$, then V is realizable, so there is a summable group G such that $f_G = f$. Since $G/p^\alpha G$ will be countable for all $\alpha < \omega_1$, this G will be a C_{ω_1}-group; this is the type of group constructed in [1]. Note that this group will not be 2-summable, since f is not 2-thin. More generally, if we define f so that $f(\alpha) = 0$ whenever α is an n-limit and $f(\alpha) = 1$ whenever α is n-isolated, then there is an n-summable group G such that $f = f_G$. However, since f_G is not $n+1$-thin, G will not be $n+1$-summable.

We now turn to a discussion of when a realizable n-summable valuated p^n-socles is, in fact, uniquely realizable. We start with the case where the Ulm function is ω_1-countable.

THEOREM 3.3. *Let V be a realizable n-summable valuated p^n-socle such that $f \stackrel{\text{def}}{=} f_V : \omega_1 \to \mathcal{C}$ is ω_1-countable. Assuming the continuum hypothesis (CH), V is uniquely realizable iff it is countable.*

PROOF. Clearly, if V is countable then any group supported by V will be countable, and such groups are classified by their Ulm functions. [This direction clearly does not depend on CH.]

Conversely, suppose V is uncountable; i.e., supp(f) is unbounded. We need to construct non-isomorphic groups G_S and G_T supported by V. Let $\{W_\lambda\}_{\lambda<\omega_1}$ be an ω_1-high tower of V. Let S_λ, T_λ, for limit ordinals $\lambda < \omega_1$, be a smoothly ascending chains of countable sets such that $S_\lambda \cap V = T_\lambda \cap V = W_\lambda$ and $|S_{\lambda+\omega} - (S_\lambda \cup V)| = |T_{\lambda+\omega} - (T_\lambda \cup V)| = \aleph_0$ and let S, T

be their unions. As sets, we will let $G_S = S$ and $G_T = T$ and we will inductively define group structures on S_λ and T_λ so that $S_\lambda[p^n] = T_\lambda[p^n] = W_\lambda$. It is important to note that since S_ω will be pure and ω-dense in S_λ, if A is a reduced group, then a homomorphism $f : S_\lambda \to A$ is uniquely determined by its restriction to S_ω.

Given CH, the cardinality of the set of functions $f : S_\omega \to T$ is ${\aleph_1}^{\aleph_0} = \aleph_1$, so we can index them by f_i for $i < \omega_1$. Let $E_i \subseteq \omega_1$ for $i < \omega_i$ be a collection of pairwise disjoint stationary subsets consisting of limit ordinals (see, for example, [12], Lemma 8.8).

Suppose λ' is a limit and for all $\lambda < \lambda'$ we have defined group structures on S_λ and T_λ supported by W_λ. If λ' is a limit of limits, then we simply take unions. Suppose then, that we have constructed our group structures on S_λ and T_λ, and $\lambda' = \lambda + \omega$. We now divide the construction into two parts.

Case 1. $\lambda \in E_i$ and $f_i(S_\omega) \subseteq T_\lambda$ extends to an isomorphism $g : S_\lambda \to T_\lambda$.

Identify S_λ and T_λ using g and call the result G_λ. Then as in the proof of Lemma 2.6, there are group structures on $S_{\lambda+\omega}$ and $T_{\lambda+\omega}$ that extend G_λ and are supported by $W_{\lambda+\omega}$, for which $S_{\lambda+\omega}/p^\lambda S_{\lambda+\omega}$ and $T_{\lambda+\omega}/p^\lambda T_{\lambda+\omega}$ map to distinct subgroups of $L_\lambda G_\lambda$. In particular, this means that the isomorphism $g : S_\lambda \to T_\lambda$ does not extend to an isomorphism $S_{\lambda+\omega}/p^\lambda S_{\lambda+\omega} \to T_{\lambda+\omega}/p^\lambda T_{\lambda+\omega}$.

Case 2. Case 1 does not apply.

