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GAGA for DQ-Algebroids

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ABSTRACT - Let X be a smooth complex projective variety with associated compact complex manifold X_{an}. If $\mathscr{A}_X$ is a DQ-algebroid on X, then there is an induced DQ-algebroid $\mathscr{A}_{X_{\mathrm{an}}}$ on X_{an}. We show that the natural functor from the derived category of bounded complexes of $\mathscr{A}_X$-modules with coherent cohomologies to the derived category of bounded complexes of $\mathscr{A}_{X_{\mathrm{an}}}$-modules with coherent cohomologies is an equivalence.

Introduction.

Let X be a projective scheme over the complex number field $\mathbb{C}$ with associated complex analytic space X_{an}. Serre's GAGA paper [11] asserts that the category of coherent sheaves on X is equivalent to the category of coherent analytic sheaves on X_{an}.

We consider the case where X is a smooth algebraic variety over $\mathbb{C}$ or a complex manifold. In [8], the authors defined a DQ-algebra $\mathscr{A}_X$ on X as a sheaf of $\mathbb{C}^{\hbar} := \mathbb{C}[[\hbar]]$-algebras locally isomorphic to $(\mathscr{O}_X[[\hbar]], \star)$ where $\star$ is a star product. The authors also defined a DQ-algebroid as a $\mathbb{C}^{\hbar}$-algebroid stack locally equivalent to the algebroid associated with a DQ-algebra. If $\mathscr{A}_X$ is a DQ-algebroid on X, then we have the notion of $\mathscr{A}_X$-modules. We denote by $\mathrm{Mod}(\mathscr{A}_X)$ the category of $\mathscr{A}_X$-modules and by $\mathrm{D^b}(\mathscr{A}_X)$ its bounded derived category. We will recall these notions and their properties from [8].

If $(X, \mathscr{A}_X)$ is a smooth variety over $\mathbb{C}$ endowed with a DQ-algebroid, then there is an induced DQ-algebroid $\mathscr{A}_{X_{\mathrm{an}}}$ on the complex manifold

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X_{an}. Then we construct a functor $f^* : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$, where $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ denotes the full triangulated subcategory of the bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_X)$ with coherent cohomologies and similarly $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ denotes the full triangulated subcategory of the bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_{\mathrm{an}}})$ with coherent cohomologies. By using Lemma 1.2, Corollary 1.4 and some results in [8], we prove the following theorem:

Main Theorem (See Theorem 4.12). *Assume that X is projective. Then the functor f^*: $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ is an equivalence.*

This paper is organized as follows: In section 1, we review Serre's GAGA theorem and translate this theorem to the derived version. In section 2, we review some notions and results of DQ-modules from [8]. In particular, Remark 2.5 and Finiteness theorem (Theorem 2.13) are crucial for the paper. In section 3, we show how to induce an analytic DQ-algebroid from an algebraic DQ-algebroid on a smooth variety. In section 4, we prove the main theorem.

Throughout this paper, all varieties (or schemes) are over $\mathbb{C}$ if not otherwise specified.

1. Review on the GAGA Theorem.

Let X be a scheme of finite type and let X_{an} be the associated complex analytic space. We denote by $\mathrm{Mod}(\mathscr{O}_X)$ (resp. $\mathrm{Mod}(\mathscr{O}_{X_{\mathrm{an}}})$) the category of sheaves on X (resp. X_{an}). We also denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X)$ and $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_{X_{\mathrm{an}}})$ the full subcategories of $\mathrm{Mod}(\mathscr{O}_X)$ and $\mathrm{Mod}(\mathscr{O}_{X_{\mathrm{an}}})$ consisting of coherent sheaves, respectively. There is a continuous map $\varphi : X_{\mathrm{an}} \to X$ of the underlying topological spaces and there is also a natural map of the structure sheaves $\varphi^{-1}\mathscr{O}_X \to \mathscr{O}_{X_{\mathrm{an}}}$. To $\mathscr{F} \in \mathrm{Mod}(\mathscr{O}_X)$, one associates its complex analytic sheaf $\mathscr{F}^{\mathrm{an}} := \mathscr{O}_{X_{\mathrm{an}}} \otimes_{\varphi^{-1}\mathscr{O}_X} \varphi^{-1}\mathscr{F} \in \mathrm{Mod}(\mathscr{O}_{X_{\mathrm{an}}})$. Hence we obtain a functor:

$$(*) \qquad \Upsilon_{\mathbf{X}} : \mathrm{Mod}(\mathscr{O}_X) \to \mathrm{Mod}(\mathscr{O}_{X_{\mathrm{an}}}).$$

If $\mathscr{F}$ is a coherent sheaf, then $\mathscr{F}^{\mathrm{an}}$ is also coherent.

The following theorem for a projective scheme is proved in Serre's famous paper GAGA (see [11]) which is generalized by Grothendieck for a proper scheme (see [6 XII]).

THEOREM 1.1. *Let X be a projective scheme. Then the functor* $(\,*\,)$ *induces an equivalence of categories*

$$\mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_{X_{\mathrm{an}}}).$$

Furthemore, for every coherent sheaf $\mathscr{F}$ on X, the natural maps

$$H^i(X;\mathscr{F}) \to H^i(X_{\mathrm{an}};\mathscr{F}^{\mathrm{an}})$$

are isomorphisms, for all $i \geq 0$. □

The following lemma is Theorem 2.2.8 in [1].

LEMMA 1.2. *Let $\mathcal{A}'$ and $\mathcal{B}'$ be thick subcategories of abelian categories $\mathcal{A}$ and $\mathcal{B}$, respectively, and let $\Phi : \mathcal{A} \to \mathcal{B}$ be an exact functor that takes $\mathcal{A}'$ to $\mathcal{B}'$. Assume furthermore that the following properties are satisfied:*

1. *$\mathcal{A}$ and $\mathcal{B}$ have enough injectives,*
2. *Φ is an equivalence of categories when restricted to $\mathcal{A}' \to \mathcal{B}'$,*
3. *Φ induces a natural isomorphism*

$$\mathrm{Ext}^i_{\mathcal{A}}(F,G) \cong \mathrm{Ext}^i_{\mathcal{B}}(\Phi(F),\Phi(G))$$

for any $F, G \in \mathcal{A}'$ and any i.

Then the natural functor $\widetilde{\Phi} : \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A}) \to \mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})$ induced by Φ is an equivalence of categories.

PROOF. (i) We prove that the functor $\widetilde{\Phi}$ is fully faithful, i.e., for any $F^\bullet$, $G^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$, $\widetilde{\Phi}$ induces an isomorphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})}(F^\bullet, G^\bullet) \cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})}(\widetilde{\Phi}(F^\bullet), \widetilde{\Phi}(G^\bullet)). \tag{1.1}$$

We'll use a technique known as d*évissage* to prove it. The d*évissage* technique is the induction on the number $n(E^\bullet)$ defined as

$$n(E^\bullet) = \max\{j - i | H^j(E^\bullet) \neq 0, H^i(E^\bullet) \neq 0\}.$$

Hence we shall prove (1.1) by induction on $N = n(F^\bullet) + n(G^\bullet)$. If $N = -\infty$, then one of $F^\bullet$ or $G^\bullet$ is the zero complex, so there is nothing to prove. If $N = 0$, then there exist $F \in \mathcal{A}', G \in \mathcal{A}'$ such that $F^\bullet = F[a]$ and $G^\bullet = G[b]$ for some $a, b \in \mathbb{Z}$. Then

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})}(F^\bullet, G^\bullet) = \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})}(F[a], G[b]) = \mathrm{Ext}^{b-a}_{\mathcal{A}}(F,G)$$

and

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})}(\widetilde{\Phi}(F^\bullet), \widetilde{\Phi}(G^\bullet)) = \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})}(\widetilde{\Phi}(F[a]), \widetilde{\Phi}(G[b]))$$

$$= \mathrm{Ext}^{b-a}_{\mathcal{B}}(\Phi(F), \Phi(G)).$$

Hence (1.1) follows from property 3 above.

