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Sectional Invariants of Scrolls Over a Smooth Projective Variety

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ABSTRACT - Let X be a smooth complex projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Then we calculate the ith sectional Euler number $e_i(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ and the ith sectional Betti number $b_i(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ for $i \geq 2n-3$ or $i = 1$, and the ith sectional Hodge number of type $(j, i-j)$ $h_i^{j,i-j}(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ for $i \geq 2n-1$ and $0 \leq j \leq i$, where $\mathbb{P}_X(\mathcal{E})$ is the projective space bundle associated with $\mathcal{E}$ and $H(\mathcal{E})$ is its tautological line bundle. Moreover we define a new invariant $v(X, \mathcal{E})$ of $(X, \mathcal{E})$ for $r \geq n-1$. This invariant is thought to be a generalization of curve genus. We will investigate some properties of this invariant.

1. Introduction.

Let X be a projective variety of dimension n defined over the field of complex numbers, and let L be an ample (resp. a nef and big) line bundle on X. Then (X, L) is called a *polarized* (*resp. quasi-polarized*) *variety*. If X is smooth, then we say that (X, L) is a polarized (resp. quasi-polarized) *manifold*. In order to study polarized varieties, it is important to use an invariant of (X, L). There are the following three invariants of (X, L) which are well-known.

- The degree L^n.
- The sectional genus $g(L)$.
- The Δ-genus $\Delta(L)$.

By using these invariants, many authors studied polarized varieties. In particular, P. Ionescu classified polarized manifolds by the degree under the assumption that L is very ample with $L^n \leq 8$ ([13], [14], and [15]), and

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T. Fujita classified polarized varieties by the Δ-genus and the sectional genus ([3]).

In [5], in order to study polarized varieties more deeply, the author introduced the notion of the ith sectional geometric genus $g_i(X,L)$ of (X,L) for every integer i with $0 \le i \le n$. This is a generalization of the degree and the sectional genus of (X,L). Namely $g_0(X,L) = L^n$ and $g_1(X,L) = g(L)$.

Here we recall the reason why this invariant is called the ith sectional geometric genus. Let (X,L) be a polarized manifold of dimension $n \ge 2$ with $\mathrm{Bs}|L| = \emptyset$, where $\mathrm{Bs}|L|$ is the base locus of the complete linear system $|L|$. Let i be an integer with $1 \le i \le n$. Let X_{n-i} be the transversal intersection of general $n-i$ members of $|L|$. In this case X_{n-i} is a smooth projective variety of dimension i. Then we can prove that $g_i(X,L) = h^i(\mathcal{O}_{X_{n-i}})$, that is, $g_i(X,L)$ is the geometric genus of X_{n-i}.

Hence we can expect that $g_i(X,L)$ has analogous properties of the geometric genus of i-dimensional varieties.

In [6] and [7], we introduced the notion of the *ith sectional H-arithmetic genus* $\chi_i^H(X,L)$ *of* (X,L). By definition we can prove that if $\mathrm{Bs}|L| = \emptyset$, then $\chi_i^H(X,L) = \chi(\mathcal{O}_{X_{n-i}})$, where X_{n-i} is the transversal intersection of general $n-i$ members of $|L|$. Namely $\chi_i^H(X,L)$ is the Euler-Poincaré characteristic of the structure sheaf of X_{n-i}. ($\chi(\mathcal{O}_{X_{n-i}})$ is called the arithmetic genus of X_{n-i} in the sense of Hirzebruch. (See [12, 15.5, Section 15, Chapter IV]. We also call $\chi(\mathcal{O}_{X_{n-i}})$ the H-arithmetic genus of X_{n-i}.)

In [8], we also introduced some ith sectional invariants of (X,L), that is, the ith sectional Euler number $e_i(X,L)$, the ith sectional Betti number $b_i(X,L)$ and the ith sectional Hodge number $h_i^{j,i-j}(X,L)$ of type $(j, i-j)$ for every integer j with $0 \le j \le i$, and we investigated some properties of these. In particular we proved that polarized manifolds' version of the Hodge duality and the Hodge decomposition hold (see [8, Theorem 3.1]).

In this paper we consider the ith sectional Euler number and the ith sectional Betti number of scrolls over a smooth projective variety. In this paper we say that a polarized manifold (P,H) is a *scroll over a smooth projective variety* X if there exists an ample vector bundle $\mathcal{E}$ on X such that $(P,H) \cong (\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$, where $H(\mathcal{E})$ is the tautological line bundle on $\mathbb{P}_X(\mathcal{E})$.

In section 3, we calculate $e_i(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ and $b_i(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ for $i \ge 2n-3$ and $i = 1$ (see Theorems 3.1 and 3.2). We also calculate $h_i^{j,i-j}(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ for $i \ge 2n-1$ and $0 \le j \le i$ (see Theorem 3.3). In particular, by using these results, we can calculate these invariants of $(\mathbb{P}_X(\mathcal{E}), H(\mathcal{E}))$ completely for $n = 1$ or 2 (see Corollaries 3.1, 3.3 and 3.4).

In section 4, we will define a new invariant of generalized polarized manifolds. Here a *generalized polarized manifold* means the pair $(X, \mathcal{E})$ where X is a smooth projective variety and $\mathcal{E}$ is an ample vector bundle on X. Let $r := \mathrm{rank}(\mathcal{E})$. Here we state the history of invariants of $(X, \mathcal{E})$. First in [2], Fujita introduced the c_1-sectional genus and the $\mathcal{O}(1)$-sectional genus of $(X, \mathcal{E})$. Next, in [1], for the case where $r = n-1$, Ballico defined an invariant of $(X, \mathcal{E})$ which is called the *curve genus* $cg(X, \mathcal{E})$ *of* $(X, \mathcal{E})$ (see Definition 2.3), and several authors (in particular Lanteri, Maeda, Sommese, and so on) studied this invariant.

As a generalization of the curve genus, for any ample vector bundle $\mathcal{E}$ with $r \leq n-1$, Ishihara ([16, Definition 1.1]) defined an invariant $g(X, \mathcal{E})$, which is called the *c_r-sectional genus of* $(X, \mathcal{E})$, and in [9] we investigated some properties about $g(X, \mathcal{E})$. We note that if $n - r = 1$, then $g(X, \mathcal{E})$ is the curve genus. This invariant means the following: If a general element of $H^0(\mathcal{E})$ has a zero locus Z which is smooth of expected dimension $n - r$, then $g(X, \mathcal{E}) = g(Z, \det\mathcal{E}|_Z)$, that is, $g(X, \mathcal{E})$ is the sectional genus of $(Z, \det\mathcal{E}|_Z)$. Recently Fusi and Lanteri generalized this invariant. See [11] for detail.

In this paper, we will introduce a new invariant $v(X, \mathcal{E})$ of generalized polarized manifolds $(X, \mathcal{E})$ with $r \geq n-1$, which is defined by using a result in section 3 (see Definition 4.1). Here we note that $v(X, \mathcal{E})$ is equal to the curve genus if $r = n-1$. We will investigate $v(X, \mathcal{E})$ and give some results about this invariant (see Theorems 4.1 and 4.2, and Proposition 4.1).

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Notation and Conventions.

