

On the Homogeneity of Global Minimizers for the Mumford-Shah Functional when K is a Smooth Cone

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ABSTRACT - We show that if (u, K) is a global minimizer for the Mumford-Shah functional in \mathbb{R}^N , and if K is a smooth enough cone, then (modulo constants) u is a homogenous function of degree $\frac{1}{2}$. We deduce some applications in \mathbb{R}^3 as for instance that an angular sector cannot be the singular set of a global minimizer, that if K is a half-plane then u is the corresponding cracktip function of two variables, or that if K is a cone that meets S^2 with an union of C^∞ curvilinear convex polygons, then it is a P, Y or T.

Introduction.

The functional of D. Mumford and J. Shah [18] was introduced to solve an image segmentation problem. If Ω is an open subset of \mathbb{R}^2 , for example a rectangle, and $g \in L^\infty(\Omega)$ is an image, one can get a segmentation by minimizing

$$J(K, u) := \int_{\Omega \setminus K} |\nabla u|^2 dx + \int_{\Omega \setminus K} (u - g)^2 dx + H^1(K)$$

over all the admissible pairs $(u, K) \in \mathcal{A}$ defined by

$$\mathcal{A} := \{(u, K); K \subset \Omega \text{ is closed, } u \in W_{loc}^{1,2}(\Omega \setminus K)\}.$$

Any solution (u, K) that minimizes J represents a “smoother” version of the image and the set K represents the edges of the image.

Existence of minimizers is a well known result (see for instance [11]) using *SBV* theory.

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The question of regularity for the singular set K of a minimizer is more difficult. The following conjecture is currently still open.

CONJECTURE 1 (Mumford-Shah). [18] *Let (u, K) be a reduced minimizer for the functional J . Then K is the finite union of C^1 arcs.*

The term “reduced” just means that we cannot find another pair (\tilde{u}, \tilde{K}) such that $K \subset \tilde{K}$ and \tilde{u} is an extension of u in $\Omega \setminus \tilde{K}$.

Some partial results are true for the conjecture. For instance it is known that K is C^1 almost everywhere (see [7], [4] and [2]). The closest to the conjecture is probably the result of A. Bonnet [4]. He proves that if (u, K) is a minimizer, then every isolated connected component of K is a finite union of C^1 -arcs. The approach of A. Bonnet is to use blow up limits. If (u, K) is a minimizer in Ω and y is a fixed point, consider the sequences (u_k, K_k) defined by

$$u_k(x) = \frac{1}{\sqrt{t_k}} u(y + t_k x), \quad K_k = \frac{1}{t_k} (K - y), \quad \Omega_k = \frac{1}{t_k} (\Omega - y).$$

When $\{t_k\}$ tends to infinity, the sequence (u_k, K_k) may tend to a pair (u_∞, K_∞) , and then (u_∞, K_∞) is called a Global Minimizers. Moreover, A. Bonnet proves that if K_∞ is connected, then (u_∞, K_∞) is one of the list below:

- 1ST CASE: $K_\infty = \emptyset$ and u_∞ is a constant.
- 2ND CASE: K_∞ is a line and u_∞ is locally constant.
- 3RD CASE: “Propeller”: K_∞ is the union of 3 half-lines meeting with 120 degrees and u_∞ is locally constant.
- 4TH CASE: “Cracktip”: $K_\infty = \{(x, 0); x \leq 0\}$ and $u_\infty(r \cos(\theta), r \sin(\theta)) = \pm \sqrt{\frac{2}{\pi}} r^{1/2} \sin \frac{\theta}{2} + C$, for $r > 0$ and $|\theta| < \pi$ (C is a constant), or a similar pair obtained by translation and rotation.

We don’t know whether the list is complete without the hypothesis that K_∞ is connected. This would give a positive answer to the Mumford-Shah conjecture.

The Mumford-Shah functional was initially given in dimension 2 but there is no restriction to define Minimizers for the analogous functional in \mathbb{R}^N . Then we can also do some blow-up limits and try to think about what should be a global minimizer in \mathbb{R}^N . Almost nothing is known in this direction and this paper can be seen as a very preliminary step. Let state some definitions.

DEFINITION 2. Let $\Omega \subset \mathbb{R}^N$, $(u, K) \in \mathcal{A}$ and B be a ball such that $\bar{B} \subset \Omega$. A competitor for the pair (u, K) in the ball B is a pair $(v, L) \in \mathcal{A}$ such that

$$\left. \begin{aligned} u &= v \\ K &= L \end{aligned} \right\} \text{ in } \Omega \setminus B$$

and in addition if x and y are two points in $\Omega \setminus (B \cup K)$ that are separated by K then they are also separated by L .

The expression “be separated by K ” means that x and y lie in different connected components of $\Omega \setminus K$.

DEFINITION 3. A global minimizer in \mathbb{R}^N is a pair $(u, K) \in \mathcal{A}$ (with $\Omega = \mathbb{R}^N$) such that for every ball B in \mathbb{R}^N and every competitor (v, L) in B we have

$$\int_{B \setminus K} |\nabla u|^2 dx + H^{N-1}(K \cap B) \leq \int_{B \setminus L} |\nabla v|^2 dx + H^{N-1}(L \cap B)$$

where H^{N-1} denotes the Hausdorff measure of dimension $N - 1$.

Proposition 9 on page 267 of [8] ensures that any blow up limit of a minimizer for the Mumford-Shah functional in \mathbb{R}^N , is a global minimizer in the sense of Definition 3. As a beginning for the description of global minimizers in \mathbb{R}^N , we can firstly think about what should be a global minimizer in \mathbb{R}^3 . If u is locally constant, then K is a minimal cone, that is, a set that locally minimizes the Hausdorff measure of dimension 2 in \mathbb{R}^3 . Then by [9] we know that K is a cone of type \mathbb{P} (hyperplane), \mathbb{Y} (three half-planes meeting with 120 degrees angles) or of type \mathbb{T} (cone over the edges of a regular tetraedron centered at the origin). Those cones became famous by the theorem of J. Taylor [20] which says that any minimal surface in \mathbb{R}^3 is locally C^1 equivalent to a cone of type \mathbb{P} , \mathbb{Y} or \mathbb{T} .

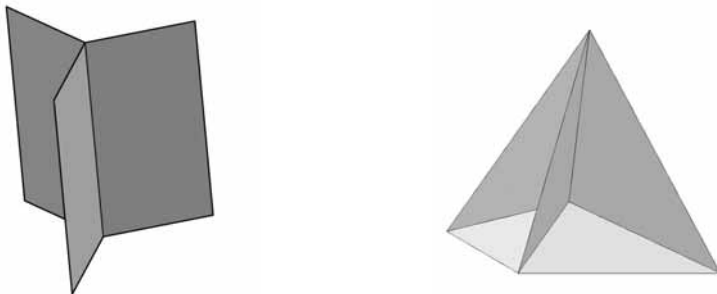


Fig. Cones of type \mathbb{Y} and \mathbb{T} in \mathbb{R}^3 .

To be clearer, this is a more precise definition of \mathbb{Y} and \mathbb{T} , as in [10].

DEFINITION 4. Define $Prop \subset \mathbb{R}^2$ by

$$\begin{aligned} Prop &= \{(x_1, x_2); x_1 \geq 0, x_2 = 0\} \\ &\cup \{(x_1, x_2); x_1 \leq 0, x_2 = -\sqrt{3}x_1\} \\ &\cup \{(x_1, x_2); x_1 \leq 0, x_2 = \sqrt{3}x_1\}. \end{aligned}$$

Then let $Y_0 = Prop \times \mathbb{R} \subset \mathbb{R}^3$. The spine of Y_0 is the line $L_0 = \{x_1 = x_2 = 0\}$. A cone of type \mathbb{Y} is a set $Y = R(Y_0)$ where R is the composition of a translation and a rotation. The spine of Y is then the line $R(L_0)$.

DEFINITION 5. Let $A_1 = (1, 0, 0)$, $A_2 = \left(-\frac{1}{3}, \frac{2\sqrt{2}}{3}, 0\right)$, $A_3 = \left(-\frac{1}{3}, -\frac{\sqrt{2}}{3}, \frac{\sqrt{6}}{3}\right)$, and $A_4 = \left(-\frac{1}{3}, -\frac{\sqrt{2}}{3}, -\frac{\sqrt{6}}{3}\right)$ the four vertices of a regular tetrahedron centered at 0. Let T_0 be the cone over the union of the 6 edges $[A_i, A_j]$ $i \neq j$. The spine of T_0 is the union of the four half-lines $[0, A_j]$. A cone of type \mathbb{T} is a set $T = R(T_0)$ where R is the composition of a translation and a rotation. The spine of T is the image by R of the spine of T_0 .

So the pairs (u, Z) where u is locally constant and Z is a minimal cone, are examples of global minimizers in \mathbb{R}^3 . Another global minimizer can be obtained with K_∞ a half-plane, by setting $u := Craktip \times \mathbb{R}$ (see [8] section 76). These examples are the only global minimizers in \mathbb{R}^3 that we know.