Assume that $\widetilde{\Phi}$ induces an isomorphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})}(F^\bullet, G^\bullet) \cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})}(\widetilde{\Phi}(F^\bullet), \widetilde{\Phi}(G^\bullet))$$

for all $F^\bullet, G^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ with $n(F^\bullet) + n(G^\bullet) < N$, and let $F^\bullet, G^\bullet$ be objects of $\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ with $n(F^\bullet) + n(G^\bullet) = N > 0$. We may assume that $n(G^\bullet) = N > 0$ and that $G^i = 0$ for $i < 0$, and $H^0(G^\bullet) \neq 0$.

Let $G'^\bullet$ be the complex with single non zero object $H^0(G^\bullet)$ in degree zero. From the morphism $G'^\bullet \to G^\bullet$, there exists a distinguished triangle $G''^\bullet \to G'^\bullet \to G^\bullet \to G''^\bullet[1]$. By the long exact cohomology sequence, one deduces $n(G''^\bullet) < n(G^\bullet)$; also, from the assumption, $n(G'^\bullet) = 0 < n(G^\bullet)$. From the long exact sequence of Hom's; the five-lemma and the induction hypothesis, we conclude that

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})}(F^\bullet, G^\bullet) \cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})}(\widetilde{\Phi}(F^\bullet), \widetilde{\Phi}(G^\bullet))$$

which is what we needed to prove that $\widetilde{\Phi}$ is fully faithful. (The case when $n(G^\bullet) = 0$ but $n(F^\bullet) > 0$ follows in a similar way.)

(ii) Next, we shall prove that the functor $\widetilde{\Phi}$ is essentially surjective, i.e., any object $G^\bullet$ of $\mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})$ is isomorphic to an object of the form $\widetilde{\Phi}(F^\bullet)$ for some $F^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$. We prove this by induction on $n = n(G^\bullet)$. The case $n = -\infty$ is trivial, and $n = 0$ follows from property 2.

So assume $n > 0$, and as before construct a distinguished triangle $G''^\bullet \to G'^\bullet \to G^\bullet \to G''^\bullet[1]$ where $G'^\bullet = H^0(G^\bullet) \neq 0$ and we assume that $G^\bullet$ is zero in degrees < 0. Since Φ is an equivalence of categories between $\mathcal{A}'$ and $\mathcal{B}'$, we can find an $F'^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ such that $\Phi(F'^\bullet) \cong G'^\bullet$. Also, by induction hypothesis, we can find an $F''^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ such that $\widetilde{\Phi}(F''^\bullet) \cong G''^\bullet$. Since we proved that $\widetilde{\Phi}$ is fully faithful, we can find a map $F''^\bullet \to F'^\bullet$ whose image by $\widetilde{\Phi}$ is the side of the distinguished triangle constructed before. Again, we have the distinguished triangle $F''^\bullet \to F'^\bullet \to F^\bullet \to F''^\bullet[1]$. Since $\widetilde{\Phi}$ is a derived functor, it is a triangulated functor. Hence, we see that $\widetilde{\Phi}(F^\bullet)$ is isomorphic to $G^\bullet$, as required. □

The following proposition is well known (see, e.g., [9]).

PROPOSITION 1.3. *Let $\mathscr{R}$ be a sheaf of rings on a topological space X. Then the category* $\mathrm{Mod}(\mathscr{R})$ *of* $\mathscr{R}$*-modules is a Grothendieck category. In particular,* $\mathrm{Mod}(\mathscr{R})$ *has enough injectives.*

Let X be a scheme of finite type. Set $\mathcal{A} := \mathrm{Mod}(\mathscr{O}_X)$, $\mathcal{B} := \mathrm{Mod}(\mathscr{O}_{X_{\mathrm{an}}})$, $\mathcal{A}' := \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X)$ and $\mathcal{B}' := \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_{X_{\mathrm{an}}})$. We also set $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) := \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X_{\mathrm{an}}) := \mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})$. Clearly, $\mathcal{A}'$ and $\mathcal{B}'$ are full thick subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively.

As an application of Lemma 1.2, we have the following.

COROLLARY 1.4. *Let X be a projective scheme. Then the functor* Υ_X *of* $(*)$ *induces an equivalence (we keep the same notation)*

$$\Upsilon_X : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X_{\mathrm{an}}).$$

PROOF. We shall apply Lemma 1.2. First, note that $\mathscr{O}_{X_{\mathrm{an}},x}$ is a flat $\mathscr{O}_{X,x}$-module for each $x \in X$. Hence the functor $\Upsilon_X : \mathcal{A} \to \mathcal{B}$ is an exact functor. Both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives by Proposition 1.3. Hence condition 1 is satisfied and condition 2 is Theorem 1.1. To check condition 3, it is sufficient to prove that

$$\mathrm{RHom}(\mathscr{F}, \mathscr{G}) \simeq \mathrm{RHom}(\mathscr{F}^{\mathrm{an}}, \mathscr{G}^{\mathrm{an}}) \text{ for } \mathscr{F}, \mathscr{G} \in \mathcal{A}'. \tag{1.2}$$

Since $\mathrm{RHom}(\mathscr{F}, \mathscr{G}) \simeq \mathrm{R}\Gamma(X, \mathscr{F}^* \overset{\mathrm{L}}{\underset{\mathscr{O}_X}{\otimes}} \mathscr{G})$ and $\mathrm{RHom}(\mathscr{F}^{\mathrm{an}}, \mathscr{G}^{\mathrm{an}}) \simeq \mathrm{R}\Gamma(X_{\mathrm{an}}, (\mathscr{F}^{\mathrm{an}})^* \overset{\mathrm{L}}{\underset{\mathscr{O}_{X_{\mathrm{an}}}}{\otimes}} \mathscr{G}^{\mathrm{an}})$, where $\mathscr{F}^* = \mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$ and $(\mathscr{F}^{\mathrm{an}})^* = \mathrm{R}\mathscr{H}om_{\mathscr{O}_{X_{\mathrm{an}}}}(\mathscr{F}^{\mathrm{an}}, \mathscr{O}_{X_{\mathrm{an}}})$, we reduce (1.2) to the following isomorphism

$$\mathrm{R}\Gamma(X, \mathscr{F}) \simeq \mathrm{R}\Gamma(X_{\mathrm{an}}, \mathscr{F}^{\mathrm{an}}), \text{ where } \mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X). \tag{1.3}$$

Now, there is a morphism $\mathrm{R}\Gamma(X, \mathscr{F}) \to \mathrm{R}\Gamma(X_{\mathrm{an}}, \mathscr{F}^{\mathrm{an}})$ for $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ and it is an isomorphism by Theorem 1.1. Hence the result follows. □

2. Review on DQ-modules (after K-S).

In this section, we recall some notions and results from [8].

Algebroid.

In this subsection, we denote by X a topological space and by $\mathbb{K}$ a commutative unital ring. If A is a ring, an A-module means a left A-module. Recall that the notion algebroid was first introduced by Kontsevich [10], see also [2] and [7]. A $\mathbb{K}$-algebroid $\mathscr{A}$ on X is a $\mathbb{K}$-linear stack locally non empty and such that for any open subset U of X, two objects of $\mathscr{A}(U)$ are locally isomorphic.

Let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open covering of X. In the sequel, we set $U_{ij} := U_i \cap U_j$, $U_{ijk} := U_i \cap U_j \cap U_k$, etc.

Consider the data of

$$\begin{cases} \text{a } \mathbb{K}\text{-algebroid } \mathscr{A} \text{ on } X \\ \sigma_i \in \mathscr{A}(U) \text{ and isomorphisms } \varphi_{ij} : \sigma_j|_{U_{ij}} \to \sigma_i|_{U_{ij}}. \end{cases} \tag{2.1}$$

To these data, we associate:

- $\mathscr{A}_i = \mathscr{E}nd_{\mathbb{K}}(\sigma_i)$,
- $f_{ij} : \mathscr{A}_j|_{U_{ij}} \to \mathscr{A}_i|_{U_{ij}}$ the $\mathbb{K}$-algebra isomorphism $a \mapsto \varphi_{ij} \circ a \circ \varphi_{ij}^{-1}$,
- a_{ijk}, the invertible element of $\mathscr{A}_i(U_{ijk})$ given by $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ik}^{-1}$.