We say that X is a *variety* if X is an integral separated scheme of finite type. In particular X is irreducible and reduced if X is a variety. Varieties are always assumed to be defined over the field of complex numbers. In this article, we shall study mainly a smooth projective variety. The words "line bundles" and "Cartier divisors" are used interchangeably. The tensor products of line bundles are denoted additively.

$\mathcal{O}(D)$: invertible sheaf associated with a Cartier divisor D on X.
$\mathcal{O}_X$: the structure sheaf of X.
$\chi(\mathcal{F})$: the Euler-Poincaré characteristic of a coherent sheaf $\mathcal{F}$.
$h^i(\mathcal{F}) := \dim H^i(X, \mathcal{F})$ for a coherent sheaf $\mathcal{F}$ on X.
$h^i(D) := h^i(\mathcal{O}(D))$ for a Cartier divisor D.
$q(X)(= h^1(\mathcal{O}_X))$: the irregularity of X.

$b_i(X) := \dim H^i(X, \mathbb{C})$.
K_X: the canonical divisor of X.
$\mathbb{P}^n$: the projective space of dimension n.
$\mathbb{Q}^n$: a smooth quadric hypersurface in $\mathbb{P}^{n+1}$.
$\sim$ (or $=$): linear equivalence.
$\det(\mathcal{E}) := \wedge^r \mathcal{E}$, where $\mathcal{E}$ is a vector bundle of rank r on X.
$\mathbb{P}_X(\mathcal{E})$: the projective space bundle associated with a vector bundle $\mathcal{E}$ on X.
$H(\mathcal{E})$: the tautological line bundle on $\mathbb{P}_X(\mathcal{E})$.
$\mathcal{E}^\vee$: the dual of a vector bundle $\mathcal{E}$.
$c_i(\mathcal{E})$: the i-th Chern class of a vector bundle $\mathcal{E}$.
$c_i(X) := c_i(\mathcal{T}_X)$, where $\mathcal{T}_X$ is the tangent bundle of a smooth projective variety X.
For a real number m and a non-negative integer n, let

$$[m]^n := \begin{cases} m(m+1)\cdots(m+n-1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

$$[m]_n := \begin{cases} m(m-1)\cdots(m-n+1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Then for n fixed, $[m]^n$ and $[m]_n$ are polynomials in m whose degree are n. For any non-negative integer n,

$$n! := \begin{cases} [n]_n & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Assume that m and n are integers with $n \geq 0$. Then we put

$$\binom{m}{n} := \frac{[m]_n}{n!}$$

We note that $\binom{m}{n} = 0$ if $0 \leq m < n$, and $\binom{m}{0} = 1$.

2. Preliminaries.

NOTATION 2.1. (1) Let Y be a smooth projective variety of dimension i, let $\mathcal{T}_Y$ be the tangent bundle of Y, and let $\Omega_Y (= \Omega_Y^1)$ be the dual bundle of $\mathcal{T}_Y$ and $\Omega_Y^j := \wedge^j \Omega_Y$. For every integer j with $0 \leq j \leq i$, we put

$$\begin{aligned} h_{i,j}(c_1(Y), \cdots, c_i(Y)) :=& \chi(\Omega_Y^j) \\ =& \int_Y \mathrm{ch}(\Omega_Y^j)\mathrm{Td}(\mathcal{T}_Y). \end{aligned}$$

(Here $\mathrm{ch}(\Omega_Y^j)$ (resp. $\mathrm{Td}(\mathcal{T}_Y)$) denotes the Chern character of Ω_Y^j (resp. the Todd class of $\mathcal{T}_Y$). See [10, Example 3.2.3 and Example 3.2.4].)

(2) Let (M,L) be a polarized manifold of dimension m. For every integers i and j with $0 \leq j \leq i \leq m$, we put

$$C_j^i(M,L) := \sum_{l=0}^{j} (-1)^l \binom{m-i+l-1}{l} c_{j-l}(M)L^l,$$

$$w_i^j(M,L) := h_{i,j}(C_1^i(M,L), \cdots, C_i^i(M,L))L^{n-i}.$$

(3) Let M be a smooth projective variety of dimension m. For every integers i and j with $0 \leq j \leq i \leq m$, we put

$$H_1(i,j) := \begin{cases} \sum_{s=0}^{i-j-1} (-1)^s h^s(\Omega_M^j) & \text{if } j \neq i, \\ 0 & \text{if } j = i, \end{cases}$$

$$H_2(i,j) := \begin{cases} \sum_{t=0}^{j-1} (-1)^{i-t} h^t(\Omega_M^{i-j}) & \text{if } j \neq 0, \\ 0 & \text{if } j = 0. \end{cases}$$

DEFINITION 2.1. (See [8, Definition 3.1].) Let (M,L) be a polarized manifold of dimension m, and let i and j be integers with $0 \leq i \leq m$ and $0 \leq j \leq i$.

(1) The *ith sectional Euler number $e_i(M,L)$ of (M,L)* is defined by the following:

$$e_i(M,L) := \sum_{l=0}^{i} (-1)^l \binom{m-i+l-1}{l} c_{i-l}(M)L^{m-i+l}.$$

(2) The *ith sectional Betti number $b_i(M,L)$ of (M,L)* is defined by the following:

$$b_i(M,L) := \begin{cases} e_0(M,L) & \text{if } i = 0, \\ (-1)^i \left(e_i(M,L) - \sum_{j=0}^{i-1} 2(-1)^j b_j(M) \right) & \text{if } 1 \leq i \leq m. \end{cases}$$

(3) The *ith sectional Hodge number $h_i^{j,i-j}(M,L)$ of type $(j,i-j)$ of*

(M,L) is defined by the following:

$$h_i^{j,i-j}(M,L) := (-1)^{i-j}\left\{w_i^j(M,L) - H_1(i,j) - H_2(i,j)\right\}.$$

REMARK 2.1. (1) If $i = 0$, then

$$e_0(M,L) = b_0(M,L) = h_0^{0,0}(M,L) = L^m.$$

(2) If $i = 1$, then

$$e_1(M,L) = 2 - 2g(L),$$

$$b_1(M,L) = 2g(L),$$

$$h_1^{1,0}(M,L) = h_1^{0,1}(M,L) = g(L).$$

(3) If $i = m$, then

$$e_m(M,L) = e(M),$$

$$b_m(M,L) = b_m(M),$$

$$h_m^{j,m-j}(M,L) = h^{j,m-j}(M),$$

$$h_m^{m-j,j}(M,L) = h^{m-j,j}(M).$$

PROPOSITION 2.1. *Let (M,L) be a polarized manifold of dimension m.*

(1) *For every integer i with $0 \leq i \leq m$, the following hold:*

(1.1) $b_i(M,L) = \sum_{k=0}^{i} h_i^{k,i-k}(M,L)$.

(1.2) $h_i^{j,i-j}(M,L) = h_i^{i-j,j}(M,L)$.

(1.3) $h_i^{i,0}(M,L) = h_i^{0,i}(M,L) = g_i(M,L)$.

(2) *Assume that L is base point free. Then for every integers i and j with $1 \leq i \leq m$ and $0 \leq j \leq i$ the following hold.*

(2.1) $b_i(M,L) \geq b_i(M)$.