Note that if (u, K) is a global minimizer in \mathbb{R}^N , then u locally minimizes the Dirichlet integral in $\mathbb{R}^N \setminus K$. As a consequence, u is harmonic in $\mathbb{R}^N \setminus K$. Moreover, if B is a ball such that $K \cap B$ is regular enough, then the normal derivative of u vanishes on $K \cap B$.

In this paper we wish to study global minimizers (u, K) for which K is a cone. It seems natural to think that any singular set of a global minimizer is a cone. But even if all known examples are cones, there is no proof of this fact. In addition, we will add some regularity on K . We denote by S^{N-1} the unit sphere in \mathbb{R}^N and, if Ω is a open set, $W^{1,2}(\Omega)$ is the Sobolev space. We will say that a domain Ω on S^{N-1} has a piecewise C^2 boundary, if the topological boundary of Ω , defined by $\partial\Omega = \bar{\Omega} \setminus \Omega$, consists of an union of $N - 2$ dimensional hypersurfaces of class C^2 . This allows some cracks, i.e. when Ω lies in each sides of its boundary. We will denote by $\tilde{\Sigma}$ the set of all the singular points of the boundary, that is

$$\tilde{\Sigma} := \{x \in \partial\Omega; \forall r > 0, B(x, r) \cap \partial\Omega \text{ is not a } C^2 \text{ hypersurface}\}.$$

DEFINITION 6. *A smooth cone is a set K of dimension $N - 1$ in \mathbb{R}^N such that K is conical, centered at the origin, and such that $S^{N-1} \setminus K$ is a domain with piecewise C^2 boundary. Moreover we assume that the embedding $W^{1,2}(S^{N-1} \setminus K) \rightarrow L^2(S^{N-1} \setminus K)$ is compact. Finally we suppose that we can strongly integrate by parts in $B(0, 1) \setminus K$. More precisely, denoting by Σ the set of singularities*

$$\Sigma := \{tx; (t, x) \in \mathbb{R}^+ \times \tilde{\Sigma}\},$$

we want that

$$\int_{B(0,1) \setminus K} \langle \nabla u, \nabla \varphi \rangle = 0$$

for every harmonic function u in $B(0, 1) \setminus K$ with $\frac{\partial}{\partial n} u = 0$ on $K \setminus \Sigma$, and for all $\varphi \in W^{1,2}(B(0, 1) \setminus K)$ with vanishing trace on $S^{N-1} \setminus K$.

REMARK 7. For instance if K is the cone over a finite union of C^2 -arcs on S^2 , then we can strongly integrate by parts in $B(0, 1) \setminus K$. Another example in \mathbb{R}^N is given by the union of admissible set of faces (as in Definition (22.2) of [5]).

Now this is the main result.

THEOREM 15. *Let (u, K) be a global minimizer in \mathbb{R}^N and assume that K is a smooth cone. Then there is a $\frac{1}{2}$ -homogenous function u_1 such that $u - u_1$ is locally constant.*

As we shall see, this result implies that if (u, K) is a global minimizer in \mathbb{R}^N , and if K is a smooth cone other than a minimal cone, then $\frac{3 - 2N}{4}$ is an eigenvalue for the spherical Laplacian in $S^{N-1} \setminus K$ with Neumann boundary conditions. In section 2 we will give some applications about global minimizers in \mathbb{R}^3 , using the estimates on the first eigenvalue that can be found in [6], [5] and [14]. More precisely, we have:

PROPOSITION 17. *Let (u, K) be a global Mumford-Shah minimizer in \mathbb{R}^3 such that K is a smooth cone. Moreover, assume that $S^2 \cap K$ is a union of convex curvilinear polygons with C^∞ sides. Then u is locally constant and K is a cone of type P, Y or T.*

Another consequence of the main result is the following.

PROPOSITION 19. *Let (u, K) be a global Mumford-Shah minimizer in \mathbb{R}^3 such that K is a half plane. Then u is equal to a function of type cracktip $\times \mathbb{R}$, that is, in cylindrical coordinates,*

$$u(r, \theta, z) = \pm \sqrt{\frac{2}{\pi}} r^{\frac{1}{2}} \sin \frac{\theta}{2} + C$$

for $0 < r < +\infty$, $-\pi < \theta < \pi$ where C is a constant.

Finally, we deduce two other consequences from Theorem 15. Let $(r, \theta, z) \in \mathbb{R}^+ \times [-\pi, \pi] \times \mathbb{R}$ be the cylindrical coordinates in \mathbb{R}^3 . For all $\omega \in [0, \pi]$ set

$$\partial\Gamma_\omega := \{(r, \theta, z) \in \mathbb{R}^3; \theta = -\omega \text{ or } \theta = \omega\}.$$

and

$$(1) \quad S_\omega := \{(r, \theta, z) \in \mathbb{R}^3; z = 0, r > 0, \theta \in [-\omega, \omega]\}$$

Observe that S_0 is a half line, $S_{\frac{\pi}{2}}$, $\partial\Gamma_0$ and $\partial\Gamma_\pi$ are half-planes, and that S_π and $\partial\Gamma_{\frac{\pi}{2}}$ are planes.

PROPOSITION 18. *There is no global Mumford-Shah minimizer in \mathbb{R}^3 such that K is wing of type $\partial\Gamma_\omega$ with $\omega \notin \left\{0, \frac{\pi}{2}, \pi\right\}$.*

PROPOSITION 23. *There is no global Mumford-Shah minimizer in \mathbb{R}^3 such that K is an angular sector of type (u, S_ω) for $\omega \notin \left\{\frac{\pi}{2}, \pi\right\}$.*

1. If K is a cone then u is homogenous.

In this section we want to prove Theorem 15. Notice that this result is only useful if the dimension $N \geq 3$. Indeed, in dimension 2, if K is a cone then it is connected thus it is in the list described in the introduction.

1.1 – Preliminary.

Let us recall a standard uniqueness result about energy minimizers.

PROPOSITION 8. *Let Ω be an open and connected set of \mathbb{R}^N and let $I \subset \partial\Omega$ be a hypersurface of class C^∞ . Suppose that u and v are two*

functions in $W^{1,2}(\Omega)$ such that $u = v$ a.e. on I (in terms of trace), solving the minimizing problem

$$\min E(w) := \int_{\Omega} |\nabla w(x)|^2 dx$$

over all the functions $w \in W^{1,2}(\Omega)$ that are equal to u and v on I . Then

$$u = v.$$

PROOF. This comes from a simple convexity argument which can be found for instance in [8], but let us write the proof since it is very short. By the parallelogram identity we have

$$(2) \quad E\left(\frac{u+v}{2}\right) = \frac{1}{2}E(u) + \frac{1}{2}E(v) - \frac{1}{4}E(u-v).$$

On the other hand, since $\frac{u+v}{2}$ is equal to u and v on I , and by minimality of u and v we have

$$E\left(\frac{u+v}{2}\right) \geq E(u) = E(v).$$

Now by (2) we deduce that $E(u-v) = 0$ and since Ω is connexe, this implies that $u-v$ is a constant. But $u-v$ is equal to 0 on I thus $u = v$. □

REMARK 9. The existence of a minimizer can also be proved using the convexity of $E(v)$.

1.2 – Spectral decomposition.

The key ingredient to obtain the main result will be the spectral theory of the Laplacian on the unit sphere. Since u is harmonic, we will decompose u as a sum of homogeneous harmonic functions just like we usually use the classical spherical harmonics. The difficulty here comes from the lack of regularity of $\mathbb{R}^N \setminus K$.

It will be convenient to work with connected sets. So let Ω be a connected component of $S^{N-1} \setminus K$, and let $A(r)$ be

$$A(r) := \{tx; (x, t) \in \Omega \times [0, r] \}.$$

We also set

$$A(\infty) := \{tx; (x, t) \in \Omega \times \mathbb{R}^+ \}.$$

All the following results are using that the embedding $W^{1,2}(\Omega)$ in $L^2(\Omega)$ is compact. Recall that this is the case by definition, since K is a smooth cone. Notice that for instance the cone property insures that the embedding is compact (see Theorem 6.2. p. 144 of [1]).