Extend the group structures on S_λ and T_λ to $S_{\lambda+\omega}$ and $T_{\lambda+\omega}$ in any way so that they are supported by $W_{\lambda+\omega}$.

Let $G_S = \bigcup_{\lambda<\omega_1} S_\lambda$ and $G_T = \bigcup_{\lambda<\omega_1} T_\lambda$. It follows that V is supported by both G_S and G_T. We claim that they are not isomorphic; assume otherwise, and let $h : G_S \to G_T$ be such an isomorphism. It follows that $C = \{\lambda < \omega_1 : h(S_\alpha) = T_\alpha\}$ is closed and unbounded in ω_1. Let $i < \omega_1$ be such that h restricted to S_ω agrees with f_i. Since E_i is stationary, it follows that there is an $\lambda \in E_i \cap C$. Note that h will induce an isomorphism from $G_S/p^\lambda G_S \cong S_{\lambda+\omega}/p^\lambda S_{\lambda+\omega}$ to $G_T/p^\lambda G_T \cong T_{\lambda+\omega}/p^\lambda T_{\lambda+\omega}$, which is an extension of the isomorphism $g \overset{\text{def}}{=} h|_{S_\lambda} : S_\lambda \to T_\lambda$. This is contrary to the construction from Case 1, which shows that G_S and G_T are not isomorphic. So V is not uniquely realizable. □

Observe that the continuum hypothesis is equivalent to the condition that $\aleph_0^{\aleph_0} \le \aleph_1 = \aleph_0^+$. By the *CEH* (for "countable exponent hypothesis) we will mean the statements that $\kappa^{\aleph_0} \leq \kappa^+$ for all infinite cardinals κ. Clearly, the generalized continuum hypothesis implies the CEH. The following result, therefore, generalizes Theorem 3.3.

THEOREM 3.4. *Suppose V is a realizable n-summable valuated p^n-socle and $f = f_V$. If $V(\omega + n - 1)$ is countable, then V is uniquely realizable. In fact, any group supported by V must be a dsc group.*

Conversely, assuming the CEH, if $V(\omega + n - 1)$ is uncountable, then V is not uniquely realizable.

PROOF. Suppose first that $V(\omega + n - 1)$ is countable and G is a group supported by V. Let H be $p^{\omega+n-1}$-high in G. Since G is n-summable, by ([4], Theorem 3.5), H must be a dsc group. Since $r(G/H) = r(V(\omega + n - 1)) \leq \aleph_0$, it follows from Wallace's Theorem (see, for example, [3], Proposition 1.1) that G is a dsc group. Therefore, G is determined by V.

Conversely, suppose $V(\omega + n - 1)$ is uncountable. Let $\kappa = \int_{\omega+n-1}^{\omega 2} f$.

CASE 1. κ is uncountable.

Let $\mu = \overline{f}(\omega)$. If $\kappa \leq \mu$, then it follows from Corollary 2.8(2) that V is not uniquely realizable. We may therefore assume that $\kappa > \mu$, so that the CEH implies $\kappa \geq \mu^{\aleph_0}$. By Theorem 2.11, $\kappa \leq \mu^{\aleph_0}$, so that $\kappa = \mu^{\aleph_0} = r(P_\omega(V))$. In this case, Corollary 2.8(1) implies that V is not uniquely realizable.

CASE 2. $\kappa = \aleph_0$.

Subcase 2a: For all α with $\omega + n - 1 \leq \alpha < \omega_1$ we have $f(\alpha) \leq \aleph_0$.

It follows from Theorem 3.3 that there are non-isomorphic groups A_0 and A_1 supported by $V(\omega + n - 1)$ (where, again, we shift values by $\omega + n - 1$). These two groups are contained in groups G_1 and G_2 supported by V, where $A_0 = p^{\omega+n-1}G_1$ and $A_1 = p^{\omega+n-1}G_2$. Since G_1 and G_2 will also not be isomorphic, it follows that V is not uniquely realizable.