(2.2) $h_i^{j,i-j}(M,L) \geq h^{j,i-j}(M)$.

PROOF. See [8, Theorem 3.1 (3.1.1), (3.1.3), (3.1.4) and Proposition 3.3 (2) and (3)]. □

PROPOSITION 2.2. *For every integers a, k, l and r with $0 \leq l$,*

$$\sum_{j=0}^{l}(-1)^j\binom{r+j-a}{j}\binom{r-k}{l-j}=(-1)^l\binom{k-a+l}{l}.$$

PROOF. See [8, Proposition 2.5]. □

NOTATION 2.2. Let X be a smooth projective variety of dimension $n \geq 1$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. We put $P := \mathbb{P}_X(\mathcal{E})$, $H := H(\mathcal{E})$ and $m := \dim P$. Then $m = n + r - 1$. In this paper we assume that $r \geq 2$.

REMARK 2.2. Let X, $\mathcal{E}$, P, H, m, n and r be as in Notation 2.2.
(1) By [18, (2.1) Proposition], we have

$$b_j(P) = b_j(X) + b_{j-2}(X) + \cdots + b_{j-2r+2}(X).$$

(2) Let i be an integer with $i \leq m$. Then $n + r - 1 \geq i$ and we obtain $r \geq i - n + 1$.

(2.1) If $i \geq 2n - 2$ and $i - 1 \geq j$, then

$$\begin{aligned} j - 2r + 2 &\leq (i-1) - 2(i-n+1) + 2 \\ &= 2n - 1 - i \\ &\leq 1. \end{aligned}$$

So by (1) above, if $i \geq 2n - 2$ and $i - 1 \geq j$, then by (1) we have

$$b_j(P) = \begin{cases} \sum_{k=0}^{l} b_{j-2k}(X) & \text{if } j = 2l, \\ \sum_{k=0}^{l} b_{j-2k}(X) & \text{if } j = 2l+1. \end{cases}$$

By the same argument as this, if $i \geq 2n - 1$, then we see that

$$b_i(P) = \begin{cases} \sum_{k=0}^{l} b_{i-2k}(X) & \text{if } i = 2l, \\ \sum_{k=0}^{l} b_{i-2k}(X) & \text{if } i = 2l+1. \end{cases}$$

(2.2) Assume that $i \le m-1$. If $i = 2n-3$ and $i-1 \ge j$, then $j - 2r + 2 \le 2$. If this equality holds, then $i = n + r - 1 = m$. But this contradicts the assumption. Hence $j - 2r + 2 \le 1$, and by (1) above we have

$$b_j(P) = \begin{cases} \sum_{k=0}^{l} b_{j-2k}(X) & \text{if } j = 2l, \\ \sum_{k=0}^{l} b_{j-2k}(X) & \text{if } j = 2l+1. \end{cases}$$

By the same argument as this, if $i \le m-1$, $n \ge 1$ (resp. $n \ge 2$) and $i = 2n-2$ (resp. $i = 2n-3$), then we see that $b_{2n-2}(P) = \sum_{k=0}^{n-1} b_{2n-2-2k}(X)$ (resp. $b_{2n-3}(P) = \sum_{k=0}^{n-2} b_{2n-3-2k}(X)$).

DEFINITION 2.2. Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be a vector bundle of rank r on X.

(1) The *Chern polynomial* $c_t(\mathcal{E})$ is defined by the following:

$$c_t(\mathcal{E}) = 1 + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 + \cdots.$$

(2) For every integer j with $j \ge 0$, the *jth Segre class* $s_j(\mathcal{E})$ *of* $\mathcal{E}$ is defined by the following equation: $c_t(\mathcal{E}^\vee)s_t(\mathcal{E}) = 1$, where $c_t(\mathcal{E}^\vee)$ is the Chern polynomial of $\mathcal{E}^\vee$ and $s_t(\mathcal{E}) = \sum_{j\ge 0} s_j(\mathcal{E})t^j$.

REMARK 2.3. (1) Let X be a smooth projective variety and let $\mathcal{F}$ be a vector bundle on X. Let $\tilde{s}_j(\mathcal{F})$ be the Segre class which is defined in [10, Chapter 3]. Then $s_j(\mathcal{F}) = \tilde{s}_j(\mathcal{F}^\vee)$.

(2) For every integer i with $1 \le i$, $s_i(\mathcal{F})$ can be written by using the Chern classes $c_j(\mathcal{F})$ with $1 \le j \le i$. (For example, $s_1(\mathcal{F}) = c_1(\mathcal{F})$, $s_2(\mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, and so on.)

DEFINITION 2.3. Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank $n-1$ on X. Then the *curve genus* $cg(X, \mathcal{E})$ *of* $(X, \mathcal{E})$ is defined as follows:

$$cg(X, \mathcal{E}) = 1 + \frac{1}{2}(K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}).$$

3. Sectional invariants of scrolls over a smooth projective manifold.

THEOREM 3.1. *Let X, $\mathcal{E}$, P, H, m, n and r be as in Notation 2.2. Then the following hold.*

(3.1.1) *If* $i \geq 2n-1$, *then* $e_i(P,H) = (i-n+1)c_n(X)$.
(3.1.2) *If* $n \geq 2$, *then* $e_{2n-2}(P,H) = (n-1)c_n(X) + c_n(\mathcal{E})$.
(3.1.3) *If* $n \geq 3$, *then* $e_{2n-3}(P,H) = (n-2)(c_n(X) - c_n(\mathcal{E})) - c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E}))$.
(3.1.4) $e_1(P,H) = -(n-2)s_n(\mathcal{E}) - (c_1(\mathcal{E}) + K_X)s_{n-1}(\mathcal{E})$.
(3.1.5) $e_0(P,H) = s_n(\mathcal{E})$.

PROOF. By [10, Example 3.2.11] and Remark 2.3 (1), for every integer l with $0 \leq l \leq m$

$$c_l(P) = \sum_{j=0}^{l} \sum_{k=0}^{j} \binom{r-k}{j-k} c_k(p^*\mathcal{E}^\vee) H(\mathcal{E})^{j-k} c_{l-j}(p^*\mathcal{T}_X).$$

(Here $p: P \to X$ denotes the projection.) Hence

$$\begin{aligned}
&e_i(P,H)\\
&= \sum_{j=0}^{i} (-1)^j \binom{m-i+j-1}{j} c_{i-j}(P) H^{m-i+j}\\
&= \sum_{j=0}^{i} (-1)^j \binom{m-i+j-1}{j}\\
&\qquad \left\{ \sum_{s=0}^{i-j} \sum_{u=0}^{s} \binom{r-u}{s-u} c_u(p^*\mathcal{E}^\vee) H(\mathcal{E})^{m-i+j-u+s} c_{i-j-s}(p^*\mathcal{T}_X) \right\}\\
&= \sum_{\substack{0 \leq k,t \\ 0 \leq k+t \leq i}} \left(\sum_{j=0}^{i-t-k} (-1)^j \binom{r+j+n-i-2}{j} \binom{r-k}{i-t-k-j} \right)\\
&\qquad c_k(p^*\mathcal{E}^\vee) c_t(p^*\mathcal{T}_X) H(\mathcal{E})^{m-k-t}.
\end{aligned}$$