Consider the quadratic form

$$Q(u) = \int_{\Omega} |\nabla u(x)|^2 dx$$

of domain $W^{1,2}(\Omega)$ dense into the Hilbert space $L^2(\Omega)$. Since Q is a positive and closed quadratic form (see for instance Proposition 10.61 p. 129 of [16]) there exists a unique selfadjoint operator denoted by $-\Delta_n$ of domain $D(-\Delta_n) \subset W^{1,2}(\Omega)$ such that

$$\forall u \in D(-\Delta_n), \forall v \in W^{1,2}(\Omega), \quad \int_{\Omega} \langle \nabla u, \nabla v \rangle = \int_{\Omega} \langle -\Delta_n u, v \rangle.$$

PROPOSITION 10. *The operator $-\Delta_n$ has a countably infinite discrete set of eigenvalues, whose eigenfunctions span $L^2(\Omega)$.*

PROOF. The proof is the same as if Ω was a regular domain. Consider the new quadratic form

$$\tilde{Q}(u) := Q(u) + \|u\|_2^2$$

with the same domain $W^{1,2}(\Omega)$. The form \tilde{Q} has the same properties than Q and the associated operator is $\text{Id} - \Delta_n$. Moreover \tilde{Q} is coercive. As a result, the operator $\text{Id} - \Delta_n$ is bijective and its inverse goes from $L^2(\Omega)$ to $D(-\Delta_n) \subset W^{1,2}(\Omega)$. By hypothesis the embedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$ is compact. Thus the resolvent $(\text{Id} - \Delta_n)^{-1}$ is a compact operator, and we conclude using the spectral theory of operators with a compact resolvent (see [19] Theorem XIII. 64 p. 245). \square

REMARK 11. The domain of $-\Delta_n$ is not known in general. If Ω was smooth, then we could show that the domain is exactly $D(-\Delta_n) = \left\{ u \in W^{2,2}(\Omega); \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}$. Here, the boundary of Ω has some singularities so this result doesn't apply directly. But knowing exactly the domain of $-\Delta_n$ will not be necessary for us.

Now we want to study the link between the abstract operator Δ_n and the classical spherical Laplacian Δ_S on the unit sphere. Recall that if we compute the Laplacian in spherical coordinates, we obtain the following

equality

$$(3) \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_S.$$

PROPOSITION 12. *For every function $f \in D(-\Delta_n)$ such that $-\Delta_n f = \lambda f$ we have*

- i) $f \in C^\infty(\Omega)$
- ii) $-\Delta_S f = -\Delta_n f = \lambda f$ in Ω
- iii) $\frac{\partial f}{\partial n}$ exists and is equal to 0 on $K \cap \overline{\Omega} \setminus \Sigma$

PROOF. Let φ be a C^∞ function with compact support in Ω and $f \in D(-\Delta_n)$. Then the Green formula in the distributional sense gives

$$\int_{\Omega} \nabla f \cdot \nabla \varphi = \langle -\Delta_S f, \varphi \rangle$$

where the left and right brackets mean the duality in the distributional sense. On the other hand, by definition of $-\Delta_n$ and since f is in the domain $D(-\Delta_n)$, we also have

$$\int_{\Omega} \nabla f \cdot \nabla \varphi = \langle -\Delta_n f, \varphi \rangle$$

where this time the brackets mean the scalar product in L^2 . Therefore

$$\Delta_n f = \Delta_S f \quad \text{in } \mathcal{D}'(\Omega).$$

In other words, $-\Delta_S f = \lambda f$ in $\mathcal{D}'(\Omega)$. But now since $f \in W^{1,2}(\Omega)$, by hypoellipticity of the Laplacian we know that f is C^∞ and that $-\Delta_S f = \lambda f$ in the classical sense. That proves *i)* and *ii)*. We even know by the elliptic theory that, since $K \setminus \Sigma$ is regular, f is regular at the boundary on $K \setminus \Sigma$.

Now consider a ball B such that the intersection with $K \cap \overline{\Omega}$ does not meet Σ . Assume that B is cut in two parts B^+ and B^- by K , and that B^+ is one part in Ω . Possibly by modifying B in a neighborhood of the intersection with K , we can assume that the boundary of B^+ and B^- is C^2 . The definition of Δ_n implies that for all function $\varphi \in C^2(\overline{\Omega})$ that vanishes out of B^+ we have

$$\int_{B^+} \langle \nabla f, \nabla \varphi \rangle dx = \int_{B^+} \langle -\Delta_n f, \varphi \rangle dx = \lambda \int_{B^+} \langle f, \varphi \rangle dx.$$

On the other hand, integrating by parts,

$$\begin{aligned} \int_{B^+} \langle \nabla f, \nabla \varphi \rangle dx &= \int_{B^+} \langle -\Delta_S f, \varphi \rangle + \int_{\partial B^+} \frac{\partial u}{\partial n} \varphi \\ &= \lambda \int_{\partial B^+} \langle f, \varphi \rangle + \int_{\partial B^+} \frac{\partial f}{\partial n} \varphi \end{aligned}$$

thus

$$\int_{\partial B^+} \frac{\partial f}{\partial n} \varphi = 0.$$

In other words the function f is a weak solution of the mixed boundary value problem

$$\begin{aligned} -\Delta_S u &= \lambda f \text{ in } B^+ \\ u &= f \text{ on } \partial B^+ \setminus K \\ \frac{\partial u}{\partial n} &= 0 \text{ on } K \cap \partial B^+ \end{aligned}$$

Therefore, some results from the elliptic theory imply that f is smooth in B and is a strong solution (see [21]). \square

Let us recapitulate what we have obtained. For all function $f \in L^2(\Omega)$, there is a sequence of numbers α_i such that

$$(4) \quad f = \sum_{i=0}^{+\infty} \alpha_i f_i$$

where the sum converges in L^2 . The functions f_i are in $C^\infty(\Omega) \cap W^{1,2}(\Omega)$, verify $-\Delta_S f_i = \lambda_i f_i$ and $\frac{\partial f_i}{\partial n} = 0$ on $K \cap \overline{\Omega} \setminus \Sigma$. Moreover, we can normalize the f_i in order to obtain an orthonormal basis on $L^2(\Omega)$, in particular we have the following Parseval formula

$$\|f\|_2^2 = \sum_{i=0}^{+\infty} |\alpha_i|^2.$$

Note that if f belongs to the kernel of $-\Delta_n$ (i.e. is an eigenfunction with eigenvalue 0), then

$$\langle \nabla f, \nabla f \rangle = \langle -\Delta_n f, f \rangle = 0$$

and since Ω is connected that means that f is a constant. Thus 0 is the first eigenvalue and the associated eigenspace has dimension 1. Then we can suppose that $\lambda_0 = 0$ and that all the λ_i for $i > 0$ are positive.

We define the scalar product in $W^{1,2}(\Omega)$ by

$$\langle u, v \rangle_{W^{1,2}} := \langle u, v \rangle_{L^2} + \langle \nabla u, \nabla v \rangle_{L^2}.$$

PROPOSITION 13. *The family $\{f_i\}$ is orthogonal in $W^{1,2}(\Omega)$. Moreover if $f \in W^{1,2}(\Omega)$ and if its decomposition in $L^2(\Omega)$ is $f = \sum_{i=0}^{+\infty} a_i f_i$, then the sum $\sum_{i=0}^{+\infty} |a_i|^2 \|\nabla f_i\|_2^2$ converges and*

$$(5) \quad \sum_{i=0}^{+\infty} |a_i|^2 \|\nabla f_i\|_2^2 = \|\nabla f\|_2^2.$$

PROOF. We know that $\{f_i\}$ is an orthogonal family in $L^2(\Omega)$. In addition if $i \neq j$ then

$$\begin{aligned} \int_{\Omega} \nabla f_i \nabla f_j &= \int_{\Omega} -\Delta_n f_i f_j \\ &= \lambda_i \int_{\Omega} f_i f_j \\ &= 0 \end{aligned}$$

thus $\{f_i\}$ is also orthogonal in $W^{1,2}(\Omega)$ and

$$\|f_i\|_{W^{1,2}}^2 := \|f_i\|_2^2 + \|\nabla f_i\|_2^2 = 1 + \lambda_i.$$

Consider now the orthogonal projection (for the scalar product of L^2)

$$P_k : f \mapsto \sum_{i=0}^k a_i f_i.$$

The operator P_k is the orthogonal projection on the closed subspace A_k generated by $\{f_0, \dots, f_k\}$. More precisely, we are interested in the restriction of P_k to the subspace $W^{1,2}(\Omega) \subset L^2(\Omega)$. Also denote by $\tilde{P}_k : W^{1,2} \rightarrow A_k$ the orthogonal projection on the same subspace but for the scalar product of $W^{1,2}$. We want to show that $P_k = \tilde{P}_k$. To prove this, it suffice to show that for all sets of coefficients $\{a_i\}_{i=1\dots k}$ and $\{b_i\}_{i=1\dots k}$,

$$\left\langle f - \sum_{i=0}^k a_i f_i, \sum_{i=0}^k b_i f_i \right\rangle_{W^{1,2}} = 0.$$

Since we already have

$$\left\langle f - \sum_{i=0}^k a_i f_i, \sum_{i=0}^k b_i f_i \right\rangle_{L^2} = 0,$$

all we have to show is that

$$\int_{\Omega} \left\langle \nabla f - \sum_{i=0}^k a_i \nabla f_i, \sum_{i=0}^k b_i \nabla f_i \right\rangle dx = 0.$$

