REND. SEM. MAT. UNIV. PADOVA, Vol. 124 (2010)

A Short Proof of the Hölder-Poincaré Duality for L_p-Cohomology

VLADIMIR GOL'DSHTEIN (*) - MARC TROYANOV (**)

ABSTRACT - We give a short proof of the duality theorem for the reduced L_p-cohomology of a complete oriented Riemannian manifold.

Let (M, g) be an oriented Riemannian manifold. For any $1 \leq p < \infty$ we denote by $L^p(M, \Lambda^k)$ the space of p-integrable differential forms on M. An element of that space is a measurable differential k-form ω such that

$$\|\omega\|_p := \left(\int_M |\omega|_x^p d\,\mathrm{vol}_g(x) \right)^{1/p} < \infty.$$

Recall that a differential form $\theta \in L^p(M, \Lambda^{k+1})$ is the *weak exterior differential* of the form $\phi \in L^p(M, \Lambda^k)$ if one has

$$\int_M \theta \wedge \omega = (-1)^{k+1} \int_M \phi \wedge d\omega$$

for any $\omega \in \mathcal{D}^{n-k}(M)$, where $\mathcal{D}^m(M)$ denotes the vector space of smooth differential m-forms with compact support in M.

One writes $d\phi = \theta$ if θ is the weak exterior differential of ϕ and $Z_p^k(M) = \ker d \cap L^p(M, \Lambda^k)$ denotes the set of weakly closed forms in $L^p(M, \Lambda^k)$. It is easy to check that $Z_p^k(M)$ is a closed linear subspace of

(*) Indirizzo dell'A.: Department of Mathematics, Ben Gurion University of the Negev, P.O.Box 653, Beer Sheva, Israel.
E-mail: vladimir@bgumail.bgu.ac.il
(**) Indirizzo dell'A.: Section de Mathematiques, École Polytechnique Féderale de Lausanne, 1015 Lausanne - Switzerland.
E-mail: marc.troyanov@epfl.ch
AMS Mathematics Subject Classification: 58A10, 58A12, 53c.

$L^p(M,\Lambda^k)$, in particular it is a Banach space (see [5, Lemma 2.2]). We then introduce the space

$$B_p^k(M) = d\big(L^p(M,\Lambda^{k-1})\big) \cap L^p(M,\Lambda^k)$$

of exact L^p-forms and we shall denote by $\overline{B}_p^k(M)$ the closure of $B_p^k(M)$ in $L^p(M,\Lambda^k)$. Because $Z_p^k(M) \subseteq L^p(M,\Lambda^k)$ is a closed subspace and $d \circ d = 0$, we have $\overline{B}_p^k(M) \subseteq Z_p^k(M)$. The reduced *$L_p$-cohomology* of (M,g) (where $1 \le p < \infty$) is defined to be the quotient

$$\overline{H}_p^k(M) = Z_p^k(M)/\overline{B}_p^k(M).$$

This is a Banach space for the natural (quotient) norm and the goal of this paper is to prove the following Theorem (here and throughout the paper, $p' = p/(p-1)$ is the conjugate number of p).

Duality Theorem. *Let (M,g) be a complete oriented Riemannian manifold of dimension n and $1 < p < \infty$. Then $\overline{H}_p^k(M)$ is isometric to the dual of $\overline{H}_{p'}^{n-k}(M)$. The duality is given by the integration pairing:*

$$\begin{array}{ccc} \overline{H}_p^k(M) \times \overline{H}_{p'}^{n-k}(M) & \to & \mathbb{R} \\ ([\omega],[\theta]) & \mapsto & \displaystyle\int_M \omega \wedge \theta. \end{array}$$

Remark. By “the dual space” X' of a Banach space X, we of course mean the topological dual, i.e. the vector space of *continuous* linear functionals together with its natural norm. The isomorphism between $\overline{H}_p^k(M)$ and the dual of $\overline{H}_{p'}^{n-k}(M)$ has first been proved in 1986 by V. M. Gol’dshtein, V.I. Kuz’minov and I.A. Shvedov, see [4]. In fact that paper also describes the dual space to the L_p-cohomology of non complete manifolds. The proof we present here is simpler and more direct than the proof in [4], although it doesn’t seem to be extendable to the non complete case. Note that this duality theorem is useful to prove vanishing or non vanishing results in L_p-cohomology, see e.g. [5, 7, 8].