Subcase 2b: For some α with $\omega + n - 1 \leq \alpha < \omega_1$ we have $f(\alpha) \geq \aleph_1$.

Let $\alpha \geq \omega$ be chosen minimally so that $f(\alpha) \geq \aleph_1$. Since $\kappa = \int_{\omega+n-1}^{\omega 2} f = \aleph_0$, we can conclude that $\alpha \geq \omega 2$. Since f is n-admissible, α must be n-isolated. If $q_\omega(\alpha) = \lambda \geq \omega 2$, then $\overline{f}(\lambda) \leq \int_{\omega+n-1}^{\lambda} f = \aleph_0$. In addition, by Theorem 2.11

$$\aleph_1 \leq \int_{\lambda+n-1}^{\lambda+\omega} f \leq r(P_\lambda(V)) = \overline{f}(\lambda)^{\aleph_0} \leq \aleph_0^{\aleph_0} = \aleph_1.$$

It follows from Corollary 2.8(1) that V is not uniquely realizable, completing the proof. □

Again, assuming the CEH, the last result states that an n-summable group G is uniquely determined by f_G iff it is a dsc group and $p^{\omega+n-1}G$ is countable. We now consider the particular case where $n = 1$.

A group G is said to be *ω-totally Σ-cyclic* if every separable subgroup of G is Σ-cyclic. These groups were studied in [2], where, for example, it was shown that they are precisely the dsc groups G for which $p^{\omega}G$ is countable. Therefore, we have the following consequence of Theorem 3.4 for $n = 1$.

Corollary 3.5. *Assuming the CEH, a summable group G is uniquely determined by f_G iff it is ω-totally Σ-cyclic.*

For example, if $f(\alpha) = \aleph_1$ for all $\alpha \leq \omega$ and $f(\alpha) = 0$ for $\alpha > \omega$, then as in Example 2.4, there are summable groups A and A' with Ulm function f such that A is a dsc group and A' is not. So, as a summable group A is *not* uniquely determined by f. However by Theorem 3.4, as a 2-summable group A *is* uniquely determined by f.

In ([4], Theorem 3.8), it was shown that if G is a group and $G[p^n]$ is n-summable for all positive integers n, then G must be a dsc group. We conclude this paper with an analogous result for Ulm functions that illustrates the power of our realization results.

Theorem 3.6. *A function $f : \omega_1 \to \mathcal{C}$ is admissible iff it is n-admissible for all positive integers n.*

Proof. If f is admissible, then there is a dsc group G such that $f = f_G$. It follows that $G[p^n]$ is n-summable and $f = f_{G[p^n]}$. So by Theorem 1.10, f is n-admissible for each positive integer n.

Conversely, suppose for each positive integer n, f is n-admissible. So by Theorem 1.10, there is an n-summable valuated p^n-socle V_n such that $f_{V_n} = f$. It is clear that $V_{n+1}[p^n]$ will also be an n-summable valuated p^n-socle and that $f_{V_{n+1}[p^n]} = f$, so that by Theorem 0.1 there is an isometry $V_n \cong V_{n+1}[p^n]$. If we identify these two, then we can define G to be the union $\bigcup_{1\leq n<\omega} V_n$. Note that G has a valuation $|\ |_G$ determined by the valuations on the various V_n. In addition, if $x \in G$ and $\alpha < |x|_G$, then $x \in V_n$ for some n, and so $x \in V_{n+1}[p^n]$. Hence, there is a $y \in V_{n+1} \subseteq G$ such that $py = x$ and $\alpha \leq |y|_G$. This means that for all $x \in G$, $|x|_G = \sup\{|y|_G + 1 : y \in G \text{ and } py = x\}$. An easy induction then implies that $|\ |_G$

is the height function on G. Since $G[p^n] = V_n$ is n-summable for each n, it follows from ([4], Theorem 3.8) that G is a dsc group. Therefore, $f = f_G$ will be admissible. □

The last result could be proven combinatorially using the definition (1.A and B), but the argument is less intuitive and significantly longer than the above.

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