Then:

$$\begin{cases} f_{ij} \circ f_{jk} = Ad(a_{ijk}) \circ f_{ik} \\ a_{ijk}a_{ikl} = f_{ij}(a_{jkl})a_{ijl}. \end{cases} \tag{2.2}$$

(Recall that $\mathrm{Ad}(a)(b) = aba^{-1}$).

Conversely, let $\mathscr{A}_i$ be sheaves of $\mathbb{K}$-algebras on U_i $(i \in I)$, let $f_{ij} : \mathscr{A}_j|_{U_{ij}} \to \mathscr{A}_i|_{U_{ij}}$ $(i,j \in I)$ be $\mathbb{K}$-algebra isomorphisms, and let a_{ijk} $(i,j,k \in I)$ be invertible sections of $\mathscr{A}_i(U_{ij})$ satisfying (2.2). One calls:

$$(\{\mathscr{A}_i\}_{i\in I}, \{f_{ij}\}_{i,j\in I}, \{a_{ijk}\}_{i,j,k\in I}) \tag{2.3}$$

a gluing datum for $\mathbb{K}$-algebroids on $\mathscr{U}$.

THEOREM 2.1 ([5]). *Assume that the topological space X is paracompact. Considering a gluing datum* (2.3) *on $\mathscr{U}$. Then there exist an algebroid $\mathscr{A}$ on X and $\{\sigma_i, \varphi_{ij}\}_{i,j\in I}$ as in* (2.1) *to which this gluing datum is associated. Moreover, the data $(\mathscr{A}, \sigma_i, \varphi_{ij})$ are unique up to an equivalence of stacks, this equivalence being unique up to a unique isomorphism.*

In general, if a topological space X is not paracompact, for example for algebraic varieties, then Theorem 2.1 may be false. Hence we need another local description of such algebraic algebroids.

DEFINITION 2.2. Let $\mathscr{A}$ and $\mathscr{A}'$ be two sheaves of $\mathbb{K}$-algebras. An $\mathscr{A} \otimes \mathscr{A}'$-module $\mathscr{L}$ is called *bi-invertible* if there exists locally a section ω of $\mathscr{L}$ such that $\mathscr{A} \ni a \mapsto (a \otimes 1)\omega \in \mathscr{L}$ and $\mathscr{A}' \ni a' \mapsto (a' \otimes 1)\omega \in \mathscr{L}$ give isomorphism of $\mathscr{A}$-modules and $\mathscr{A}'$-modules, respectively.

Let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open covering of X. Consider the data of

$$(2.4)\qquad \begin{cases} a\mathbb{K}\text{-algebroid } \mathscr{A} \text{ on } X \\ \sigma_i \in \mathscr{A}(U). \end{cases}$$

To these data, we associate:

- $\mathscr{A}_i = \mathscr{E}nd_{\mathbb{K}}(\sigma_i)$,
- $\mathscr{L}_{ij} := \mathscr{H}om_{\mathscr{A}_i|_{U_{ij}}}(\sigma_j|_{U_{ij}}, \sigma_i|_{U_{ij}})$, hence $\mathscr{L}_{ij}$ is a bi-invertible $\mathscr{A}_i \otimes \mathscr{A}_j^{\mathrm{op}}$-module on U_{ij}.
- the natural isomorphisms $a_{ijk} : \mathscr{L}_{ij} \otimes_{\mathscr{A}_j} \mathscr{L}_{jk} \xrightarrow{\sim} \mathscr{L}_{ik}$.

Hence we obtain:

$$(2.5)\qquad (\{\mathscr{A}_i\}_{i\in I}, \{\mathscr{L}_{ij}\}_{i,j\in I}, \{a_{ijk}\}_{i,j,k\in I})$$

an algebraic gluing datum for $\mathbb{K}$-algebroids on $\mathscr{U}$.

THEOREM 2.3 ([8] Proposition 2.1.13). *Consider an algebraic gluing datum* (2.5) *on* $\mathscr{U}$. *Then there exist an algebroid* $\mathscr{A}$ *on* X *and* $\{\sigma_i, \varphi_{ij}\}_{i,j\in I}$ *as in* (2.1) *to which this gluing datum is associated. Moreover, the data* $(\mathscr{A}, \sigma_i, \varphi_{ij})$ *are unique up to an equivalence of stacks, this equivalence being unique up to a unique isomorphism.*

For an algebroid $\mathscr{A}$, one defines the Grothendieck $\mathbb{K}$-linear abelian category $\mathrm{Mod}(\mathscr{A})$, whose objects are called $\mathscr{A}$-modules, by setting:

$$\mathrm{Mod}(\mathscr{A}) := \mathrm{Fct}_{\mathbb{K}}(\mathscr{A}, \mathfrak{Mod}(\mathbb{K}_X)).$$

Here $\mathfrak{Mod}(\mathbb{K}_X)$ is the $\mathbb{K}$-linear stack of sheaves of $\mathbb{K}$-modules on X, and $\mathrm{Fct}_{\mathbb{K}}$ is the category of $\mathbb{K}$-linear functors of stacks.

We have the well defined notion of tensor product for two $\mathbb{K}$-algebroids $\mathscr{C}$ and $\mathscr{C}'$, say $\mathscr{C} \otimes_{\mathbb{K}} \mathscr{C}'$. For a $\mathbb{K}$-algebroid $\mathscr{A}$, $\mathrm{Mod}(\mathscr{A} \otimes_{\mathbb{K}} \mathscr{A}^{\mathrm{op}})$ has a canonical object given by

$$\mathscr{A} \otimes_{\mathbb{K}} \mathscr{A}^{\mathrm{op}} \ni (\sigma, \sigma'^{\mathrm{op}}) \mapsto \mathrm{H}om_{\mathscr{A}}(\sigma', \sigma) \in \mathfrak{Mod}(\mathbb{K}_X).$$

We denote this object by the same letter $\mathscr{A}$.

For $\mathbb{K}$-algebroids $\mathscr{A}_i$ $(i = 1, 2, 3)$, we have functors:

$$\cdot \otimes_{\mathscr{A}_2} \cdot : \mathrm{Mod}(\mathscr{A}_1 \otimes_{\mathbb{K}} \mathscr{A}_2^{\mathrm{op}}) \times \mathrm{Mod}(\mathscr{A}_2 \otimes_{\mathbb{K}} \mathscr{A}_3^{\mathrm{op}}) \to \mathrm{Mod}(\mathscr{A}_1 \otimes_{\mathbb{K}} \mathscr{A}_3^{\mathrm{op}})$$

and

$$\mathscr{H}om_{\mathscr{A}_1}(\cdot,\cdot) : \mathrm{Mod}(\mathscr{A}_1 \otimes_{\mathbb{K}} \mathscr{A}_2^{\mathrm{op}})^{\mathrm{op}} \times \mathrm{Mod}(\mathscr{A}_1 \otimes_{\mathbb{K}} \mathscr{A}_3^{\mathrm{op}}) \to \mathrm{Mod}(\mathscr{A}_2 \otimes_{\mathbb{K}} \mathscr{A}_3^{\mathrm{op}}).$$

In particular, we have

$$\cdot \otimes_{\mathscr{A}} \cdot : \mathrm{Mod}(\mathscr{A}^{\mathrm{op}}) \times \mathrm{Mod}(\mathscr{A}) \to \mathrm{Mod}(\mathbb{K}_X)$$

and

$$\mathscr{H}om_{\mathscr{A}}(\cdot,\cdot) : \mathrm{Mod}(\mathscr{A})^{\mathrm{op}} \times \mathrm{Mod}(\mathscr{A}) \to \mathrm{Mod}(\mathbb{K}_X).$$

Let Y be another topological space. Let $f : X \mapsto Y$ be a continuous map and let $\mathscr{A}$ be a $\mathbb{K}$-algebroid on Y. We denote by $f^{-1}\mathscr{A}$ the $\mathbb{K}$-linear stack associated with the prestack $\mathfrak{S}$ given by

$$\mathfrak{S}(U) = \{(\sigma, V);\ V \text{ is an open subset of } Y \text{ such that } f(U) \subset V \text{ and } \sigma \in \mathscr{A}(V)\} \text{ for any open subset } U \text{ of } X,$$
$$\mathrm{Hom}_{\mathfrak{S}(U)}((\sigma, V), (\sigma', V')) = \Gamma(U, f^{-1}\mathscr{H}om_{\mathscr{A}}(\sigma, \sigma')).$$

Then $f^{-1}\mathscr{A}$ is a $\mathbb{K}$-algebroid on X.