By Proposition 2.2 and [3, (3.7) in Chapter 0], we get

$$\begin{aligned}
(1) e_i(P,H)
&= \sum_{\substack{0 \leq k,t \\ 0 \leq k+t \leq i}} (-1)^{i-t-k} \binom{k+(n-i-2)+i-t-k}{i-t-k} c_k(p^*\mathcal{E}^\vee) c_t(p^*\mathcal{T}_X) H(\mathcal{E})^{m-k-t}\\
&= \sum_{\substack{0 \leq k,t \\ 0 \leq k+t \leq i}} (-1)^{i-t-k} \binom{n-t-2}{i-t-k} c_k(p^*\mathcal{E}^\vee) c_t(p^*\mathcal{T}_X) H(\mathcal{E})^{m-k-t}\\
&= \sum_{\substack{0 \leq k,t \\ 0 \leq k+t \leq i}} (-1)^{i-t-k} \binom{n-t-2}{i-t-k} c_k(\mathcal{E}^\vee) c_t(\mathcal{T}_X) s_{n-k-t}(\mathcal{E}).
\end{aligned}$$

Here we put

$$E(i,k,t) := (-1)^{i-t-k}\binom{n-t-2}{i-t-k}c_k(\mathcal{E}^\vee)c_t(\mathcal{T}_X)s_{n-k-t}(\mathcal{E}).$$

In order to calculate $e_i(P,H)$, we have only to consider the case where $E(i,k,t)\neq 0$. We note that if $k+t>n$, then $c_k(\mathcal{E}^\vee)c_t(\mathcal{T}_X)s_{n-k-t}(\mathcal{E})=0$. So we may assume that $k+t\leq n$.

(a) The case where $i\geq 2n-1$.

First we note that if $(n,i)=(1,1)$, then

$$e_i(P,H)=e_1(P,H)=2-b_1(P,H)=2-2g_1(P,H)=2-2q(X)=c_1(X).$$

So we assume that $(n,i)\neq(1,1)$.

Here we note that $(n-t-2)-(i-t-k)=n-i-2+k\leq -n+k-1\leq -1$ and $k+t\leq i$. Hence $\binom{n-t-2}{i-t-k}\neq 0$ if and only if one of the following holds.

(a.1) $i-t-k=0$.

(a.2) $i-t-k>0$ and $n-t-2<0$.

(a.1) If $i-t-k=0$, then $2n-1\leq i=t+k\leq n$. Hence $n=1$ and $i=1$. But this contradicts the assumption here.

(a.2) If $i-t-k>0$ and $n-t-2<0$, then $t=n-1$ or n because $k+t\leq n$ and $k\geq 0$. Hence $(t,k)=(n-1,0)$, $(n-1,1)$ or $(n,0)$.
If $(t,k)=(n-1,0)$, then

$$\binom{n-t-2}{i-t-k}=\binom{-1}{i-n+1}=(-1)^{i-n+1}.$$

If $(t,k)=(n-1,1)$, then

$$\binom{n-t-2}{i-t-k}=\binom{-1}{i-n}=(-1)^{i-n}.$$

If $(t,k)=(n,0)$, then

$$\binom{n-t-2}{i-t-k}=\binom{-2}{i-n}=(-1)^{i-n}(i-n+1).$$

Hence if $(n,i)\neq(1,1)$, then

$$\begin{aligned}
&e_i(P,H)\\
&=(-1)^{2(i-n+1)}c_{n-1}(\mathcal{T}_X)s_1(\mathcal{E})+(-1)^{2(i-n)}c_1(\mathcal{E}^\vee)c_{n-1}(\mathcal{T}_X)\\
&\quad+(-1)^{2(i-n)}(i-n+1)c_n(\mathcal{T}_X)\\
&=(i-n+1)c_n(X).
\end{aligned}$$

Therefore in any case

$$e_i(P,H)=(i-n+1)c_n(X)$$

if $i\geq 2n-1$.

(b) The case where $i=2n-2$. Here we note that $n\geq 2$ in this case.

Then $\binom{n-t-2}{i-t-k}=\binom{n-t-2}{2n-t-2-k}$. Here we note that $(n-t-2)\leq 2n-t-2-k$. Hence $\binom{n-t-2}{2n-t-2-k}\neq 0$ if and only if one of the following holds.

(b.1) $k=n$.

(b.2) $k<n$ and $n-t-2<0$.

(b.1) If $k=n$, then $t=0$ because $t+k\leq n$ and $t\geq 0$, and $\binom{n-t-2}{2n-t-2-k}=1$. Here we note that $n-t-2\geq 0$ in this case because $n\geq 2$ and $t=0$. Hence

$$E(2n-2,n,0)=(-1)^{2n-2-n}c_n(\mathcal{E}^\vee)=c_n(\mathcal{E}).$$

(b.2) If $k<n$ and $n-t-2<0$, then $t=n-1$ or n. Hence $(t,k)=(n-1,0)$, $(n-1,1)$ or $(n,0)$. By the same argument as (a.2), we obtain

$$\begin{aligned}
&E(2n-2,0,n-1)+E(2n-2,1,n-1)+E(2n-2,0,n)\\
&=(-1)^{2(2n-2-n+1)}c_{n-1}(X)s_1(\mathcal{E})+(-1)^{2(2n-2-n)}c_1(\mathcal{E}^\vee)c_{n-1}(X)\\
&\quad+(-1)^{2(2n-2-n)}(n-1)c_n(X)\\
&=(n-1)c_n(X).
\end{aligned}$$

Hence we get

$$e_{2n-2}(P,H) = (n-1)c_n(X) + c_n(\mathcal{E}).$$

(c) Assume that $i = 2n-3$. Here we note that $n \geq 3$ in this case. Then $\binom{n-t-2}{2n-t-k-3} \neq 0$ if and only if one of the following holds.

(c.1) $k = n$.

(c.2) $k = n-1$.

(c.3) $k \leq n-2$ and $t > n-2$.

First we consider the case (c.1). Then $t = 0$ because $k + t \leq n$ and $t \geq 0$. So we have

$$\begin{aligned} E(2n-3, n, 0) =& (-1)^{2n-3-0-n}\binom{n-0-2}{2n-3-0-n} c_n(\mathcal{E}^\vee) \\ =& -\binom{n-2}{n-3} c_n(\mathcal{E}) \\ =& -(n-2)c_n(\mathcal{E}). \end{aligned}$$

Next we consider the case (c.2). Then $t = 0$ or 1, and $\binom{n-t-2}{2n-t-k-3} = 1$. Hence we have $E(2n-3, n-1, 0) = -c_{n-1}(\mathcal{E})s_1(\mathcal{E}) = = -c_{n-1}(\mathcal{E})c_1(\mathcal{E})$ and $E(2n-3, n-1, 1) = c_{n-1}(\mathcal{E})c_1(X)$.