Now

$$\begin{aligned} \int_{\Omega} \left\langle \nabla f - \sum_{i=0}^k a_i \nabla f_i, \sum_{i=0}^k b_i \nabla f_i \right\rangle &= \int_{\Omega} \left\langle \nabla f, \sum_{i=0}^k b_i \nabla f_i \right\rangle - \sum_{i=0}^k a_i b_i \|\nabla f_i\|_2^2 \\ &= \sum_{i=0}^k b_i \langle -\Delta_n f_i, f \rangle_{L^2} - \sum_{i=0}^k a_i b_i \lambda_i \\ &= \sum_{i=0}^k a_i b_i \lambda_i - \sum_{i=0}^k a_i b_i \lambda_i \\ &= 0 \end{aligned}$$

thus $P_k = \tilde{P}_k$ and therefore, by Pythagoras

$$\|P_k(f)\|_{W^{1,2}}^2 \leq \|f\|_{W^{1,2}}^2.$$

By letting k tend to infinity we obtain

$$(6) \quad \sum_{i=0}^{+\infty} a_i^2 \|\nabla f_i\|_2^2 \leq \|\nabla f\|_2^2.$$

From this inequality we deduce that the sum is absolutely converging in $W^{1,2}(\Omega)$. Therefore, the sequence of partial sum $\sum_{i=0}^K a_i f_i$ is a Cauchy sequence for the norm $W^{1,2}(\Omega)$. Thus, since the sum $\sum a_i f_i$ already converges to f in $L^2(\Omega)$, by uniqueness of the limit the sum converges to f in $W^{1,2}(\Omega)$, so we deduce that (6) is an equality and the proof is over. \square

Once we have a basis $\{f_i\}$ on $\Omega \subset S^{N-1}$, we consider for a certain $r_0 > 0$, the functions

$$h_i(x) = r_0^{z_i} f_i\left(\frac{x}{r_0}\right)$$

defined on $r_0\Omega$. The exponent α_i is defined by

$$(7) \quad \alpha_i = \frac{-(N-2) + \sqrt{(N-2)^2 + 4\lambda_i}}{2}.$$

The functions h_i form a basis of $W^{1,2}(r_0\Omega)$. Indeed, if $f \in W^{1,2}(r_0\Omega)$, then $f(r_0x) \in W^{1,2}(\Omega)$ thus applying the decomposition on Ω we obtain

$$f(r_0x) = \sum_{i=0}^{+\infty} b_i f_i(x)$$

thus

$$f(x) = \sum_{i=0}^{+\infty} a_i h_i(x)$$

with

$$(8) \quad a_i = b_i r_0^{-\alpha_i}.$$

Notice that since $\|h_i\|_2^2 = r_0^{2\alpha_i + N - 1}$ we also have

$$(9) \quad \sum_{i=0}^{\infty} a_i^2 \|h_i\|_2^2 = \sum_{i=0}^{\infty} a_i^2 r_0^{2\alpha_i + N - 1} = \|f\|_{L^2(r_0\Omega)}^2 < +\infty.$$

Moreover, applying Proposition 13 we have that

$$(10) \quad \sum_{i=0}^{\infty} b_i^2 \|\nabla f_i\|_2^2 = \|\nabla f(r_0x)\|_2^2 < +\infty.$$

We are now able to state our decomposition in $A(r_0)$.

PROPOSITION 14. *Let K be a smooth cone in \mathbb{R}^N , centered at the origin and let Ω be a connected component of $S^{N-1} \setminus K$. Then there exist some harmonic homogeneous functions g_i , orthogonal in $W^{1,2}(A(1))$, such that for every function $u \in W^{1,2}(A(1))$ harmonic in $A(1)$ with $\frac{\partial u}{\partial n} = 0$ on $K \cap A(1) \setminus \Sigma$, and for every $r_0 \in]0, 1[$, we have that*

$$u = \sum_{i=0}^{+\infty} a_i g_i \quad \text{in } A(r_0)$$

where the a_i do not depend on radius r_0 and are unique. The sum converges in $W^{1,2}(A(r_0))$ and uniformly on all compact sets of $A(1)$. Moreover

$$(11) \quad \|u\|_{W^{1,2}(A(r_0))}^2 = \sum_{i=0}^{+\infty} a_i^2 \|g_i\|_{W^{1,2}(A(r_0))}^2.$$

PROOF. Since $u \in W^{1,2}(A(1))$ then for almost every r_0 in $]0, 1]$ we have that

$$u|_{r_0\Omega} \in W^{1,2}(r_0\Omega).$$

Thus we can apply the decomposition on $r_0\Omega$ and say that

$$u = \sum_{i=0}^{+\infty} a_i h_i \quad \text{on } r_0\Omega.$$

Define g_i by

$$g_i(x) := \|x\|^{\alpha_i} f_i \left(\frac{x}{\|x\|} \right)$$

where α_i is defined by (7). Since the f_i are eigenfunctions for $-\Delta_S$, we deduce from (3) that

$$\begin{aligned} \Delta g_i &= \frac{\partial^2}{\partial r^2} g_i + \frac{N-1}{r} \frac{\partial}{\partial r} g_i + \frac{1}{r^2} \Delta_S g_i \\ &= \alpha_i(\alpha_i - 1)r^{\alpha_i-2}f_i + \frac{N-1}{r} \alpha_i r^{\alpha_i-1}f_i - r^{\alpha_i-2} \lambda_i f_i \\ &= (\alpha_i^2 + (N-2)\alpha_i - \lambda_i)r^{\alpha_i-2}f_i \\ &= 0 \end{aligned}$$

by definition of α_i , thus the g_i are harmonic in $A(+\infty)$. Notice that the g_i are orthogonal in $L^2(A(1))$ because they are homogeneous and orthogonal in $L^2(\Omega)$. Note also that h_i is equal to g_i on $r_0\Omega$. Moreover for all $0 < r \leq 1$ we have

$$\begin{aligned} (12) \quad \|g_i\|_{L^2(A(r))}^2 &= \int_{A(r)} |g_i|^2 = \int_0^r \int_{\partial B(t) \cap A(1)} |g_i(w)|^2 dw dt \\ &= \int_0^r \int_{\Omega} t^{N-1} |g_i(ty)|^2 dy dt = \int_0^r t^{2\alpha_i+N-1} \int_{\Omega} |g_i(y)|^2 dy dt \\ &= \frac{r^{2\alpha_i+N}}{2\alpha_i+N} \|f_i\|_{L^2(\Omega)}^2 = \frac{r^{2\alpha_i+N}}{2\alpha_i+N} \leq 1. \end{aligned}$$

On the other hand, since the f_i and their tangential gradients are orthogonal in $L^2(\Omega)$, we deduce that the gradients of g_i are orthogonal in $A(1)$.

Then, by a computation similar to (12) we obtain for all $0 < r \leq 1$

$$\begin{aligned}
 (13) \quad \|\nabla g_i\|_{L^2(A(r))}^2 &= \int_0^r \int_{\partial B(t) \cap A(1)} \left| \frac{\partial g_i}{\partial r} \right|^2 + |\nabla_\tau g_i|^2 \, dwdt \\
 &= \int_0^r \int_{\partial B(t) \cap A(1)} \left| \alpha_i t^{\alpha_i - 1} f_i \left(\frac{w}{t} \right) \right|^2 + \left| t^{\alpha_i} \nabla_\tau f_i \left(\frac{w}{t} \right) \frac{1}{t} \right|^2 \, dwdt \\
 &= \alpha_i^2 \int_0^r t^{2(\alpha_i - 1)} \int_{\partial B(t) \cap A(1)} \left| f_i \left(\frac{w}{t} \right) \right|^2 \, dwdt \\
 &\quad + \int_0^r t^{2(\alpha_i - 1)} \int_{\partial B(t) \cap A(1)} \left| \nabla_\tau f_i \left(\frac{w}{t} \right) \right|^2 \, dwdt \\
 &= \alpha_i^2 \int_0^r t^{2(\alpha_i - 1)} \int_\Omega |f_i(w)|^2 t^{N-1} \, dwdt \\
 &\quad + \int_0^r t^{2(\alpha_i - 1)} \int_\Omega |\nabla_\tau f_i(w)|^2 t^{N-1} \, dwdt \\
 &= \alpha_i^2 \frac{\gamma^{2(\alpha_i - 1) + N}}{2(\alpha_i - 1) + N} \|f_i\|_{L^2(\Omega)}^2 + \frac{\gamma^{2(\alpha_i - 1) + N}}{2(\alpha_i - 1) + N} \|\nabla_\tau f_i\|_{L^2(\Omega)}^2 \\
 &= \frac{\gamma^{2(\alpha_i - 1) + N}}{2(\alpha_i - 1) + N} (\alpha_i^2 + \lambda_i) \|f_i\|_{L^2(\Omega)}^2 \\
 &\leq C \gamma^{2\alpha_i} (\alpha_i^2 + \lambda_i)
 \end{aligned}$$

because $\|\nabla_\tau f_i\|_2^2 = \lambda_i \|f_i\|_2^2$, $r \leq 1$ and $\alpha_i \geq 0$. Moreover the constant C depends on the dimension N but does not depend on i .