Notations: For the rest of this section, we denote by X a complex manifold or a smooth variety and by $\mathbb{C}^{\hbar} := \mathbb{C}[[\hbar]]$ the power series algebra.

Invertible $\mathscr{O}_X$-algebroids.

DEFINITION 2.4. A $\mathbb{C}$-algebroid $\mathscr{P}$ on X is called an invertible $\mathscr{O}_X$-algebroid if for any open subset U of X and any $\sigma \in \mathscr{P}(U)$, there is a $\mathbb{C}$-algebra isomorphism $\mathscr{E}nd_{\mathscr{P}}(\sigma) \simeq \mathscr{O}_U$.

We shall state some properties for invertible $\mathscr{O}_X$-algebroids.

Let $\mathscr{P}$ be an invertible $\mathscr{O}_X$-algebroid. Then for any $\sigma, \sigma' \in \mathscr{P}(U)$, $\mathscr{H}om(\sigma, \sigma')$ is an invertible $\mathscr{O}_U$-module.

For two invertible $\mathscr{O}_X$-algebroids $\mathscr{P}_1$ and $\mathscr{P}_2$. We denote by $\mathscr{P}_1 \otimes_{\mathscr{O}_X} \mathscr{P}_2$ the $\mathbb{C}$-linear stack associated with the prestack whose objects over an open set U is $\mathscr{P}_1(U) \times \mathscr{P}_2(U)$, and $\mathscr{H}om((\sigma_1, \sigma_2), (\sigma_1', \sigma_2')) = \mathscr{H}om(\sigma_1, \sigma_1') \otimes_{\mathscr{O}_X} \mathscr{H}om(\sigma_2, \sigma_2')$. Then $\mathscr{P}_1 \otimes_{\mathscr{O}_X} \mathscr{P}_2$ is an invertible $\mathscr{O}_X$-algebroid. Note that the set of equivalence classes of invertible $\mathscr{O}_X$-algebroids has structure of an additive group by the operation $\cdot \otimes_{\mathscr{O}_X} \cdot$, and this group is isomorphic to $H^2(X, \mathscr{O}_X^{\times})$ ([4], [9]).

The following remark is due to Prof. Joseph Oesterle and is crucial for the paper.

REMARK 2.5. For a smooth algebraic variety X as Zariski topology over $\mathbb{C}$, the group $H^2(X, \mathscr{O}_X^{\times})$ is trivial. Hence any invertible $\mathscr{O}_X$-algebroid $\mathscr{P}$ is equivalent to $\mathscr{O}_X$.

We sketch the proof of it. Let K be the field of rational functions on X

and let $K_X^\times$ be the constant sheaf with stalk the abelian group $K^\times$. Denote by $X_1 = \{x \in X | \dim \mathscr{O}_{X,x} = 1\}$ (or the set of closed irreducible hypersurfaces of X). Let $x \in X_1$, since X is a variety, the ring $\mathscr{O}_{X,x}$ is a DVR with valuation υ_x and quotient field K. Let $\mathbb{Z}_x = (i_x)_*(\mathbb{Z})$ where $i_x : x \to X$ and let $U \subset X$ be an open set, then $\mathbb{Z}_x(U) = 0$ if $x \notin U$ and $\mathbb{Z}_x(U) = \mathbb{Z}$ if $x \in U$. Consider the sheaf $\bigoplus_{x \in X_1} \mathbb{Z}_x$, then $\Big(\bigoplus_{x \in X_1} \mathbb{Z}_x\Big)(U) = \bigoplus_{x \in X_1} \mathbb{Z}_x(U) = \mathbb{Z}^{U \cap X_1}$. Hence we can define a morphism of sheaves

$$\upsilon : K_X^\times \to \bigoplus_{x \in X_1} \mathbb{Z}_x$$

by: $\upsilon(f) = (\upsilon_x(f))_{x \in X_1 \cap U}$ where U is a nonempty open subset of X and $f \in K_X^\times(U) = K^\times$. Then one has an exact sequence

$$0 \to \mathscr{O}_X^\times \xrightarrow{u} K_X^\times \xrightarrow{\upsilon} \bigoplus_{x \in X_1} \mathbb{Z}_x \to 0$$

where u is the natural morphism. Since $K_X^\times$ is constant, it is a flabby sheaf for the Zariski topology. On the other hand, the sheaf $\bigoplus_{x \in X_1} \mathbb{Z}_x$ is also flabby. It follows that $H^j(X; \mathscr{O}_X^\times)$ is zero for $j > 1$.

Let $f : X \to Y$ be a morphism of complex manifolds or smooth varieties. For an invertible $\mathscr{O}_Y$-algebroid $\mathscr{P}_Y$, we denote by $f^*\mathscr{P}_Y$ the $\mathbb{C}$-linear stack on X associated with the prestack whose objects on U are the objects of $(f^{-1}\mathscr{P}_Y)(U)$ and $\mathscr{H}om(\sigma, \sigma') = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{H}om_{f^{-1}\mathscr{P}_Y}(\sigma, \sigma')$. Then $f^*\mathscr{P}_Y$ is an invertible $\mathscr{O}_X$-algebroid.

Star-products.

Let X be a complex manifold (or a smooth variety). We denote by $\delta_X : X \hookrightarrow X \times X$ the diagonal embedding and we set $\triangle_X = \delta_X(X)$. We denote by $\mathscr{O}_X$ the structure sheaf on X, by Ω_X the sheaf of differential forms of maximal degree and by Θ_X the sheaf of vector fields. As usual, we denote by $\mathscr{D}_X$ the sheaf of rings of differential operators on X. Recall that a bi-differential operator P on X is a $\mathbb{C}$-bilinear morphism $\mathscr{O}_X \times \mathscr{O}_X \to \mathscr{O}_X$ which is obtained as the composition $\delta_X^{-1} \circ \tilde{P}$ where $\tilde{P}$ is a differential operator on $X \times X$ defined on a neighborhood of the diagonal and δ^{-1} is the restriction to the diagonal:

$$P(f,g)(x) = (\tilde{P}(x_1, x_2; \partial_{x_1}, \partial_{x_2})(f(x_1)g(x_2))|_{x_1=x_2=x}.$$

Hence the sheaf of bi-differential operators is isomorphic to $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{D}_X$, where the both $\mathscr{D}_X$ are regarded as $\mathscr{O}_X$-modules by the left multiplications.

Definition 2.6. A *star algebra* on $\mathscr{O}_X[[\hbar]]$ is a $\mathbb{C}^\hbar$-bilinear sheaf morphism

$$\star : \mathscr{O}_X[[\hbar]] \times \mathscr{O}_X[[\hbar]] \to \mathscr{O}_X[[\hbar]]$$

satisfying the following conditions:

(i) the star product makes $\mathscr{O}_X[[\hbar]]$ into a sheaf of associated unital $\mathbb{C}^\hbar$-algebra with unit $1 \in \mathscr{O}_X$.