Finally we consider the case (c.3). Then $(k,t) = (0, n-1)$, $(1, n-1)$ or $(0, n)$. Hence by the same argument as above

$$E(2n-3, 0, n-1) + E(2n-3, 1, n-1) + E(2n-3, 0, n) = (n-2)c_n(X).$$

Therefore

$$\begin{aligned} & e_{2n-3}(P,H) \\ & = -(n-2)c_n(\mathcal{E}) - c_{n-1}(\mathcal{E})c_1(\mathcal{E}) + c_{n-1}(\mathcal{E})c_1(X) + (n-2)c_n(X) \\ & = (n-2)(c_n(X) - c_n(\mathcal{E})) + c_{n-1}(\mathcal{E})(c_1(X) - c_1(\mathcal{E})) \\ & = (n-2)(c_n(X) - c_n(\mathcal{E})) - c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})). \end{aligned}$$

(d) The case where $i = 1$. Then by (1) we get

$$
\begin{aligned}
& e_1(P,H) \\
& = \sum_{\substack{0\le k,t \\ 0\le k+t\le 1}} (-1)^{1-t-k}\binom{n-t-2}{1-t-k} c_k(\mathcal{E}^\vee)c_t(\mathcal{T}_X)s_{n-k-t}(\mathcal{E}) \\
& = -(n-2)s_n(\mathcal{E}) + (c_1(\mathcal{E}^\vee) + c_1(\mathcal{T}_X))s_{n-1}(\mathcal{E}) \\
& = -(n-2)s_n(\mathcal{E}) - (c_1(\mathcal{E}) + K_X)s_{n-1}(\mathcal{E}).
\end{aligned}
$$

(e) The case where $i = 0$. Then by (1) we get $e_0(P,H) = s_n(\mathcal{E})$.
We get the assertion of Theorem 3.1. □

By Theorem 3.1, we get the following for $n = 1$ and 2.

COROLLARY 3.1. *Let X, $\mathcal{E}$, P, H and n be as in Notation* 2.2.
(3.1.1) *Assume that $n = 1$. Then we get the following:*

$$
e_i(P,H) = \begin{cases} i(2-2g(X)) & \text{if } i \ge 1, \\ \deg \mathcal{E} & \text{if } i = 0. \end{cases}
$$

(3.1.2) *Assume that $n = 2$. Then we get the following*:

$$
e_i(P,H) = \begin{cases} (i-1)c_2(X) & \text{if } i \ge 3, \\ c_2(X) + c_2(\mathcal{E}) & \text{if } i = 2, \\ -(c_1(\mathcal{E}) + K_X)c_1(\mathcal{E}) & \text{if } i = 1, \\ s_2(\mathcal{E}) & \text{if } i = 0. \end{cases}
$$

THEOREM 3.2. *Let X, $\mathcal{E}$, P, H, m, n and r be as in Notation 2.2. Then the following hold.*

(3.2.1) *If $m \ge i \ge 2n-1$, then $b_i(P,H) = b_i(P)$.*

(3.2.2) *If $n \ge 2$ and $m > 2n-2$, then $b_{2n-2}(P,H) = b_{2n-2}(P) + c_n(\mathcal{E}) - 1$.*

(3.2.3) *If $n \ge 3$ and $m > 2n-3$, then*

$$
b_{2n-3}(P,H) = b_{2n-3}(P) + (n-2)c_n(\mathcal{E}) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})) + 2 - 2q(X).
$$

(3.2.4) $b_1(P,H) = 2 + (n-2)s_n(\mathcal{E}) + (c_1(\mathcal{E}) + K_X)s_{n-1}(\mathcal{E})$.

(3.2.5) $b_0(P,H) = s_n(\mathcal{E})$.

PROOF. Since $e_i(P,H)$ has been calculated in Theorem 3.1, we need to study $\sum_{k=0}^{i-1}(-1)^k b_k(P)$. By using Remark 2.2, we calculate this.

(a) The case where $i \geq 2n-1$. Since $b_m(P,H)=b_m(P)$, we assume that $i<m$.

(a.1) Assume that i is even. Then $i \geq 2n$. Here we note that $b_j(X)=0$ if $j>2n$. Then

$$\begin{aligned}
&\sum_{k=0}^{i-1}(-1)^k b_k(P)\\
&=\sum_{k=0}^{n}\left(\frac{i}{2}-k\right)b_{2k}(X)-\sum_{k=1}^{n}\left(\frac{i}{2}-k+1\right)b_{2k-1}(X)\\
&=\sum_{k=0}^{n}(n-k+1)b_{2k}(X)-\sum_{k=1}^{n}(n-k+2)b_{2k-1}(X)+\frac{i-2n-2}{2}e(X).
\end{aligned}$$

Here we note that by Remark 2.2 (2.1), we have $b_i(P)=\sum_{k=0}^{n}b_{2k}(X)$ because $i \geq 2n$ in this case. Since i is even, we have

$$\begin{aligned}
&b_i(P,H)-b_i(P)\\
&=(-1)^i\left(e_i(P,H)-2\sum_{j=0}^{i-1}(-1)^j b_j(P)\right)-\sum_{k=0}^{n}b_{2k}(X)\\
&=(i-n+1)c_n(X)-2\sum_{k=0}^{n}(n-k+1)b_{2k}(X)\\
&\quad+2\sum_{k=1}^{n}(n-k+2)b_{2k-1}(X)-(i-2n-2)e(X)-\sum_{k=0}^{n}b_{2k}(X)\\
&=(n+3)e(X)+\sum_{k=0}^{n}(2k-2n-3)b_{2k}(X)+\sum_{k=1}^{n}(2n-2k+4)b_{2k-1}(X)\\
&=\sum_{k=0}^{n}(2k-n)b_{2k}(X)-\sum_{k=1}^{n}(2k-n-1)b_{2k-1}(X).
\end{aligned}$$

Here we prove the following.

CLAIM 3.1. (1) $\sum_{k=0}^{n}(2k-n)b_{2k}(X)=0$.

(2) $\sum_{k=1}^{n}(2k-n-1)b_{2k-1}(X)=0$.

PROOF. (1) Assume that $n = 2l$. Then we note that $(2k - n)b_{2k}(X) = 0$ if $k = l$. So by Poincaré duality

$$\begin{aligned}
&\sum_{k=0}^{n}(2k-n)b_{2k}(X)\\
&= \sum_{k=0}^{l-1}(2k-n)b_{2k}(X) + \sum_{k=l+1}^{n}(2k-n)b_{2k}(X)\\
&= \sum_{k=0}^{l-1}(2k-n)b_{2k}(X) + \sum_{k=0}^{l-1}(n-2k)b_{2n-2k}(X)\\
&= \sum_{k=0}^{l-1}(2k-n)b_{2k}(X) - \sum_{k=0}^{l-1}(2k-n)b_{2k}(X)\\
&= 0.
\end{aligned}$$

If $n = 2l + 1$, then by Poincaré duality

$$\begin{aligned}
&\sum_{k=0}^{n}(2k-n)b_{2k}(X)\\
&= \sum_{k=0}^{l}(2k-n)b_{2k}(X) + \sum_{k=l+1}^{n}(2k-n)b_{2k}(X)\\
&= \sum_{k=0}^{l}(2k-n)b_{2k}(X) + \sum_{k=0}^{l}(n-2k)b_{2n-2k}(X)\\
&= \sum_{k=0}^{l}(2k-n)b_{2k}(X) - \sum_{k=0}^{l}(2k-n)b_{2k}(X)\\
&= 0.
\end{aligned}$$

Hence we obtain the assertion of (1).