We denote by g the function defined in $A(\infty)$ by

$$g := \sum_{i=0}^{+\infty} a_i g_i.$$

Then g lies in $L^2(A(r_0))$ because using (12) and (9)

$$\|g\|_{L^2(A(r_0))}^2 = \sum_{i=0}^{+\infty} |a_i|^2 \|g_i\|_{L^2(A(r_0))}^2 \leq \sum_{i=0}^{+\infty} |a_i|^2 \gamma_0^{2\alpha_i + N} < +\infty.$$

We want now to show that $g = u$.

• *First step:* We claim that g is harmonic in $A(r_0)$. Indeed, since the g_i are all harmonic in $A(r_0)$, the sequence of partial sums $s_k := \sum_{i=0}^k a_i g_i$ is a sequence of harmonic functions, uniformly bounded for the L^2 norm in each compact set of $A(r_0)$. By the Harnack inequality we deduce that the sequence of partial sums is uniformly bounded for the uniform norm in each compact set. Thus there is a subsequence that converges uniformly to a harmonic function, which in fact is equal to g by uniqueness of the limit. Therefore, g is harmonic in $A(r_0)$.

• *Second step:* We claim that g belongs to $W^{1,2}(A(r_0))$. Firstly, since $u \in W^{1,2}(r_0\Omega)$, by (8) and (10) we have that

$$(14) \quad \sum_{i=0}^{+\infty} a_i^2 r_0^{2\alpha_i} \|\nabla_{\tau} f_i\|_{L^2(\partial B(0,1)\setminus K)}^2 < +\infty.$$

In addition, since $\|\nabla_{\tau} f_i\|_2^2 = \lambda_i \|f_i\|_2^2$ and $\|f_i\|_2 = 1$, we deduce

$$(15) \quad \sum_{i=0}^{+\infty} a_i^2 r_0^{2\alpha_i} \lambda_i < +\infty$$

and since α_i and λ_i are linked by the formula (7) we also have that

$$(16) \quad \sum_{i=0}^{+\infty} a_i^2 r_0^{2\alpha_i} \alpha_i^2 < +\infty.$$

Now, since $\sum a_i g_i$ converges absolutely on every compact set, we can say that

$$\nabla g = \sum_{i=0}^{+\infty} a_i \nabla g_i$$

thus using (13), (15), (16), and orthogonality,

$$\begin{aligned} \|\nabla g\|_{L^2(A(r_0))}^2 &= \sum_{i=0}^{+\infty} a_i^2 \|\nabla g_i\|_{L^2}^2 \\ &\leq C \sum_{i=0}^{+\infty} a_i^2 r_0^{2\alpha_i} (\alpha_i^2 + \lambda_i) < +\infty. \end{aligned}$$

Therefore, $g \in W^{1,2}(A(r_0))$.

• *Third step:* We claim that $\frac{\partial g}{\partial n} = 0$ on $K \cap \overline{A(r_0)} \setminus \Sigma$. We already know that $\frac{\partial g_i}{\partial n} = 0$ on $K \setminus \Sigma$ (because the f_i have this property). We want to show

that g is so regular that we can exchange the order of $\frac{\partial}{\partial n}$ and \sum . So let x_0 be a point of $K \cap \overline{A(r_0)} \setminus \Sigma$ and let B be a neighborhood of x_0 in \mathbb{R}^N that doesn't meet Σ and such that K separates B in two parts B^+ and B^- . Assume that B^+ is a part in $A(r_0)$. The sequence of partial sums $s_k := \sum_{i=0}^k a_i g_i$ is a sequence of harmonic functions in B^+ . Since $\partial B^+ \cap K$ is C^2 we can do a reflection to extend s_k in B^- . For all k , this new function s_k is the solution of a certain elliptic equation whose operator become from the composition of the Laplacian with the application that makes $\partial B^+ \cap K$ flat. Thus since $\sum a_i g_i$ converges absolutely for the L^2 norm, by the Harnack inequality $\sum a_i g_i$ converges absolutely for the uniform norm in a smaller neighborhood $B' \subset B$ that still contains x_0 . Thus s_k converges to a C^1 function denoted by s , which is equal to g on B^+ . And since $\frac{\partial s_k}{\partial n}(x_0) = 0$, by the absolute convergence of the sum we can exchange the order of the derivative and the symbol \sum so we deduce that $\frac{\partial s}{\partial n}(x_0) = 0$. Finally, since s is equal to g on B^+ we deduce that g is C^1 at the boundary and $\frac{\partial g}{\partial n} = 0$ at x_0 .

• *Fourth step:* we claim that g is equal to u on $r_0\Omega$. Let r be a radius such that $r < r_0$. Then the function $x \mapsto g_r(x) := g\left(r \frac{x}{r_0}\right)$ is well defined for $x \in r_0\Omega$, and since the g_i are homogeneous we have

$$g\left(r \frac{x}{r_0}\right) = \sum_{i=0}^{+\infty} a_i g_i\left(r \frac{x}{r_0}\right) = \sum_{i=0}^{+\infty} \left(\frac{r}{r_0}\right)^{\alpha_i} a_i g_i(x) = \sum_{i=0}^{+\infty} \left(\frac{r}{r_0}\right)^{\alpha_i} a_i h_i(x).$$

We deduce that the function $x \mapsto g\left(\frac{r}{r_0}x\right)$ is in $L^2(r_0\Omega)$ and its coefficients in the basis $\{h_i\}$ are $\left\{\left(\frac{r}{r_0}\right)^{\alpha_i} a_i\right\}$. We want to show that $\|g_r - u\|_{L^2(r_0\Omega)}$ tend to 0. Indeed, writing u in the basis $\{h_i\}$

$$u = \sum_{i=0}^{+\infty} a_i h_i,$$

we obtain

$$\|g_r - u\|_2^2 = \sum_{i=0}^{+\infty} \left(\left(\frac{r}{r_0}\right)^{\alpha_i} - 1\right)^2 a_i^2 \|h_i\|_2^2$$

which tends to zero when r tends to r_0 by the dominated convergence theorem because $\left(\left(\frac{r}{r_0}\right)^{\alpha_i} - 1\right)^2 \leq 1$. Therefore, there is a subsequence for which g_r tends to u almost everywhere. On the other hand, since g is harmonic, the limit of g_r exists and is equal to g . That means that g tends to u radially at almost every point of $r_0\Omega$.

• *Fifth step:* The functions u and g are harmonic functions in $A(r_0)$, with finite energy, with a normal derivative equal to zero on $K \cap \overline{A(r_0)} \setminus \Sigma$ and that coincide on $\partial A(r_0) \setminus K$. To show that $u = g$ in $A(r_0)$ we shall prove that g is an energy minimizer. Proposition 8 will then give the uniqueness.

Let $\varphi \in W^{1,2}(A(r_0) \setminus K)$ have a vanishing trace on $\partial B(0, r_0)$. Then, setting $J(v) := \int_{A(r_0)} |\nabla v|^2$ for $v \in W^{1,2}(A(r_0))$ we have

$$J(g + \varphi) = J(g) + \int_{A(r_0)} \nabla g \nabla \varphi + J(\varphi).$$

Now since g is harmonic with Neumann condition on $K \setminus \Sigma$ and since φ vanishes on $r_0\Omega$, integrating by parts we obtain

$$J(g + \varphi) = J(g) + J(\varphi).$$

Since J is non negative and $g + \varphi$ describes all the functions in $W^{1,2}(A(r_0))$ with trace equal to u on $r_0\Omega$, we deduce that g minimizes J . We can do the same with u thus u and g are two energy minimizers with same boundary conditions. Therefore, by Proposition 8 we know that $g = u$.

• *Sixth step:* The decomposition do not depends on r_0 . Indeed, let r_1 be a second choice of radius. Then we can do the same work as before to obtain a decomposition

$$u(x) := \sum_{i=0}^{+\infty} b_i g_i(x) \quad \text{in } B(0, r_1) \setminus K.$$

Now by uniqueness of the decomposition in $B(0, \min(r_0, r_1))$ we deduce that $b_i = a_i$ for all i .

In addition, r_0 was initially chosen almost everywhere in $]0, 1[$. But since the decomposition does not depend on the choice of radius, r_0 can be chosen anywhere in $]0, 1[$, by choosing a radius almost everywhere in $]r_0, 1[$. \square

THEOREM 15. *Let (u, K) be a global minimizer in \mathbb{R}^N such that K is a smooth cone. Then for each connected component of $\mathbb{R}^N \setminus K$ there is a constant u_k such that $u - u_k$ is $\frac{1}{2}$ -homogenous.*

PROOF. Let Ω be a connected component of $\mathbb{R}^N \setminus K$. We apply the preceding proposition to u . Thus

$$u(x) = \sum_{i=0}^{+\infty} a_i g_i(x) \quad \text{in } A(r_0).$$

for a certain radius r_0 chosen in $]0, 1[$. Let us prove that the same decomposition is true in $A(\infty)$. Applying Proposition 14 to the function $u_R(x) = u(Rx)$ we know that there are some coefficients $a_i(R)$ such that

$$u_R(x) = \sum_{i=0}^{+\infty} a_i(R) g_i(x) \quad \text{in } A(r_0).$$

Now since $u_R\left(\frac{x}{R}\right) = u(x)$ we can use the homogeneity of the g_i to identify the terms in $B(0, r_0)$ thus $a_i(R) = a_i R^{\alpha_i}$. Now we fix $y = Rx$ and we obtain that

$$u(y) = \sum_{i=0}^{+\infty} a_i g_i(y) \quad \text{in } A(Rr_0).$$

Since R is arbitrary the decomposition is true in $A(\infty)$.