Let us also mention that Gromov deduced the above theorem from the simplicial version of the L_p-cohomology, see [7]. Gromov’s argument works only for Riemannian manifolds with bounded geometry, while the proof we give here works for any complete manifold. Our proof can also be extended to the more general $L_{q,p}$-cohomology, see [6].

The proof will rest on a few auxiliary facts. Recall first that a *pairing* between two Banach spaces X_0 and X_1 is simply a continuous bilinear map

$I : X_0 \times X_1 \to \mathbb{R}$. Such a pairing defines two continuous linear maps $\lambda : X_0 \to X_1'$, and $\mu : X_1 \to X_0'$ defined by

$$\lambda_\xi(\eta) = \mu_\eta(\xi) = I(\xi, \eta),$$

for any $\xi \in X_0$ and $\eta \in X_1$.

DEFINITION 1. An *isometric duality* between two Banach spaces X_0 and X_1 is a pairing $I : X_0 \times X_1 \to \mathbb{R}$ such that the associated maps $\lambda : X_0 \to X_1'$, and $\mu : X_1 \to X_0'$ are bijective isometries.

Observe that if an isometric duality exists between two Banach spaces, then these spaces are reflexive. The classic L^p–$L^{p'}$ duality for function spaces extends to the case of differential forms, see [4]:

PROPOSITION 2. *If* $1 < p < \infty$, *then the pairing* $L^p(M, \Lambda^k) \times \times L^{p'}(M, \Lambda^{n-k}) \to \mathbb{R}$ *defined by*

$$\langle \omega, \varphi \rangle = \int_M \omega \wedge \varphi \tag{1}$$

is an isometric duality. In particular, $L^p(M, \Lambda^k)$ *is a reflexive Banach space.*

We will also need the following density result whose proof is based on regularization methods, see e.g. [3, 5]:

PROPOSITION 3. *Let* $\theta \in L^p(M, \Lambda^{k-1})$ *be a* $(k-1)$*-form whose weak exterior differential is* p*-integrable,* $d\theta \in L^p(M, \Lambda^k)$*. Then there exists a sequence* $\theta_j \in C^\infty(M, \Lambda^{k-1})$ *such that* $\theta = \lim_{j\to\infty} \theta_j$ *and* $d\theta = \lim_{j\to\infty} d\theta_j$ *in* $L^p(M)$*.*

The next lemma is the place where the completeness hypothesis enters:

LEMMA 4. *If* (M, g) *is complete, then* $d\mathcal{D}^{k-1}(M)$ *is dense in* $B_p^k(M)$*.*

PROOF. Because M is complete, one can find a sequence of smooth functions with compact support $\{\eta_j\} \subseteq C_0^\infty(M)$ such that $0 \le \eta_j \le 1$, $\limsup_{j\to\infty} |d\eta_j| = 0$ and $\eta_j \to 1$ uniformly on every compact subset of M. Let $\omega \in B_p^k(M)$, then there exists $\theta \in L^p(M, \Lambda^{k-1})$ such that $d\theta = \omega$. Choose a sequence $\{\theta_j\} \subseteq C^\infty(M, \Lambda^{k-1})$ as in Proposition 3, i.e. $\theta_j \to \theta$ and $d\theta_j \to d\theta =$