(2) This can be proved by the same argument as (1). □

By Claim 3.1, $b_i(P,H) = b_i(P)$ if $i \geq 2n - 1$ and i is even.

(a.2) Assume that i is odd. Then

$$\begin{aligned}
&\sum_{j=0}^{i-1}(-1)^j b_j(P) \\
&= \sum_{k=0}^{n}\left(\frac{i+1}{2}-k\right)b_{2k}(X) - \sum_{k=1}^{n}\left(\frac{i-1}{2}-k+1\right)b_{2k-1}(X) \\
&= \sum_{k=0}^{n}(n-k+1)b_{2k}(X) - \sum_{k=1}^{n}(n-k+1)b_{2k-1}(X) \\
&\quad + \left(\frac{i+1}{2}-n-1\right)e(X).
\end{aligned}$$

By Remark 2.2 (2.1), we have $b_i(P)=\sum_{k=1}^{n} b_{2k-1}(X)$ because $i \geq 2n-1$ in this case. Hence by Theorem 3.1 and Claim 3.1, we have

$$\begin{aligned}
&b_i(P,H) - b_i(P) \\
&= -\left((i-n+1)e(X) - 2\sum_{j=0}^{i-1}(-1)^j b_j(P)\right) - b_i(P) \\
&= -(n+2)e(X) + \sum_{k=0}^{n}(2n-2k+2)b_{2k}(X) - \sum_{k=1}^{n}(2n-2k+3)b_{2k-1}(X) \\
&= \sum_{k=0}^{n}(n-2k)b_{2k}(X) - \sum_{k=1}^{n}(n-2k+1)b_{2k-1}(X) \\
&= 0.
\end{aligned}$$

Therefore $b_i(P,H) = b_i(P)$ if $i \geq 2n-1$ and i is odd.

In any case, if $m \geq i \geq 2n-1$, then $b_i(P,H) = b_i(P)$.

(b) The case where $i = 2n-2$ and $2n-2 < m$. Then by Remark 2.2 (2.1) we obtain

$$b_j(P) = \begin{cases} \sum_{k=0}^{l} b_{j-2k}(X) & \text{if } j = 2l, \\ \sum_{k=0}^{l} b_{j-2k}(X) & \text{if } j = 2l+1 \end{cases}$$

for every integer j with $j < 2n - 2$. Hence

$$\sum_{j=0}^{2n-3} (-1)^j b_j(P) = \sum_{k=0}^{n-2} (n-k-1) b_{2k}(X) - \sum_{k=1}^{n-1} (n-k) b_{2k-1}(X).$$

By assumption, we have $n + r - 1 = m \geq i + 1 = 2n - 1$. So by Remark 2.2 (2.2) we obtain

$$b_{2n-2}(P) = \sum_{k=0}^{n-1} b_{2k}(X).$$

Therefore

$$\begin{aligned}
& b_{2n-2}(P, H) - b_{2n-2}(P) \\
&= \left(e_{2n-2}(P, H) - 2 \sum_{j=0}^{2n-3} (-1)^j b_j(P) \right) - b_{2n-2}(P) \\
&= (n-1)e(X) + c_n(\mathcal{E}) - \sum_{k=0}^{n-2} (2n-2k-2) b_{2k}(X) \\
&\quad + \sum_{k=1}^{n-1} (2n-2k) b_{2k-1}(X) - \sum_{k=0}^{n-1} b_{2k}(X) \\
&= (n-1)e(X) + c_n(\mathcal{E}) - \sum_{k=0}^{n-2} (2n-2k-1) b_{2k}(X) \\
&\quad + \sum_{k=1}^{n-1} (2n-2k) b_{2k-1}(X) - b_{2n-2}(X) \\
&= (n-1)e(X) + c_n(\mathcal{E}) + e(X) - 1 - \sum_{k=0}^{n-1} (2n-2k) b_{2k}(X) \\
&\quad + \sum_{k=1}^{n-1} (2n-2k+1) b_{2k-1}(X) + b_{2n-1}(X) \\
&= ne(X) + c_n(\mathcal{E}) - 1 - \sum_{k=0}^{n-1} (2n-2k) b_{2k}(X) \\
&\quad + \sum_{k=1}^{n} (2n-2k+1) b_{2k-1}(X).
\end{aligned}$$

Since

$$\sum_{k=0}^{n-1}(2n-2k)b_{2k}(X)=\sum_{k=0}^{n}(2n-2k)b_{2k}(X),$$

we obtain

$$\begin{aligned}
&b_{2n-2}(P,H)-b_{2n-2}(P)\\
&=ne(X)+c_n(\mathcal{E})-1-\sum_{k=0}^{n}(2n-2k)b_{2k}(X)\\
&\quad+\sum_{k=1}^{n}(2n-2k+1)b_{2k-1}(X)\\
&=c_n(\mathcal{E})-1+\sum_{k=0}^{n}(2k-n)b_{2k}(X)\\
&\quad-\sum_{k=1}^{n}(2k-n-1)b_{2k-1}(X).
\end{aligned}$$

By Claim 3.1, we get

$$b_{2n-2}(P,H)=b_{2n-2}(P)+c_n(\mathcal{E})-1.$$

(c) The case where $i=2n-3$ and $i<m$. Then by Remark 2.2 (2.2) we obtain

$$b_j(P)=\begin{cases}\displaystyle\sum_{k=0}^{l}b_{j-2k}(X) & \text{if } j=2l,\\ \displaystyle\sum_{k=0}^{l}b_{j-2k}(X) & \text{if } j=2l+1\end{cases}$$

for every integer j with $j<2n-3$. Hence

$$\sum_{j=0}^{2n-4}b_j(P)=\sum_{k=0}^{n-2}(n-k-1)b_{2k}(X)-\sum_{k=1}^{n-2}(n-k-1)b_{2k-1}(X).$$

By assumption we have $n+r-1=m\geq i+1=2n-2$. Hence by Remark 2.2 (2.2) we obtain

$$b_{2n-3}(P)=\sum_{k=1}^{n-1}b_{2k-1}(X).$$

Therefore

$$
\begin{aligned}
& b_{2n-3}(P,H) - b_{2n-3}(P) \\
& = -e_{2n-3}(P,H) + 2\sum_{j=0}^{2n-4}(-1)^j b_j(P) - b_{2n-3}(P) \\
& = -(n-2)(e(X) - c_n(\mathcal{E})) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})) \\
& \quad + \sum_{k=0}^{n-2}(2n-2k-2)b_{2k}(X) - \sum_{k=1}^{n-2}(2n-2k-2)b_{2k-1}(X) \\
& \quad - \sum_{k=1}^{n-1} b_{2k-1}(X) \\
& = -(n-2)(e(X) - c_n(\mathcal{E})) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})) \\
& \quad + \sum_{k=0}^{n-2}(2n-2k-2)b_{2k}(X) - \sum_{k=1}^{n-2}(2n-2k-1)b_{2k-1}(X) \\
& \quad - b_{2n-3}(X) \\
& = -(n-2)(e(X) - c_n(\mathcal{E})) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})) \\
& \quad - 2e(X) + 2b_{2n-2}(X) + 2b_{2n}(X) - 3b_{2n-3}(X) - 2b_{2n-1}(X) \\
& \quad + \sum_{k=0}^{n-2}(2n-2k)b_{2k}(X) - \sum_{k=1}^{n-2}(2n-2k+1)b_{2k-1}(X) \\
& = -(n-2)(e(X) - c_n(\mathcal{E})) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})) \\
& \quad + \sum_{k=0}^{n}(2n-2k)b_{2k}(X) - \sum_{k=1}^{n}(2n-2k+1)b_{2k-1}(X) \\
& \quad - 2e(X) + 2b_{2n}(X) - b_{2n-1}(X) \\
& = -(n-2)e(X) + (n-2)c_n(\mathcal{E}) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E})) \\
& \quad + \sum_{k=0}^{n}(n-2k)b_{2k}(X) - \sum_{k=1}^{n}(n-2k+1)b_{2k-1}(X) \\
& \quad + (n-2)e(X) + 2b_{2n}(X) - b_{2n-1}(X).
\end{aligned}
$$