In addition for every radius R we know that

$$(17) \quad \|\nabla u\|_{L^2(A(R))}^2 = \sum_{i=0}^{+\infty} a_i^2 \|\nabla g_i\|_{L^2(A(R))}^2$$

and since g_i is α_i -homogenous,

$$\|\nabla g_i\|_{L^2(A(R))}^2 = R^{2(\alpha_i-1)+N} \|\nabla g_i\|_{L^2(A(1))}^2.$$

Now, since u is a global minimizer, a classical estimate on the gradient obtained by comparing (u, K) with (v, L) where $v = \mathbf{1}_{B(0,R)^c} u$ and $L = \partial B(0, R) \cup (K \setminus B(0, R))$ gives that there is a constant C such that for all radius R

$$\|\nabla u\|_{L^2(B(0,R) \setminus K)}^2 \leq CR^{N-1}.$$

We deduce

$$\sum_{i=0}^{+\infty} a_i^2 R^{2(\alpha_i-1)+N} \|\nabla g_i\|_{L^2(A(1))}^2 \leq CR^{N-1}.$$

Thus

$$\sum_{i=0}^{+\infty} a_i^2 R^{2\alpha_i-1} \|\nabla g_i\|_{L^2(A(1))}^2 \leq C.$$

This last quantity is bounded when R goes to infinity if and only if $a_i = 0$ whenever $\alpha_i > 1/2$. On the other hand, this quantity is bounded when R goes to 0, if and only if $a_i = 0$ whenever $0 < \alpha_i < 1/2$. Therefore, $u - a_0$ is a finite sum of terms of degree $\frac{1}{2}$. \square

REMARK 16. In Chapter 65 of [8], we can find a variational argument that leads to a formula in dimension 2 that links the radial and tangential derivatives of u . For all $\xi \in K \cap \partial B(0, r)$, we call $\theta_\xi \in [0, \frac{\pi}{2}]$ the non oriented angle between the tangent to K at point ξ and the radius $[0, \xi]$. Then we have the following formula

$$\int_{\partial B(0,r) \setminus K} \left(\frac{\partial u}{\partial r}\right)^2 dH^1 = \int_{\partial B(0,r) \setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2 dH^1 + \sum_{\xi \in K \cap \partial B(0,r)} \cos \theta_\xi - \frac{1}{r} H^1(K \cap B(0, r)).$$

Notice that for a global minimizer in \mathbb{R}^2 with K a centered cone we find

$$(18) \quad \int_{\partial B(0,r) \setminus K} \left(\frac{\partial u}{\partial r}\right)^2 dH^1 = \int_{\partial B(0,r) \setminus K} \left(\frac{\partial u}{\partial \tau}\right)^2 dH^1.$$

Now suppose that (u, K) is a global minimizer in \mathbb{R}^N with K a smooth cone centered at 0. Then by Theorem 15 we know that u is harmonic and $\frac{1}{2}$ -homogenous. Its restriction to the unit sphere is an eigenfunction for the spherical Laplacian with Neumann boundary condition and associated to the eigenvalue $\frac{2N - 3}{4}$. We deduce that

$$\|\nabla_\tau u\|_{L^2(\partial B(0,1))}^2 = \frac{2N - 3}{4} \|u\|_{L^2(\partial B(0,1))}^2.$$

On the other hand

$$\frac{\partial u}{\partial r}(x) = \frac{1}{2} \|x\|^{-\frac{1}{2}} u\left(\frac{x}{\|x\|}\right)$$

thus

$$\left\| \frac{\partial u}{\partial r} \right\|_{L^2(\partial B(0,1))}^2 = \frac{1}{4} \|u\|_{L^2(\partial B(0,1))}^2.$$

So

$$\|\nabla_\tau u\|_{L^2(\partial B(0,1))}^2 = (2N - 3) \left\| \frac{\partial u}{\partial r} \right\|_{L^2(\partial B(0,1))}^2.$$

In particular, for $N = 2$ we have the same formula as (18).

2. Some applications.

As it was claimed in the introduction, here is some few applications of Theorem 15.

PROPOSITION 17. *Let (u, K) be a global minimizer in \mathbb{R}^3 such that K is a smooth cone. Moreover, assume that $S^2 \cap K$ is a union of convex curvilinear polygons with C^∞ sides. Then u is locally constant and K is a cone of type \mathbb{P} , \mathbb{Y} or \mathbb{T} .*

PROOF. In each polygon we know by Proposition 4.5. of [6] that the smallest positive eigenvalue for the operator minus Laplacian with Neumann boundary conditions is greater than or equal to 1. Thus it cannot be $\frac{3}{4}$ and u is locally constant. Then K is a minimal cone in \mathbb{R}^3 and we know from [9] that it is a cone of type \mathbb{P} , \mathbb{Y} or \mathbb{T} . □

Let $(r, \theta, z) \in \mathbb{R}^+ \times [-\pi, \pi] \times \mathbb{R}$ be the cylindrical coordinates in \mathbb{R}^3 . For every $\omega \in [0, \pi]$ set

$$\Gamma_\omega := \{(r, \theta, z) \in \mathbb{R}^3; -\omega < \theta < \omega\}$$

of boundary

$$\partial\Gamma_\omega := \{(r, \theta, z) \in \mathbb{R}^3; \theta = -\omega \text{ or } \theta = \omega\}.$$

Consider $\Omega_\omega = \Gamma_\omega \cap S^2$ and let λ_1 be the smallest positive eigenvalue of $-\Delta_S$ in Ω_ω with Neumann conditions on $\partial\Omega_\omega$. Then by Lemma 4.1. of [6] we have that

$$\lambda_1 = \min(2, \lambda_\omega)$$

where

$$\lambda_\omega = \left(\frac{\pi}{2\omega} + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

In particular for the cone of type \mathbb{Y} , $\omega = \frac{\pi}{3}$ thus $\lambda_1 = 2$.

Observe that for $\omega \neq \pi$, $\lambda_\omega \neq \frac{3}{4}$. So we get this following proposition.

PROPOSITION 18. *There is no global Mumford-Shah minimizer in \mathbb{R}^3 such that K is wing of type $\partial\Gamma_\omega$ with $\omega \notin \left\{0, \frac{\pi}{2}, \pi\right\}$.*

Another consequence of Theorem 15 is the following. Let P be the half plane

$$P := \{(r, \theta, z) \in \mathbb{R}^3; \theta = \pi\}.$$

PROPOSITION 19. *Let (u, K) be a global Mumford-Shah minimizer in \mathbb{R}^3 such that $K = P$. Then u is equal to $\text{cracktip} \times \mathbb{R}$, that is in cylindrical coordinates*

$$u(r, \theta, z) = \pm \sqrt{\frac{2}{\pi}} r^{\frac{1}{2}} \sin \frac{\theta}{2} + C$$

for $0 < r < +\infty$ and $-\pi < \theta < \pi$.

REMARK 20. In Section 3 we will give a second proof of Proposition 19.

REMARK 21. We already know that $u = \text{cracktip} \times \mathbb{R}$ is a global minimizer in \mathbb{R}^3 (see [8]).

To prove Proposition 19 we will use the following well known result.

PROPOSITION 22 ([5], [13]). *The smallest positive eigenvalue for $-\Delta_n$ in $S^2 \setminus P$ is $\frac{3}{4}$, the corresponding eigenspace is of dimension 1 generated by the restriction on S^2 of the following function in cylindrical coordinates*

$$u(r, \theta, z) = r^{\frac{1}{2}} \sin \frac{\theta}{2}$$

for $0 < r < +\infty$ and $-\pi < \theta < \pi$.

Now the proof of Proposition 19 can be easily deduce from Proposition 22 and Theorem 15.