(ii) there is a sequence $P_i : \mathscr{O}_X \times \mathscr{O}_X \to \mathscr{O}_X$ of bi-differential operators, such that for any two local sections $f, g \in \mathscr{O}_X$ one has

$$f \star g = fg + \sum_{i=1}^{\infty} P_i(f, g)\hbar^i.$$

Note that $f \star g \equiv fg \mod \hbar$, and $P_i(f, 1) = P_i(1, f) = 0$ for all f and $i > 0$. We call $(\mathscr{O}_X[[\hbar]], \star)$ a star algebra.

DQ-algebras.

Definition 2.7. A DQ-algebra $\mathscr{A}$ on X is a $\mathbb{C}^\hbar$-algebra locally isomorphic to a star-algebra $(\mathscr{O}_X[[\hbar]], \star)$ as a $\mathbb{C}^\hbar$-algebra.

Clearly, a DQ-algebra is a sheaf of $\hbar$-adically complete flat $\mathbb{C}^\hbar$-algebra on X satisfying $\mathscr{A}/\hbar\mathscr{A} \simeq \mathscr{O}_X$. Note also that for an algebraic variety X, a DQ-algebra $\mathscr{A}$ is called *deformation quantization* of $\mathscr{O}_X$ in [3] and [12].

Remark 2.8. For a smooth projective variety X, there exists a DQ-algebra $\mathscr{A}_X$ on X. For details, one refers to [3].

DQ-algebroids.

Definition 2.9. A DQ-algebroid $\mathscr{A}$ on X is a $\mathbb{C}^\hbar$-algebroid such that for each open set $U \subset X$ and each $\sigma \in \mathscr{A}(U)$, the $\mathbb{C}^\hbar$-algebra $\mathscr{H}om_{\mathscr{A}}(\sigma, \sigma)$ is a DQ-algebra on U.

Let $\mathscr{A}_X$ be a DQ-algebroid on X. For an $\mathscr{A}_X$-module $\mathscr{M}$, the local notions of being coherent or locally free, etc. make sense.

The category $\mathrm{Mod}(\mathscr{A}_X)$ is a Grothendieck category and we denote by $\mathrm{D}(\mathscr{A}_X)$ its derived category and by $\mathrm{D^b}(\mathscr{A}_X)$ its bounded derived category. We also denote by $\mathrm{D^b_{coh}}(\mathscr{A}_X)$ the full triangulated subcategory of $\mathrm{D^b}(\mathscr{A}_X)$ consisting of objects with coherent cohomologies.

Graded modules.

Let $\mathscr{A}_X$ be a DQ-algebroid on X. Let us denote by $\mathrm{gr}(\mathscr{A}_X)$ the $\mathbb{C}$-algebroid associated with the prestack $\mathfrak{S}$ given by

$$\mathrm{Ob}(\mathfrak{S}(U)) = \mathrm{Ob}(\mathscr{A}_X(U)) \text{ for an open subset } U \text{ of } X,$$
$$\mathrm{Hom}_{\mathfrak{S}}(\sigma, \sigma') = \mathrm{Hom}_{\mathscr{A}_X}(\sigma, \sigma')/\hbar\mathrm{Hom}_{\mathscr{A}_X}(\sigma, \sigma') \text{ for } \sigma, \sigma' \in \mathscr{A}_X(U).$$

Then it is easy to see that $\mathrm{gr}(\mathscr{A}_X)$ is an invertible $\mathscr{O}_X$-algebroid and the left derived functor of the right exact functor $\mathrm{Mod}(\mathscr{A}_X) \to \mathrm{Mod}(\mathrm{gr}\,(\mathscr{A}_X))$ given by $\mathscr{M} \to \mathscr{M}/\hbar\mathscr{M}$ is denoted by $\mathrm{gr} : \mathrm{D^b}(\mathscr{A}_X) \to \mathrm{D^b}(\mathrm{gr}\,(\mathscr{A}_X))$.

The functor gr induces a functor (we keep the same notation):

$$\mathrm{gr} : \mathrm{D^b_{coh}}(\mathscr{A}_X) \to \mathrm{D^b_{coh}}(\mathrm{gr}\,(\mathscr{A}_X)). \tag{2.6}$$

The following lemma is in [8].

LEMMA 2.10. *The functor* gr *of* (2.6) *is conservative (i.e., a morphism in* $\mathrm{D^b_{coh}}(\mathscr{A}_X)$ *is an isomorphism as soon as its image by* gr *is an isomorphism in* $\mathrm{D^b_{coh}}(\mathrm{gr}\,(\mathscr{A}_X))$.

Denote by $\mathrm{D}^\mathrm{b}_f(\mathbb{C}^\hbar) := \mathrm{D^b_{coh}}(\mathbb{C}^\hbar)$ and $\mathrm{D}^\mathrm{b}_f(\mathbb{C}) := \mathrm{D^b_{coh}}(\mathbb{C})$ the full triangulated subcategories of $\mathrm{D^b}(\mathbb{C}^\hbar)$ and $\mathrm{D^b}(\mathbb{C})$ consisting of objects with finitely generated cohomologies, respectively.

Hence we have a well defined functor $\mathbb{C} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar} \cdot : \mathrm{D}^\mathrm{b}_f(\mathbb{C}^\hbar) \to \mathrm{D}^\mathrm{b}_f(\mathbb{C})$. As an application of Lemma 2.10, we get the following.

COROLLARY 2.11. *The functor* $\mathbb{C} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar} \cdot : \mathrm{D}^\mathrm{b}_f(\mathbb{C}^\hbar) \to \mathrm{D}^\mathrm{b}_f(\mathbb{C})$. *is conservative.*

PROOF. Applying the functor gr in Lemma 2.10 to $X = \{\mathrm{pt}\}$. □

The following proposition is in [8] which will be used in Theorem 4.2.

PROPOSITION 2.12. *Let* $(X_i, \mathscr{A}_{X_i})$ *be complex manifolds or smooth varieties endowed with* DQ-*algebroids* $\mathscr{A}_{X_i}$ $(i = 1, 2, 3)$.

(i) *Let* $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_i \times X^a_{i+1}})$ $(i = 1, 2)$. *Then*

$$\mathrm{gr}\,(\mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_2} \mathscr{K}_2) \simeq \mathrm{gr}\,(\mathscr{K}_1) \overset{\mathrm{L}}{\otimes}_{\mathrm{gr}(\mathscr{A}_2)} \mathrm{gr}\,(\mathscr{K}_2).$$

(ii) *Let* $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_i \times X_{i+1}})$ $(i = 1, 2)$. *Then*

$$\mathrm{gr}\,\mathrm{R}\mathscr{H}om_{\mathscr{A}_2}(\mathscr{K}_1, \mathscr{K}_2) \simeq \mathrm{R}\mathscr{H}om_{\mathrm{gr}\,(\mathscr{A}_2)}(\mathrm{gr}\,(\mathscr{K}_1), \mathrm{gr}\,(\mathscr{K}_2)).$$

Finiteness for DQ-kernels.

Recall that we have the following Finiteness theorem.

FINITENESS THEOREM 2.13 ([8]). *Let* $(X, \mathscr{A}_X)$ *be a compact complex manifold or a smooth projective variety endowed with a* DQ*-algebroid* $\mathscr{A}_X$. *Let* $\mathscr{M}$ *and* $\mathscr{N}$ *be two objects of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$. *Then the object* $\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M}, \mathscr{N})$ *belongs to* $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$.

3. Analytization of a DQ-algebroid.

In this section, we denote by X a smooth algebraic variety. Let X_{an} be the corresponding complex analytic manifold of X with continuous map $f : X_{\mathrm{an}} \to X$.