Since $b_{2n}(X)=1$ and $b_{2n-1}(X)=b_1(X)=2q(X)$, by Claim 3.1 we obtain

$$\begin{aligned}&b_{2n-3}(P,H)-b_{2n-3}(P)\\&=(n-2)c_n(\mathcal{E})+c_{n-1}(\mathcal{E})(K_X+c_1(\mathcal{E}))+2-2q(X).\end{aligned}$$

(d) The case where $i=1$.
Then by Theorem 3.1 (3.1.4) and the definition of $b_1(P,H)$

$$\begin{aligned}b_1(P,H)=&-e_1(P,H)+2b_0(P)\\&=2+(n-2)s_n(\mathcal{E})+(c_1(\mathcal{E})+K_X)s_{n-1}(\mathcal{E}).\end{aligned}$$

(e) The case where $i=0$.
Then by Theorem 3.1 (3.1.5) and the definition of $b_0(P,H)$,

$$b_0(P,H)=e_0(P,H)=s_n(\mathcal{E}).$$

Therefore we get the assertion. □

COROLLARY 3.2. *Let X, $\mathcal{E}$, P, H, m and n be as in Notation* 2.2. *If $n\geq 2$ and $m>i\geq 2n-2$, then $b_i(P,H)\geq b_i(P)$.*

PROOF. If $i\geq 2n-1$, then by Theorem 3.2 (3.2.1) we get the assertion. Next we consider the case where $i=2n-2$. Since $\mathcal{E}$ is ample, we have $c_n(\mathcal{E})\geq 1$. Therefore by Theorem 3.2 (3.2.2), we see $b_{2n-2}(P,H)\geq b_{2n-2}(P)$. This completes the proof. □

COROLLARY 3.3. *Let X, $\mathcal{E}$, P, H, m and n be as in Notation* 2.2.
(3.3.1) *Assume that $n=1$. Then we get the following*:

$$b_i(P,H)=\begin{cases}b_i(P) & \textit{if } m\geq i\geq 1,\\ \deg\mathcal{E} & \textit{if } i=0.\end{cases}$$

(3.3.2) *Assume that $n=2$. Then we get the following*:

$$b_i(P,H)=\begin{cases}b_i(P) & \textit{if } m\geq i\geq 3,\\ b_2(P)+c_2(\mathcal{E})-1 & \textit{if } i=2,\\ c_1(\mathcal{E})(c_1(\mathcal{E})+K_X)+2 & \textit{if } i=1,\\ s_2(\mathcal{E}) & \textit{if } i=0.\end{cases}$$

PROOF. By Theorem 3.2, we get the assertion. (Here we note that if $n=2$, then $2n-2=2<n+r-1=m$ because we assume $r\geq 2$ in this paper.) □

REMARK 3.1. Since $\mathcal{E}$ is ample, we see that if $n = 1$ or 2, then $b_i(P, H) \geq 0$ for any i.

Here we calculate $h_i^{j,i-j}(P, H)$ for the case where $m \geq i \geq 2n - 1$.

THEOREM 3.3. *Let X, $\mathcal{E}$, P, H, m and n be as in Notation* 2.2. *If $m \geq i \geq 2n - 1$ and $0 \leq j \leq i$, then $h_i^{j,i-j}(P, H) = h^{j,i-j}(P)$.*

PROOF. First we note that we can take an ample line bundle A on X such that $\mathcal{E} \otimes A^{\otimes t}$ is ample and spanned for every positive integer t. Hence $H(\mathcal{E} \otimes A^{\otimes t})$ is ample and spanned. We also note that there exists an isomorphism $\phi : \mathbb{P}_X(\mathcal{E} \otimes A^{\otimes t}) \to P$ with $H(\mathcal{E} \otimes A^{\otimes t}) = \phi^*(H \otimes p^*(A^{\otimes t}))$, where $p : P \to X$ is the projection. Therefore $H \otimes p^*(A^{\otimes t})$ is also ample and spanned. By this ϕ, we identify $\mathbb{P}_X(\mathcal{E} \otimes A^{\otimes t})$ and P. By Theorem 3.2 we have

$$b_i(P, H(\mathcal{E} \otimes A^{\otimes t})) = b_i(P).$$

Hence

$$\begin{aligned} & b_i(P, H \otimes p^*(A^{\otimes t})) \\ & = b_i(P, H(\mathcal{E} \otimes A^{\otimes t})) \\ & = b_i(P). \end{aligned}$$

On the other hand by Proposition 2.1 (1.1) and (2) we have

$$h_i^{j,i-j}(P, H \otimes p^*(A^{\otimes t})) = h^{j,i-j}(P)$$

because $H \otimes p^*(A^{\otimes t})$ is ample and spanned. But since $F(t) :=$ $:= h_i^{j,i-j}(P, H \otimes p^*(A^{\otimes t})) - h^{i,i-j}(P)$ is a polynomial in t and $F(t) = 0$ for every positive integer t, we see that $F(0) = 0$, that is,

$$h_i^{j,i-j}(P, H) = h^{i,i-j}(P).$$

This completes the proof. □

COROLLARY 3.4. *Let X, $\mathcal{E}$, P, H, m and n be as in Notation* 2.2.
(3.4.1) *Assume that $n = 1$. Then we get the following*:

$$h_i^{j,i-j}(P, H) = \begin{cases} h^{j,i-j}(P) & \text{if } m \geq i \geq 1 \text{ and } 0 \leq j \leq i, \\ \deg \mathcal{E} & \text{if } i = 0 \text{ and } j = 0. \end{cases}$$

(3.4.2) *Assume that $n = 2$. Then we get the following*:

$$h_i^{j,i-j}(P,H) = \begin{cases} h^{j,i-j}(P) & \text{if } m \geq i \geq 3 \text{ and } 0 \leq j \leq i, \\ h^{0,2}(P) & \text{if } i = 2 \text{ and } j = 0, \\ h^{2,0}(P) & \text{if } i = 2 \text{ and } j = 2, \\ h^{1,1}(P) + c_2(\mathcal{E}) - 1 & \text{if } i = 2 \text{ and } j = 1, \\ \frac{1}{2}(c_1(\mathcal{E})(c_1(\mathcal{E}) + K_X)) + 1 & \text{if } i = 1 \text{ and } j = 0, 1, \\ s_2(\mathcal{E}) & \text{if } i = 0 \text{ and } j = 0. \end{cases}$$

PROOF. If $n = 1$, then by Corollary 3.3 and Theorem 3.3 we get the assertion.