PROOF OF PROPOSITION 19. If (u, P) is a global minimizer, we know that after removing a constant the restriction of u to the unit sphere is an eigenfunction for $-\Delta_n$ in $S^2 \setminus P$ associated to the eigenvalue $\frac{3}{4}$. Therefore, from Proposition 22 we know that

$$u(r, \theta, z) = Cr^{\frac{1}{2}} \sin \frac{\theta}{2}$$

so we just have to determinate the constant C . But by a well known argument about Mumford-Shah minimizers we prove that C must be equal to

$\pm \sqrt{\frac{2}{\pi}}$ (see [8] Section 61 for more details). □

Now set

$$S_\omega := \{(r, \theta, 0); r > 0, \theta \in [-\omega, \omega]\}$$

PROPOSITION 23. *There is no global Mumford-Shah minimizer in \mathbb{R}^3 such that K is an angular sector of type (u, S_ω) for $0 < \omega < \frac{\pi}{2}$ or $\frac{\pi}{2} < \omega < \pi$.*

PROOF. According to Theorem 15, if (u, S_ω) is a global minimizer, then $u - u_0$ is a homogenous harmonic function of degree $\frac{1}{2}$, thus its restriction to $S^2 \setminus S_\omega$ is an eigenfunction for $-\Delta_n$ associated to the eigenvalue $\frac{3}{4}$. Now if $\lambda(\omega)$ denotes the smallest eigenvalue on $\partial B(0, 1) \setminus S_\omega$, we know by Theorem 2.3.2. p. 47 of [14] that $\lambda(\omega)$ is non decreasing with respect to ω . Since $\lambda\left(\frac{\pi}{2}\right) = \frac{3}{4}$, we deduce that for $\omega < \frac{\pi}{2}$, we have

$$(19) \quad \lambda(\omega) \geq \frac{3}{4}.$$

In [14] page 53 we can find the following asymptotic formula near $\omega = \frac{\pi}{2}$

$$(20) \quad \lambda(\omega) = \frac{3}{4} + \frac{2}{\pi} \cos \omega + O(\cos^2 \omega).$$

this proves that the case when (19) is a equality only arises when $\omega = \frac{\pi}{2}$. Thus such eigenfunction u doesn't exist.

Consider now the case $\omega > \frac{\pi}{2}$. For $\omega = \pi$ there are tow connected components. Thus 0 is an eigenvalue of multiplicity 2. The second eigenvalue is equal to 2. Therefore, for $\omega = \pi$ the spectrum is

$$0 \leq 0 \leq 2 \leq \lambda_3 \leq \dots \quad \omega = \pi$$

By monotonicity, when ω decreases, the eigenvalues increase. Since the domain becomes connexe, 0 become of multiplicity 1 thus the second eigenvalue become positive. The spectrum is now

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \quad \omega < \pi$$

with $\lambda_i \geq 2$ for $i \geq 2$. Thus the only eigenvalue that could be equal to $\frac{3}{4}$ is λ_2 which is increasing from from 0 to $\frac{3}{4}$, reached for $\omega = \frac{\pi}{2}$. Now (20) says that the increasing is strict near $\omega = \frac{\pi}{2}$. Therefore there is no eigenvalue equal to $\frac{3}{4}$ for $\omega > \frac{\pi}{2}$ and there is no possible global minimizer. \square

3. Second proof of Propositions 19 and 22.

Here we want to give a second proof of Proposition 19, without using Theorem 15, and which do not use Proposition 22. In a remark at the end of this section, we will briefly explain how to use this proof of Proposition 19 in order to obtain a new proof of Proposition 22 as well.

Let assume that K is a half plane in \mathbb{R}^3 . We can suppose for instance that

$$(21) \quad K = P := \{x_2 = 0\} \cap \{x_1 \leq 0\}$$

We begin by studying the harmonic measure in $\mathbb{R}^3 \setminus P$.

Let $B := B(0, R)$ be a ball of radius R and let γ be the trace operator on $\partial B(0, R) \setminus P$. We denote by T the image of $W^{1,2}(B \setminus K)$ by γ . We also denote by $C_b^0(\partial B \setminus K)$ the set of continuous and bounded functions on $\partial B(0, 1) \setminus P$. Finally set $A := T \cap C_b^0$. Obviously A is not empty. To every function $f \in A$, Proposition 15.6. of [8] associates a unique energy minimizing function $u \in W^{1,2}(B \setminus K)$ such that $\gamma(u) = f$ on $\partial B \setminus P$. Since u is harmonic we know that it is C^∞ in $B \setminus K$. Let $y \in B \setminus K$ be a fixed point and consider the linear form μ_y defined by

$$(22) \quad \begin{aligned} \mu_y : A &\rightarrow \mathbb{R} \\ f &\mapsto u(y). \end{aligned}$$

By the maximum principle for energy minimizers, we know that for all $f \in A$ we have

$$|\mu_y(f)| \leq \|f\|_\infty$$

thus μ_y is a continuous linear form on A for the norm $\|\cdot\|_\infty$. We identify μ_y with its representant in the dual space of A and we call it *harmonic measure*.

Moreover, the harmonic measure is positive. That is, if $f \in A$ is a non negative function, then (by the maximum principle) $\mu_y(f)$ is non negative. By positivity of μ_y , if $f \in A$ is a non negative function and $g \in A$ is such that $fg \in A$, then since $(\|g\|_\infty + g)f$ and $(\|g\|_\infty - g)f$ are two non negative functions of A we deduce that

$$(23) \quad |\langle fg, \mu_y \rangle| \leq \|g\|_\infty \langle f, \mu_y \rangle.$$

Now here is an estimate on the measure μ_y^R .

LEMMA 24. *There is a dimensional constant C_N such that the following holds. Let R be a positive radius. For $0 < \lambda < \frac{R}{2}$ consider the spherical domain*

$$C_\lambda := \{x \in \mathbb{R}^3 ; |x| = R \text{ and } d(x, P) \leq \lambda\}.$$

Let $\varphi_\lambda \in C^\infty(\partial B(0, R))$ be a function between 0 and 1, that is equal to 1 on C_λ and 0 on $\partial B(0, R) \setminus C_{2\lambda}$ and that is symmetrical with respect to P . Then for every $y \in B\left(0, \frac{R}{2}\right) \setminus P$ we have

$$\mu_y^R(\varphi_\lambda) \leq C \frac{\lambda}{R}.$$

PROOF. Since φ_λ is continuous and symmetrical with respect to P , by the reflection principle, its harmonic extension φ in $B(0, R)$ has a normal derivative equal to zero on P in the interior of $B(0, R)$. Moreover φ_λ is clearly in the space A . Thus by definition of μ_y ,

$$\varphi(y) = \langle \varphi_\lambda, \mu_y^R \rangle.$$

On the other hand, since φ_λ is continuous on the entire sphere, we also have the formula with the classical Poisson kernel

$$\varphi(y) = \frac{R^2 - |y|^2}{N\omega_N R} \int_{\partial B_R} \frac{\varphi_\lambda(x)}{|x - y|^3} ds(x)$$

with ω_N equal to the measure of the unit sphere. In other words

$$\mu_y^R(\varphi_\lambda) = \frac{R^2 - |y|^2}{N\omega_N R} \int_{\partial B_R} \frac{\varphi_\lambda(x)}{|x - y|^3} ds(x).$$

For $x \in \partial B_R$ we have

$$\frac{1}{2}R \leq |x| - |y| \leq |x - y| \leq |x| + |y| \leq \frac{3}{2}R.$$

We deduce that

$$\mu_y^R(\varphi_\lambda) \leq C_N \frac{1}{R^2} \int_{C_{2\lambda}} ds.$$

Now integrating by parts,

$$\begin{aligned} \int_{C_\lambda} ds &= 2 \int_0^\lambda 2\pi \sqrt{R^2 - w^2} dw \\ &= 4\pi \frac{\lambda}{2} \sqrt{R^2 - \lambda^2} + R^2 \arcsin\left(\frac{\lambda}{R}\right) \\ &\leq CR\lambda \end{aligned}$$

because $\arcsin(x) \leq \frac{\pi}{2}x$. The proposition follows. \square

Now we can prove the uniqueness of *cracktip* $\times \mathbb{R}$.

SECOND PROOF OF PROPOSITION 19. Let us show that u is vertically constant. Let t be a positive real. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ set $x_t := (x_1, x_2, x_3 + t)$. We also set

$$u_t(x) := u(x) - u(x_t).$$

Since u is a function associated to a global minimizer, and since K is regular, we know that for all $R > 0$, the restriction of u to the sphere $\partial B(0, R) \setminus K$ is continuous and bounded on $\partial B(0, R) \setminus K$ with finite limits on each sides of K . It is the same for u_t . Thus for all $x \in \mathbb{R}^3 \setminus P$ and for all $R > 2\|x\|$ we can write

$$u_t(x) := \langle u_t|_{\partial B(0,R) \setminus P}, \mu_x^R \rangle$$

where μ_x is the harmonic measure defined in (22). We want to prove that for $x \in \mathbb{R}^3 \setminus P$, $\langle u_t|_{\partial B(0,R) \setminus P}, \mu_x^R \rangle$ tends to 0 when R goes to infinity. This will prove that $u_t = 0$.