Let $\mathscr{A}_X$ be a DQ-algebroid on X and let $\mathscr{U} = \{U_i\}_{i=1,\dots,n}$ be a finite affine open covering of X. Consider the data:

$$\begin{cases} a\ \mathbb{C}\text{-algebroid } \mathscr{A}_X \text{ on } X \\ \sigma_i \in \mathscr{A}_X(U_i). \end{cases}$$

Then by Theorem 2.3, we have the following gluing data:

- $\mathscr{A}_i := \mathscr{E}nd_{\mathscr{A}}(\sigma_i) = (\mathscr{O}_{U_i}[[\hbar]], \star_i)$,
- $f_{ij} : \mathscr{A}_j|_{U_{ij}} \to \mathscr{A}_i|_{U_{ij}}$ the $\mathbb{C}^\hbar$-algebra isomorphism,
- a_{ijk}: invertible elements of $\mathscr{A}_i(U_{ijk})$

which satisfies:

$$\begin{cases} f_{ij} \circ f_{jk} = Ad(a_{ijk}) \circ f_{ik} \\ a_{ijk} a_{ikl} = f_{ij}(a_{jkl}) a_{ijl}. \end{cases}$$

Since $\mathscr{A}_i = (\mathscr{O}_{U_i}[[\hbar]], \star_i)$ is a star algebra for each $i = 1, \cdots, n$, by definition, we have

$$f_i \star_i g_i = f_i g_i + \sum_{j=1}^{\infty} \beta_j(f_i, g_i)\hbar^j$$

for $f_i, g_i \in C_i := \Gamma(U_i, \mathscr{O}_X)$and $\beta_j : \mathscr{O}_{U_i} \times \mathscr{O}_{U_i} \to \mathscr{O}_{U_i}$ is a bi-differential operators for each j.

From the inclusion $C_i \hookrightarrow C_i^{\mathrm{an}} := \Gamma(U_i, \mathscr{O}_{X_{\mathrm{an}}})$, we can define a star product $\star_i^{\mathrm{an}}$ on the analytic sheaf $\mathscr{O}_{U_i}$ to be

$$f_i^{\mathrm{an}} \star_i^{\mathrm{an}} g_i^{\mathrm{an}} = f_i^{\mathrm{an}} g_i^{\mathrm{an}} + \sum_{j=1}^{\infty} \beta_j^{\mathrm{an}}(f_i^{\mathrm{an}}, g_i^{\mathrm{an}})\hbar^j \text{ for } f_i^{\mathrm{an}}, g_i^{\mathrm{an}} \in C_i^{\mathrm{an}},$$

where $\beta_j^{\mathrm{an}} : \mathscr{O}_{U_i} \times \mathscr{O}_{U_i} \to \mathscr{O}_{U_i}$ is a bi-differential operators on the analytic sheaf $\mathscr{O}_{U_i}$ for each j. Hence, we obtain the (analytic) star algebra $\mathscr{A}_i^{\mathrm{an}} = (\mathscr{O}_{U_i}[[\hbar]], \star_i^{\mathrm{an}})$ for each i.

Therefore, we get the corresponding descent data on X_{an}:

- $\mathscr{A}_i^{\mathrm{an}} = (\mathscr{O}_{U_i}[[\hbar]], \star_i^{\mathrm{an}})$,
- $f_{ij}^{\mathrm{an}} : \mathscr{A}_j^{\mathrm{an}}|_{U_{ij}} \to \mathscr{A}_i^{\mathrm{an}}|_{U_{ij}}$ the $\mathbb{C}^{\hbar}$-algebra isomorphism,
- a_{ijk}^{an}, the invertible element of $\mathscr{A}_i^{\mathrm{an}}(U_{ijk})$

and we obtain the DQ-algebroid $\mathscr{A}_{X_{\mathrm{an}}}$ on X_{an} by Theorem 2.1 (note that X_{an} is paracompact).

Hence for a DQ-algebroid $\mathscr{A}_X$ on X, we have the induced analytic DQ-algebroid $\mathscr{A}_{X_{\mathrm{an}}}$ on X_{an}.

Furthemore,

$$\mathscr{A}_{X_{\mathrm{an}}} \in \mathrm{Mod}(f^{-1}\mathscr{A}_X \otimes_{\mathbb{C}^{\hbar}} \mathscr{A}_{X_{\mathrm{an}}}^{\mathrm{op}}).$$

Hence for a DQ-algebroid $\mathscr{A}_X$ on a smooth variety X, we have the functor $f^* := \mathscr{A}_{X_{\mathrm{an}}} \otimes_{f^{-1}(\mathscr{A}_X)} f^{-1}(\,\cdot\,) : \mathrm{Mod}(\mathscr{A}_X) \to \mathrm{Mod}(\mathscr{A}_{X_{\mathrm{an}}})$ which sends $\mathscr{M}$ to $\mathscr{A}_{X_{\mathrm{an}}} \otimes_{f^{-1}(\mathscr{A}_X)} f^{-1}(\mathscr{M})$. Denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$ and $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ the categories consisting of coherent $\mathscr{A}_X$-modules and $\mathscr{A}_{X_{\mathrm{an}}}$-modules, respectively. If $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$, then $f^*(\mathscr{M}) \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$.

4. The main theorem.

In this section, we prove the main theorem of this paper. Let $\mathscr{A}_X$ be a DQ-algebroid on a smooth algebraic variety X.

Flatness.

Let X_{an} be the corresponding complex analytic manifold of X with continuous map $f : X_{\mathrm{an}} \to X$. First, we need the following lemma. The following lemma over one point as a corollary of Theorem 1.6.5 of [8].

LEMMA 4.1. *The functor* $f^* : \mathrm{Mod}(\mathscr{A}_X) \to \mathrm{Mod}(\mathscr{A}_{X_{\mathrm{an}}})$ *constructed above is exact.*

PROOF. We may assume that $\mathscr{A}_X$ and $\mathscr{A}_{X_{\mathrm{an}}}$ are DQ-algebras. We need to show that $B := \mathscr{A}_{X_{\mathrm{an}},x}$ is flat over $R := \mathscr{A}_{X,x}$ for each $x \in X$. Note that:

(a) B has no $\hbar$-torsion,

(b) $B_0 := B/\hbar B = \mathscr{O}_{X_{\mathrm{an}},x}$ is a flat $R_0 := R/\hbar R = \mathscr{O}_{X,x}$-module,

(c) $B \simeq \varprojlim_n B/\hbar^n B$.

Hence applying Theorem 1.6.5 of [8] to $X = \{\mathrm{pt}\}$, one gets the result. □

From Lemma 4.1, one can see that the functor $f^* : \mathrm{Mod}(\mathscr{A}_X) \to \to \mathrm{Mod}(\mathscr{A}_{X_{\mathrm{an}}})$ induces a functor (we keep the same notation):

$$f^* : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \;\to\; \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}).$$

Fully faithfulness.

Now we can prove the following theorem.

THEOREM 4.2. *Let X be a smooth projective variety, then the functor* f^**:* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ *is fully faithful.*

PROOF. For any $\mathscr{M},\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$, we need to show that the morphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)}(\mathscr{M},\mathscr{N}) \;\to\; \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})}(f^*(\mathscr{M}),f^*(\mathscr{N})) \tag{4.1}$$

is a bijection. In order to show that the morphism of (4.1) is a bijection, it is sufficient to show that the morphism

$$\mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)}(\mathscr{M},\mathscr{N}) \to \mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})}(f^*(\mathscr{M}),f^*(\mathscr{N})) \tag{4.2}$$

is an isomorphism. Since X is projective, by Theorem 2.13, the complexes $\mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)}(\mathscr{M},\mathscr{N})$ and $\mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})}(f^*(\mathscr{M}),f^*(\mathscr{N}))$ belong to $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$. Moreover, since X is a smooth variety and $\mathrm{gr}(\mathscr{A}_X)$ is an invertible $\mathscr{O}_X$-algebroid, $\mathrm{gr}(\mathscr{A}_X)$ is equivalent to $\mathscr{O}_X$ by Remark 2.5 and hence $\mathrm{gr}(\mathscr{A}_{X_{\mathrm{an}}})$ is equivalent to $\mathscr{O}_{X_{\mathrm{an}}}$. Thus, the equivalence $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X_{\mathrm{an}}})$ (Corollary 1.4) implies that the following morphism

$$\mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{gr}(\mathscr{A}_X))}(\mathrm{gr}\mathscr{M}, \mathrm{gr}\mathscr{N}) \to \mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{gr}(\mathscr{A}_{X_{\mathrm{an}}}))}(f^*(\mathrm{gr}\mathscr{M}),f^*(\mathrm{gr}\mathscr{N})) \tag{4.3}$$

is an isomorphism in $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C})$. Applying the functor $\mathbb{C} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar} \cdot$ to (4.2) and using Proposition 2.12, we get (4.3). Since the functor $\mathbb{C} \overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar} \cdot$ is conservative by Corollary 2.11, the morphism of (4.2) is an isomorphism and the result follows. □

COROLLARY 4.3. *Let X be a smooth projective variety, then the natural functor* $f^* : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ *is exact and fully faithful.*

For each $n > 0$, we denote by $\mathrm{Mod}(\mathscr{A}_X/\hbar^n\mathscr{A}_X)$ (resp. $\mathrm{Mod}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$) the full subcategory of $\mathrm{Mod}(\mathscr{A}_X)$ (resp. $\mathrm{Mod}(\mathscr{A}_{X_{\mathrm{an}}})$) consisting of objects $\mathscr{M}$ such that $\hbar^n : \mathscr{M} \to \mathscr{M}$ is the zero morphism.