$= \omega$ in $L^p(M)$ and set $\tilde{\theta}_j = \eta_j\theta_j \in \mathcal{D}^{k-1}(M)$. We first claim that $(\tilde{\theta}_j - \theta_j) \to 0$ in $L^p(M, \Lambda^{k-1})$. Indeed, fix $\varepsilon > 0$ and choose a compact set Q such that $\|\theta\|_{L^p(M\setminus Q)} < \varepsilon$. Since $|\eta_j - 1| < 1$, we have

$$\begin{aligned}\|\tilde{\theta}_j - \theta_j\|_{L^p(M)} &\leq \|(\eta_j - 1)\theta_j\|_{L^p(Q)} + \|\theta_j\|_{L^p(M\setminus Q)} \\ &\leq \|(\eta_j - 1)\theta_j\|_{L^p(Q)} + \|\theta_j - \theta\|_{L^p(M\setminus Q)} + \|\theta\|_{L^p(M\setminus Q)}.\end{aligned}$$

The first term converges to zero because $\eta_j \to 1$ uniformly on Q and $\{\|\theta_j\|_{L^p}\}$ is bounded. The second term converges to zero because $\theta_j \to \theta$ in $L^p(M, \Lambda^{k-1})$ and the last term is bounded by ε, hence

$$\limsup_{j\to\infty} \|\tilde{\theta}_j - \theta_j\|_{L^p(M)} \leq \varepsilon.$$

Since ε is arbitrary, the limit is zero and we obtain

$$\lim_{j\to\infty} \|\tilde{\theta}_j - \theta\|_{L^p(M)} \leq \lim_{j\to\infty} \|\tilde{\theta}_j - \theta_j\|_{L^p(M)} + \lim_{j\to\infty} \|\theta_j - \theta\|_{L^p(M)} = 0.$$

We similarly have

$$\begin{aligned}\|d\tilde{\theta}_j - d\theta_j\|_p &\leq \|(\eta_j - 1)d\theta_j\|_p + \|d\eta_j \wedge \theta_j\|_p \\ &\leq \|(\eta_j - 1)d\theta_j\|_p + \sup|d\eta_j| \cdot \|\theta_j\|_p \to 0.\end{aligned}$$

This implies that $\omega = \lim_{j\to\infty} d\tilde{\theta}_j$ in L^p. □

Definition 5. Given an isometric duality $I : X_0 \times X_1 \to \mathbb{R}$ and a nonempty subset B of X_0, we define the *annihilator* $B^\perp \subseteq X_1$ of B to be the set of all elements $\eta \in X_1$ such that $I(\xi, \eta) = 0$ for all $\xi \in B$.

For any $B \subseteq X_0$ the annihilator $B^\perp$ is a closed linear subspace of X_1. The Hahn-Banach Theorem implies that if B is a linear subspace of X_0 then $(B^\perp)^\perp = \overline{B}$.

For these and further facts on the notion of annihilator, we refer to the books [1, 2].

The proof of the duality Theorem is based on the following lemma about annihilators:

Lemma 6. *Let $I : X_0 \times X_1 \to \mathbb{R}$ be an isometric duality between two Banach spaces. Let B_0, A_0, B_1, A_1 be linear subspaces such that*

$$B_0 \subseteq A_0 = B_1^\perp \subseteq X_0 \qquad \text{and} \qquad B_1 \subseteq A_1 = B_0^\perp \subseteq X_1.$$

Then the pairing $\overline{I} : \overline{H}_0 \times \overline{H}_1 \to \mathbb{R}$ of $\overline{H}_0 := A_0/\overline{B}_0$ and $\overline{H}_1 := A_1/\overline{B}_1$ is well defined and induces an isometric duality between $\overline{H}_0$ and $\overline{H}_1$.

PROOF. Observe first that $A_i \subseteq X_i$ is a closed subspace since the annihilator of any subset of a Banach space is always a closed linear subspace.