So let $x \in \mathbb{R}^3 \setminus P$ be fixed. We can suppose that $R > 100(\|x\| + t)$. Let C_λ and φ_λ be as in Lemma 24. Then write

$$u_t(x) = \langle u_t|_{\partial B(0,R) \setminus P} \varphi_\lambda, \mu_x^R \rangle + \langle u_t|_{\partial B(0,R) \setminus P} (1 - \varphi_\lambda), \mu_x^R \rangle.$$

Now by a standard estimate on Mumford-Shah minimizers (that comes from Campanato's Theorem, see [3] p. 371) we have for all $x \in \mathbb{R}^N \setminus P$,

$$|u_t(x)| \leq C\sqrt{t}.$$

Then, using Lemma 24 we obtain

$$|\langle u_t|_{\partial B(0,R) \setminus P} \varphi_\lambda, \mu_x^R \rangle| \leq C\sqrt{t} \frac{\lambda}{R}.$$

On the other hand, for the points y such that $d(y, P) \geq \lambda$, since $\tilde{u} : u(\cdot) - u(y)$ is harmonic in $B(y, d(y, P))$ we have, by a classical estimation on harmonic functions (see the introduction of [12])

$$|\nabla \tilde{u}(y)| \leq C \frac{1}{d(y, P)} \|\tilde{u}\|_{L^\infty(\partial B(y, \frac{1}{2}d(y, P)))}$$

Now using Campanato's Theorem again we know that

$$\|\tilde{u}\|_{L^\infty(\partial B(y, \frac{1}{2}d(y, P)))} \leq Cd(y, P)^{\frac{1}{2}}$$

thus

$$|\nabla u(y)| \leq C \frac{1}{d(y, P)^{\frac{1}{2}}}$$

and finally by the mean value theorem we deduce that for all the points y such that $d(y, P) \geq \lambda$,

$$|u_t(y)| \leq C \sup_{z \in [y, y_t]} |\nabla u(z)| \cdot |y - y_t| \leq t \frac{1}{\lambda^{\frac{3}{2}}}$$

Therefore,

$$|\langle u_t |_{\partial B(0, R) \setminus P} (1 - \varphi_\lambda), \mu_x^R \rangle| \leq Ct \frac{1}{\lambda^{\frac{3}{2}}}$$

So

$$|u_t(x)| \leq C\sqrt{t} \frac{\lambda}{R} + Ct \frac{1}{\lambda^{\frac{3}{2}}}$$

thus by setting $\lambda = R^{\frac{1}{2}}$ and by letting R go to $+\infty$ we deduce that $u_t(x) = 0$ thus $z \mapsto u(x, y, z)$ is constant.

Now we fix $z_0 = 0$ and we introduce $P_0 := P \cap \{z = 0\}$. We want to show that $(u(x, y, 0), P_0)$ is a global minimizer in \mathbb{R}^2 . Let $(v(x, y), \Gamma)$ be a competitor for $u(x, y, 0)$ in the 2-dimensional ball B of radius ρ . Let \mathcal{C} be the cylinder $\mathcal{C} := B \times [-R, R]$. Define \tilde{v} and $\tilde{\Gamma}$ in \mathbb{R}^3 by

$$\tilde{v}(x, y, z) = \begin{cases} v(x, y) & \text{if } (x, y, z) \in \mathcal{C} \\ u(x, y, z) & \text{if } (x, y, z) \notin \mathcal{C} \end{cases}$$

$$\tilde{\Gamma} := (\mathcal{C} \cap [\Gamma \times [-R, R]]) \cup (P \setminus \mathcal{C}) \cup (B \times \{\pm R\}).$$

It is a topological competitor because $\mathbb{R}^3 \setminus P$ is connected (thus P doesn't separate any points). Now finally let \tilde{B} be a ball that contains \mathcal{C} . Then $(\tilde{v}, \tilde{\Gamma})$

is a competitor for (u, P) in \tilde{B} . By minimality we have:

$$\int_{\tilde{B}} |\nabla u|^2 + H^2(P \cap \tilde{B}) \leq \int_{\tilde{B}} |\nabla \tilde{v}|^2 + H^2(\tilde{I} \cap \tilde{B}).$$

In the other hand u is equal to \tilde{v} in $\tilde{B} \setminus C$ and it is the same for Γ and \tilde{I} . We deduce

$$\int_C |\nabla u|^2 dx dy dz + H^2(P \cap C) \leq \int_C |\nabla \tilde{v}|^2 dx dy dz + H^2(\tilde{I} \cap C).$$

Now, since u and \tilde{v} are vertically constant, $\nabla_z u = \nabla_z \tilde{v} = 0$, and $\nabla_x u, \nabla_y u$ are also constant with respect to the variable z (as for \tilde{v}). Thus

$$2R \int_B |\nabla u(x, y, 0)|^2 dx dy + H^2(P \cap C) \leq 2R \int_B |\nabla v(x, y)|^2 dx dy + H^2(\tilde{I} \cap C).$$

To conclude we will use the following lemma.

LEMMA 25. *If Γ is rectifiable and contained in a plane Q then*

$$H^2(\Gamma \times [-R, R]) = 2RH^1(\Gamma).$$

PROOF. We will use the coarea formula (see Theorem 2.93 of [3]). We take $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ the orthogonal projection on the coordinate orthogonal to Q . By this way, if $E := \Gamma \times [-R, R]$, we have $E \cap f^{-1}(t) = \Gamma$ for all $t \in [-R, R]$. E is rectifiable (because Γ is by hypothesis). So we can apply the coarea formula. To do this we have to calculate the jacobian $c_k d^E f_x$. By construction, the approximate tangente plane in each point of E is orthogonal to Q . We deduce that if T_x is a tangent plane, then there is a basis of T_x (\vec{b}_1, \vec{b}_2) such that \vec{b}_1 is orthogonal to Q . Since the function f is the projection on \vec{b}_1 , and its derivative as well (because f is linear) we obtain that the matrix of $d^E f_x : T_x \rightarrow \mathbb{R}$ in the basis (\vec{b}_1, \vec{b}_2) is

$$d^E f_x = (1, 0)$$

thus

$$c_k d^E f_x = \sqrt{\det[(1, 0).^t(1, 0)]} = 1.$$

Therefore

$$H^2(E) = \int_{-R}^R H^1(\Gamma) = 2RH^1(\Gamma). \quad \square$$

Here we can suppose that Γ is rectifiable. Indeed, the definition of Mumford-Shah minimizers is equivalent if we only allow rectifiable competitors. This is because the jump set of a *SBV* function is rectifiable and in [11] it is proved that the relaxed functional on the *SBV* space has same minimizers.

So we have

$$2R \int_B |\nabla u(x, y, 0)|^2 dx dy + 2RH^1(P \cap B) \leq 2R \int_B |\nabla v(x, y)|^2 dx dy + 2RH^1(\Gamma \cap B) + H^2(B \times \{\pm R\}).$$

Then, dividing by $2R$,

$$\int_B |\nabla u(x, y, 0)|^2 dx dy + H^1(P \cap B) \leq \int_B |\nabla v(x, y)|^2 dx dy + H^1(\Gamma \cap B) + \pi \frac{\rho^2}{R}$$

thus, letting R go to infinity,

$$\int_B |\nabla u(x, y, 0)|^2 dx dy + H^1(P \cap B) \leq \int_B |\nabla v(x, y)|^2 dx dy + H^1(\Gamma \cap B).$$

This last inequality proves that $(u(x, y, 0), P_0)$ is a global minimizer in \mathbb{R}^2 , and since P_0 is a half-line, u is a *cracktip*. \square

REMARK 26. Using a similar argument as the preceding proof, we can show that the first eigenvalue for $-\Delta$ in $S^2 \setminus P$ with Neumann boundary conditions (where P is still a half-plane), is equal to $\frac{3}{4}$. Moreover we can prove that the eigenspace is of dimension 1, generated by a function of type *cracktip* $\times \mathbb{R}$, thus we have a new proof of Proposition 22. The argument is to take an eigenfunction f in $S^2 \setminus P$, then to consider $u(x) := \|x\|^\alpha f\left(\frac{x}{\|x\|}\right)$ with a good coefficient $\alpha \in \left]0, \frac{1}{2}\right]$ that makes u harmonic. Finally we use the same sort of estimates on the harmonic measure to prove that u is vertically constant. Thus we have reduced the problem in dimension 2 and we conclude using that we know the eigenfunctions on the circle. A detailed proof is done in [15].

4. Open questions.

As it is said in the introduction, this paper is a very short step in the discovering of all the global minimizers in \mathbb{R}^N . This final goal seems rather far but nevertheless some open questions might be accessible in a more reasonable time. All the following questions were pointed out by Guy David in [8], and unfortunately they are still open after this paper.

- Suppose that (u, K) is a global minimizer in \mathbb{R}^N . Is it true that K is conical?
- Suppose that (u, K) is a global minimizer in \mathbb{R}^N , and K is a cone. Is it true that $\frac{3-2N}{4}$ is the smallest eigenvalue of the Laplacian on $S^{N-1} \setminus K$?
- Suppose that (u, K) is a global minimizer in \mathbb{R}^3 , and suppose that K is contained in a plan (and not empty). Is it true that K is a plane or a half-plane?
- Could one find an extra global minimizer in \mathbb{R}^3 by blowing up the minimizer described in section 76.c. of [8] (see also [17])?

One can find other open questions on global minimizers in the last page of [8].

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