Similarly, we denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X)$ (resp. $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$) the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$ (resp. $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$) consisting of objects $\mathscr{M}$ such that $\hbar^n : \mathscr{M} \to \mathscr{M}$ is the zero morphism for each $n > 0$. Therefore, we have a functor $f_n^* := f^*|_{\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X)} : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X) \to$ $\to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$ for each $n > 0$.

Note that for $n = 1$, the category $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^1\mathscr{A}_X) \simeq \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X)$ is equivalent to the category $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^1\mathscr{A}_{X_{\mathrm{an}}}) \simeq \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_{X_{\mathrm{an}}})$ by Theorem 1.1.

COROLLARY 4.4. *Let X be a smooth projective variety, then the functor* $f_n^* : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$ *is exact and fully faithful for each* $n > 0$.

Essential surjectivity.

Denote by X a smooth projective variety. Next, we shall prove that the functor f^*: $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ is essentially surjective.

We first prove that the functor f_n^*: $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X /, \hbar^n\mathscr{A}_X) \to$

$\to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$ is essentially surjective for each $n > 0$. We need the following lemma.

LEMMA 4.5. *Let $\mathcal{A}'$ and $\mathcal{B}'$ be thick subcategories of abelian categories $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be an exact functor which takes $\mathcal{A}'$ to $\mathcal{B}'$ and such that the natural functor (we keep the same notation) $\Phi : \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A}) \to \mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})$ induced by Φ is fully faithful. Consider an exact sequence in $\mathcal{B}$*

$$(*) \qquad 0 \to \Phi(M') \to N \to \Phi(M'') \to 0,$$

with $M', M'' \in \mathcal{A}'$ and $N \in \mathcal{B}'$.

Then there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi(M') & \longrightarrow & \Phi(M) & \longrightarrow & \Phi(M'') & \longrightarrow & 0 \\ & & \| & & \downarrow\wr & & \| & & \\ 0 & \longrightarrow & \Phi(M') & \longrightarrow & N & \longrightarrow & \Phi(M'') & \longrightarrow & 0 \end{array}$$

for some $M \in \mathcal{A}'$ (note that the middle arrow is an isomorphism).

PROOF. Since $(*)$ is an exact sequence, we get the morphism $v : \Phi(M'') \to \Phi(M')[1] = \Phi(M'[1])$ in $\mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})$. Since Φ is fully faithful, there exists a morphism $u : M'' \to M'[1]$ in $\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ such that $v = \Phi(u)$. Consider the distinguished triangle

$$M'' \xrightarrow{u} M'[1] \longrightarrow L \xrightarrow{+1}$$

in $\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ induced by u with $L \in \mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$. Then from the long exact sequence

$$\cdots \longrightarrow H^i(M'') \longrightarrow H^i(M'[1]) \longrightarrow H^i(L) \longrightarrow H^i(M''[1]) \longrightarrow \cdots,$$

we get $H^i(L) = 0$ for $i \neq -1$. Hence $L[-1]$ is isomorphic to $H^0(L[-1]) \in \mathcal{A}'$ in $\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$. Denote by $M = H^0(L[-1])$, then from the morphism of distinguished triangles

$$\begin{array}{ccccccc} \Phi(M') & \longrightarrow & \Phi(M) & \longrightarrow & \Phi(M'') & \longrightarrow & \Phi(M'[1]) \\ \| & & \downarrow & & \| & & \| \\ \Phi(M') & \longrightarrow & N & \longrightarrow & \Phi(M'') & \longrightarrow & \Phi(M'[1]) \end{array}$$

we obtain $N \simeq \Phi(M)$ and the result follows. □

Set $\mathcal{A} := \mathrm{Mod}(\mathscr{A}_X/\hbar^n\mathscr{A}_X)$, $\mathcal{A}' := \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X)$, $\mathcal{B} :=$ $:= \mathrm{Mod}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$ and $\mathcal{B}' = \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$. We shall apply Lemma 4.5.

THEOREM 4.6. *The functor*

$$f_n^* : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$$

is essentially surjective for each $n > 0$.

PROOF. We shall prove this by induction.
When $n = 1$, it is Theorem 1.1.
We shall prove the theorem for $n > 1$.
For any $\mathscr{M}^{\mathrm{an}} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$, consider the exact sequence

$$0 \to \hbar\mathscr{M}^{\mathrm{an}} \to \mathscr{M}^{\mathrm{an}} \to \mathscr{M}^{\mathrm{an}}/\hbar\mathscr{M}^{\mathrm{an}} \to 0$$

where $\hbar\mathscr{M}^{\mathrm{an}} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^{n-1}\mathscr{A}_{X_{\mathrm{an}}})$ and $\mathscr{M}^{\mathrm{an}}/\hbar\mathscr{M}^{\mathrm{an}} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_{X_{\mathrm{an}}})$. Denote by $\mathscr{M}_1^{\mathrm{an}} = \hbar\mathscr{M}^{\mathrm{an}}$ and by $\mathscr{M}_2^{\mathrm{an}} = \mathscr{M}^{\mathrm{an}}/\hbar\mathscr{M}^{\mathrm{an}}$. By induction hypothesis, there exists $\mathscr{M}_1 \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^{n-1}\mathscr{A}_X)$ such that $f_{n-1}^*(\mathscr{M}_1) \simeq$ $\simeq \mathscr{M}_1^{\mathrm{an}}$. On the other hand, by Theorem 1.1, there exists $\mathscr{M}_2 \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_X)$ such that $f_1^*(\mathscr{M}_2) \simeq \mathscr{M}_2^{\mathrm{an}}$. Since $f^*|_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ is exact by Lemma 4.1 and the functor $\mathrm{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A}) \to \mathrm{D}^{\mathrm{b}}_{\mathcal{B}'}(\mathcal{B})$ induced by $f^*|_{\mathcal{A}}$ is fully faithful by the proof of Theorem 4.2, applying Lemma 4.5, we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & f_n^*(\mathscr{M}_1) & \longrightarrow & f_n^*(\mathscr{M}) & \longrightarrow & f_n^*(\mathscr{M}_2) & \longrightarrow & 0 \\
 & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\
0 & \longrightarrow & \mathscr{M}_1^{\mathrm{an}} & \longrightarrow & \mathscr{M}^{\mathrm{an}} & \longrightarrow & \mathscr{M}_2^{\mathrm{an}} & \longrightarrow & 0
\end{array}$$

for some $\mathscr{M} \in \mathcal{A}'$. Hence f_n^* is essentially surjective. □

From Corollary 4.4 and Theorem 4.6, we obtain the following.

THEOREM 4.7. *The functor*

$$f_n^* : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$$

is an equivalence for each $n > 0$.

In order to prove that the functor $f^*\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ is essentially surjective, we need the notion of projective limit in the 2-category **Cat**. For its definition, we refer to [9] Definition 19.1.6.