The bounded bilinear map $I : A_0 \times A_1 \to \mathbb{R}$ is defined by restriction. It gives rise to a well defined bounded bilinear map $\overline{I} : A_0/\overline{B}_0 \times A_1/\overline{B}_1 \to \mathbb{R}$ because we have the inclusions $B_0 \subseteq B_1^{\perp}$ and $B_1 \subseteq B_0^{\perp}$.

We denote by $\lambda : X_0 \to X_1'$ the isometry induced by the pairing I, by $\overline{\lambda} : \overline{H}_0 \to \overline{H}_1'$ the map defined by the pairing $\overline{I}$ and by $\pi_i : A_i \to \overline{H}_i$ $(i = 1, 2)$, the canonical projections.

We first prove that $\|\overline{\lambda}_\xi\|_{\overline{H}_1'} \leq \|\xi\|_{\overline{H}_0}$ for any $\xi \in \overline{H}_0$. Indeed, let us choose $\hat{\xi} \in A_0$ such that $\pi_0(\hat{\xi}) = \xi$, we have

$$\begin{aligned} \|\overline{\lambda}_\xi\|_{\overline{H}_1'} &= \sup\{\overline{I}(\xi,\eta) \,|\, \eta \in \overline{H}_1, \|\eta\|_{\overline{H}_1} \leq 1\} \\ &\leq \sup\{I(\hat{\xi},\hat{\eta}) \,|\, \hat{\eta} \in A_1, \|\hat{\eta}\|_{A_1} \leq 1\} \\ &\leq \sup\{I(\hat{\xi},\hat{\eta}) \,|\, \hat{\eta} \in X_1, \|\hat{\eta}\|_{X_1} \leq 1\} = \|\lambda_{\hat{\xi}}\|_{X_1'}. \end{aligned}$$

By hypothesis, we have $\|\lambda_{\hat{\xi}}\|_{X_1'} = \|\hat{\xi}\|_{X_0}$, therefore

$$\|\xi\|_{\overline{H}_0} = \inf_{\hat{\xi} \in \pi_0^{-1}(\xi)} \|\hat{\xi}\|_{X_0} = \inf_{\hat{\xi} \in \pi_0^{-1}(\xi)} \|\lambda_{\hat{\xi}}\|_{X_1'} \geq \|\overline{\lambda}_\xi\|_{\overline{H}_1'}.$$

We then prove that for any $\theta \in \overline{H}_1'$, there exists an element $\xi \in \overline{H}_0$ such that $\theta = \overline{\lambda}_\xi$ and $\|\theta\|_{\overline{H}_1'} \geq \|\xi\|_{\overline{H}_0}$. This implies that $\overline{\lambda}$ is surjective and $\|\overline{\lambda}_\xi\|_{\overline{H}_1'} \geq \|\xi\|_{\overline{H}_0}$.

Indeed, for any $\theta \in \overline{H}_1'$, the linear form $\hat{\theta} = \theta \circ \pi_1 : A_1 \to \mathbb{R}$ satisfies $\hat{\theta}(b) = 0$ for any $b \in B_1$ and $\|\hat{\theta}\|_{A_1'} = \|\theta\|_{\overline{H}_1'}$. By the Hahn-Banach Theorem, there exists a continuous extension $\hat{\varphi} : X_1 \to \mathbb{R}$ of $\hat{\theta}$ such that $\|\hat{\varphi}\|_{X_1'} = \|\hat{\theta}\|_{A_1'}$. Since $\lambda : X_0 \to X_1'$ is an isometry, one can find $\hat{\xi} \in X_0$ such that $\lambda_{\hat{\xi}} = \hat{\varphi}$ and

$$\|\hat{\xi}\|_{X_0} = \|\hat{\varphi}\|_{X_1'} = \|\hat{\theta}\|_{A_1'} = \|\theta\|_{\overline{H}_1'}.$$

For any $b \in B_1$, we have $I(\hat{\xi}, b) = \lambda_{\hat{\xi}}(b) = \hat{\theta}(b) = 0$, thus $\hat{\xi} \in B_1^{\perp} = A_0$. Let us set $\xi = \pi_0(\hat{\xi})$, we have

$$\overline{I}(\xi,\eta) = I(\hat{\xi},\hat{\eta}) = \hat{\theta}(\hat{\eta}) = \theta(\eta)$$

for any $\eta \in \overline{H}_1$ and $\hat{\eta} \in A_1$, that is $\theta = \overline{\lambda}_\xi$. We also have

$$\|\xi\|_{\overline{H}_0} \leq \|\hat{\xi}\|_{X_0} = \|\theta\|_{\overline{H}_1'} = \|\overline{\lambda}_\xi\|_{\overline{H}_1'}.$$

In conclusion, we have have proved that $\overline{\lambda} : \overline{H}_0 \to \overline{H}_1'$ is norm preserving and surjective: it is an isometry. The proof that $\overline{\mu} : \overline{H}_1 \to \overline{H}_0'$ is also an isometry is the same. □

PROOF OF THE DUALITY THEOREM. Let $\phi \in L^p(M, \Lambda^k)$, then $d\phi = 0$ in the weak sense if and only if $\int_M \phi \wedge d\omega = 0$ for any $\omega \in \mathcal{D}^{n-k-1}(M)$. This precisely means that $Z_p^k(M) \subseteq L^p(M, \Lambda^k)$ is the annihilator of $d\mathcal{D}^{n-k-1}(M) \subseteq \subseteq L^{p'}(M, \Lambda^{n-k})$ for the pairing (1):

$$Z_p^k(M) = (d\mathcal{D}^{n-k-1})^{\perp}(M).$$

By lemma 1, $d\mathcal{D}^{n-k-1}(M)$ and $B_{p'}^{n-k}$ have the same annihilator, thus

$$B_p^k \subseteq Z_p^k = (B_{p'}^{n-k})^{\perp} \subseteq L^p(M, \Lambda^k).$$

Similarly, we also have

$$B_{p'}^{n-k} \subseteq Z_{p'}^{n-k} = (B_p^k)^{\perp} \subseteq L^{p'}(M, \Lambda^{n-k}),$$

and Lemma 1 says that the duality (1) induces an isometric duality between $Z_{p'}^{n-k}/\overline{B}_{p'}^{n-k}$ and $Z_p^k/\overline{B}_p^k$. □

Acknowledgments. We thank the anonymous referee for helpful comments and for pointing out that the duality in our main theorem is an isometric duality.

REFERENCES

[1] H. BREZIS, *Analyse fonctionnelle, Théorie et applications.* Dunod, Paris 1999.
[2] J. CONWAY, *A course in functional analysis.* Second edition. Graduate Texts in Mathematics, 96. Springer-Verlag, New York, 1990.
[3] V. M. GOL′DSHTEIN - V. I. KUZ′MINOV - I. A. SHVEDOV, *A Property of de Rham Regularization Operators*, Siberian Math. Journal, **25**, No 2 (1984).
[4] V. M. GOL′DSHTEIN - V. I. KUZ′MINOV - I. A. SHVEDOV, *Dual spaces of Spaces of Differential Forms*, Siberian Math. Journal, **54**, No 1 (1986).
[5] V. GOL′DSHTEIN - M. TROYANOV, *Sobolev Inequality for Differential forms and L_p-cohomology.* Journal of Geom. Anal., **16,** No 4 (2006), pp. 597–631.
[6] V. GOL′DSHTEIN - M. TROYANOV, *The Hölder-Poincaré Duality for $L_{q,p}$-cohomology* Preprint.
[7] M. GROMOV, *Asymptotic invariants of infinite groups.* In “Geometric Group Theory”, ed. G. Niblo and M. Roller, Cambridge: Cambridge University Press, (1993).
[8] P. PANSU, *Cohomologie L_p et pincement.* Comment. Math. Helv., **83** (2008), pp. 327–357.

Manoscritto pervenuto in redazione il 14 marzo 2010.