Recall that a presite X is nothing but a category which we denote by $\mathcal{C}_X$. If $\mathfrak{S}$ is a prestack on X, then we have the morphism $u : U_1 \to U_2$ in $\mathcal{C}_X$, and the functor $r_u : \mathfrak{S}(U_2) \to \mathfrak{S}(U_1)$ for any $U_1, U_2 \in \mathcal{C}_X$.

Denote by $\mathbb{N}$ the set of positive integers, viewed as a category defined by

$$\mathrm{Ob}(\mathbb{N}) = \mathbb{N}$$

$$\mathrm{Hom}_{\mathbb{N}}(i,j) = \begin{cases} \{\mathrm{pt}\} & \text{if } i \leq j, \\ \emptyset & \text{otherwise.} \end{cases}$$

We define prestacks $\mathfrak{S}$ and $\mathfrak{S}_{\mathrm{an}}$ on $\mathbb{N}$ as follows:

- $\mathfrak{S}(n) := \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X/\hbar^n\mathscr{A}_X)$ for any $n \in \mathbb{N}$,
- $r_u : \mathfrak{S}(j) \to \mathfrak{S}(i)$ is the functor for any $i \leq j$ and $u \in \mathrm{Hom}_{\mathbb{N}}(i,j)$,

and

- $\mathfrak{S}_{\mathrm{an}}(n) := \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}/\hbar^n\mathscr{A}_{X_{\mathrm{an}}})$ for any $n \in \mathbb{N}$,
- $r_u : \mathfrak{S}_{\mathrm{an}}(j) \to \mathfrak{S}_{\mathrm{an}}(i)$ is the functor for any $i \leq j$ and $u \in \mathrm{Hom}_{\mathbb{N}}(i,j)$.

The following lemma shows that coherent $\mathscr{A}$-modules are $\hbar$-complete.

LEMMA 4.8 ([8]). *Let $(X, \mathscr{A}_X)$ be a complex manifold or a smooth variety endowed with a* DQ*-algebroid $\mathscr{A}_X$. Let $\{\mathscr{M}_n\}_{n\geq 0}$ be a projective system of coherent $\mathscr{A}_X$-modules. Assume that $\hbar^{n+1}\mathscr{M}_n = 0$ and the induced morphism $\mathscr{M}_{n+1}/\hbar^{n+1}\mathscr{M}_{n+1} \to \mathscr{M}_n$ is an isomorphism for any $n \geq 0$. Then $\mathscr{M} := \varprojlim_n \mathscr{M}_n$ is a coherent $\mathscr{A}_X$-module and $\mathscr{M}/\hbar^{n+1}\mathscr{M} \to \mathscr{M}_n$ is an isomorphism for any $n \geq 0$.*

We need the following theorem.

THEOREM 4.9. *We have the following equivalences:*

(1) $\varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$,

(2) $\varprojlim_{n\in\mathbb{N}} \mathfrak{S}_{\mathrm{an}}(n) \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$.

PROOF. We only need to prove (1) and (2) can be proved similarly. Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$, then we obtain the family $\{(F_n, \varphi_u)\}$ where:

(i) $F_n := \mathscr{M}/\hbar^n\mathscr{M} \in \mathfrak{S}(n)$ for any $n \in \mathbb{N}$,

(ii) $\varphi_u : r_u F_j \xrightarrow{\sim} F_i$ for any $i \leq j$ and $u \in \mathrm{Hom}_{\mathbb{N}}(i,j)$ and $r_u : \mathfrak{S}(j) \to$ $\to \mathfrak{S}(i)$ is defined by sending $\mathscr{M}$ to $\mathscr{M}/\hbar^i\mathscr{M}$.

It is easy to check that $\{(F_n, \varphi_u)\}$ satisfies the cocycle condition (a) of [9] 19.1.6 and hence $\{(F_n, \varphi_u)\} \in \varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)$.

Let $\mathscr{M}, \mathscr{M}' \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$, then these define two objects $F = \{(F_n, \varphi_u)\}$ and $F' = \{(F'_n, \varphi'_u)\}$ in $\varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)$. Let $f : \mathscr{M} \to \mathscr{M}' \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$, then we have the set of families $\{f_n\}_{n\in\mathbb{N}}$ where $f_n : \mathscr{M}/\hbar^n\mathscr{M} \to \mathscr{M}'/\hbar^n\mathscr{M}' \in \mathrm{Hom}_{\mathfrak{S}(n)}(\mathscr{M}/\hbar^n\mathscr{M}, \mathscr{M}'/\hbar^n\mathscr{M}')$. It is easy to check that $\{f_n\}$ satisfies the commutative diagram of definition (b) of [9] 19.1.6 and hence $\{f_n\}_{n\in\mathbb{N}} \in \mathrm{Hom}_{\varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)}(F, F')$. Hence we can define a functor $\Phi : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X) \to \varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)$ by sending $\mathscr{M}$ to $\{(F_n, \varphi_u)\}$ and $f \in \mathrm{Hom}_{\mathscr{A}_X}(\mathscr{M}, \mathscr{M}')$ to $\{f_n\}_{n\in\mathbb{N}}$.

On the other hand, if $\{(F_n, \varphi_u)\} \in \varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)$, then by definition (a) of [9] 19.1.6, we have

(i) $F_n \in \mathfrak{S}(n)$ for any $n \in \mathbb{N}$,

(ii) $\varphi_u : r_u F_j \xrightarrow{\sim} F_i$ for any $i \le j$ and $u \in \mathrm{Hom}_{\mathbb{N}}(i, j)$ and $r_u : \mathfrak{S}(j) \to \mathfrak{S}(i)$.

Hence the system $\{F_n\}_{n\in\mathbb{N}}$ is a projective system and $\varprojlim_{n\in\mathbb{N}} F_n \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$ by Lemma 4.8.

For two objects $F = \{(F_n, \varphi_u)\}$ and $F' = \{(F'_n, \varphi'_u)\}$ in $\varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)$. By definition (b) of [9] 19.1.6, $\mathrm{Hom}_{\varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)}(F, F')$ is the set of families $f = \{f_n\}_{n\in\mathbb{N}}$ such that $f_n \in \mathrm{Hom}_{\mathfrak{S}(n)}(F_n, F'_n)$ and the following diagram commutes for any $u : i \to j$ and $i \le j$,

$$\begin{array}{ccc} r_u F_j & \xrightarrow{\varphi_u} & F_i \\ {\scriptstyle r_u(f_j)}\downarrow & & \downarrow{\scriptstyle f_i} \\ r_u F'_j & \xrightarrow{\varphi'_u} & F'_i. \end{array}$$

Hence the system $f = \{f_n\}_{n\in\mathbb{N}}$ is a projective system and we get that the morphism $\varprojlim_{n\in\mathbb{N}} f_n$ belongs to $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$. Therefore, we can define a functor $\Psi : \varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$ by $\Psi(\{(F_n, \varphi_u)\}) = \varprojlim_{n\in\mathbb{N}} F_n$ and $\Psi(f = \{f_n\}_{n\in\mathbb{N}}) = \varprojlim_{n\in\mathbb{N}} f_n$. Now it is easy to check that $\Phi \circ \Psi \simeq \mathrm{id}_{\varprojlim_{n\in\mathbb{N}} \mathfrak{S}(n)}$ and $\Psi \circ \Phi \simeq \mathrm{id}_{\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)}$. Therefore, the result follows. $\square$

COROLLARY 4.10. *The functor* $f^*: \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ *is an equivalence.*

PROOF. This follows from Theorem 4.7 and Theorem 4.9. □

From Theorem 4.2, Corollary 4.10 and the proof of Lemma 1.2 for essentially surjectivity, we obtain what we want mentioned above.

COROLLARY 4.11. *The natural functor* f^**:* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ *is essentially surjective.*

Equivalence.

Therefore, we obtain the main theorem of this paper.

MAIN THEOREM 4.12. *The natural functor* f^**:* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}})$ *is an equivalence.*

PROOF. This follows from Theorem 4.2 and Corollary 4.11. □

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