Assume that $n = 2$. Then by Corollary 3.3 and Theorem 3.3 we get the assertion for the case where $m \geq i \geq 3$ and $0 \leq j \leq i$. If $(i,j) = (2,0)$ (resp. $(2,2)$), then by Proposition 2.1 (1.3) and [5, Example 2.10 (8)] we have $h_2^{0,2}(P,H) = g_2(P,H) = h^2(\mathcal{O}_P) = h^{0,2}(P)$ (resp. $h_2^{2,0}(P,H) = g_2(P,H) = = h^2(\mathcal{O}_P) = h^{0,2}(P) = h^{2,0}(P)$). Moreover by Corollary 3.3 (3.3.2) and Proposition 2.1 (1.1) we get $h_2^{1,1}(P,H) = h^{1,1}(P) + c_2(\mathcal{E}) - 1$.

Assume that $i = 1$. Then by Proposition 2.1 (1.1) we have $b_1(P,H) = h_1^{1,0}(P,H) + h_1^{0,1}(P,H)$. Moreover by Proposition 2.1 (1.2) we have $h_1^{1,0}(P,H) = h_1^{0,1}(P,H)$. Therefore by Corollary 3.3 we get the assertion for the case $i = 1$.

By Corollary 3.3 we get the assertion for the case where $i = 0$. □

REMARK 3.2. Since $\mathcal{E}$ is ample, we see that if $n = 1$ or 2, then $h_i^{j,i-j}(P,H) \geq 0$ for any i and j with $0 \leq j \leq i \leq m$.

4. A new invariant of generalized polarized manifolds.

Here we use Notation 2.2. Assume that $i = 2n - 3$, $n \geq 3$ and $i < m$. Then by Theorem 3.2 (3.2.3) we see that $(n-2)c_n(\mathcal{E}) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E}))$ is even because $b_{2n-3}(P,H)$ and $b_{2n-3}(P)$ are even (see [8, Theorem 3.1 (3.1.2)]). We put

$$v(X,\mathcal{E}) := 1 + \frac{1}{2}((n-2)c_n(\mathcal{E}) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E}))).$$

Here we note that $r \geq n - 1$ since $n + r - 1 = m \geq i + 1 = 2n - 2$.

If $r = n - 1$, then

$$v(X,\mathcal{E}) = 1 + \frac{1}{2}(K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}).$$

So $v(X,\mathcal{E})$ is thought to be a generalization of the curve genus $cg(X,\mathcal{E})$ of $(X,\mathcal{E})$ (see Definition 2.3). Here we define $v(X,\mathcal{E})$ again.

DEFINITION 4.1. Let $(X,\mathcal{E})$ be a generalized polarized manifold of dimension $n \geq 3$. Assume that $r \geq n-1$. Then the invariant $v(X,\mathcal{E})$ of $(X,\mathcal{E})$ is defined as follows.

$$v(X,\mathcal{E}) := 1 + \frac{1}{2}((n-2)c_n(\mathcal{E}) + c_{n-1}(\mathcal{E})(K_X + c_1(\mathcal{E}))).$$

Since $b_{2n-3}(P,H) - b_{2n-3}(P) = 2v(X,\mathcal{E}) - 2q(X)$ by Theorem 3.2 (3.2.3), we can propose the following conjecture.

CONJECTURE 4.1. *Let $(X,\mathcal{E})$ be a generalized polarized manifold of dimension $n \geq 3$. Assume that $r \geq n-1$. Then $v(X,\mathcal{E}) \geq q(X)$.*

Here we study the non-negativity of $v(X,\mathcal{E})$.

THEOREM 4.1. *Let $(X,\mathcal{E})$ be a generalized polarized manifold of dimension $n \geq 3$. Assume that $r \geq n-1$. Then $v(X,\mathcal{E}) \geq 0$.*

PROOF. If $r = n-1$, then $v(X,\mathcal{E})$ is the curve genus and then $v(X,\mathcal{E}) \geq 0$ by [19, Theorem 1]. So we may assume that $r \geq n$.

If $r \geq n$ and $K_X + c_1(\mathcal{E})$ is nef, then $(K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}) \geq 0$ because $\mathcal{E}$ is ample. Furthermore $c_n(\mathcal{E}) \geq 1$. So we obtain $v(X,\mathcal{E}) \geq 2$.

If $r \geq n$ and $K_X + c_1(\mathcal{E})$ is not nef, then $(X,\mathcal{E}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$ by [22, Theorem 1 and Theorem 2]. In this case $(K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}) = -n$ and $c_n(\mathcal{E}) = 1$. Hence $v(X,\mathcal{E}) = 0$. Therefore we get the assertion. □

THEOREM 4.2. *Let $(X,\mathcal{E})$ be a generalized polarized manifold of dimension $n \geq 3$. Assume that $r \geq n-1$.*

(1) *If $v(X,\mathcal{E}) = 0$, then $(X,\mathcal{E})$ is one of the following.*
- (1.1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$.
- (1.2) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-1})$.
- (1.3) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-2} \oplus \mathcal{O}_{\mathbb{P}^n}(2))$.
- (1.4) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus n-1})$.
- (1.5) *$X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{F})$ for some vector bundle $\mathcal{F}$ of rank n on $\mathbb{P}^1$, and $\mathcal{E} \cong \oplus_{j=1}^{n-1}(H(\mathcal{F}) + \pi^*\mathcal{O}_{\mathbb{P}^1}(b_j))$, where $\pi : X \to \mathbb{P}^1$ is the bundle projection.*

(2) *If* $v(X,\mathcal{E})=1$, *then* $(X,\mathcal{E})$ *is one of the following.*
  (2.1) $X\cong\mathbb{P}_C(\mathcal{F})$ *for some vector bundle* $\mathcal{F}$ *of rank* n *on a smooth elliptic curve* C, *and* $\mathcal{E}_F\cong\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus n-1}$ *for any fiber* F *of the bundle projection* $X\to C$.
  (2.2) $K_X+\det(\mathcal{E})=\mathcal{O}_X$ and $r=n-1$.

PROOF. If $v(X,\mathcal{E})\leq 1$, then by the proof of Theorem 4.1, one of the following cases occurs:

(4.2.1) $r=n-1$.

(4.2.2) $r\geq n$ and $K_X+c_1(\mathcal{E})$ is not nef.

If $r=n-1$ and $v(X,\mathcal{E})=0$ (resp. $v(X,\mathcal{E})=1$), then $0=v(X,\mathcal{E})= =cg(X,\mathcal{E})$ (resp. $1=v(X,\mathcal{E})=cg(X,\mathcal{E})$) and by [19, Theorem 1 and Theorem 2] we get the above types.

If $r\geq n$ and $K_X+c_1(\mathcal{E})$ is not nef, then $(X,\mathcal{E})\cong(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$ by [22, Theorem 1 and Theorem 2] and then $v(X,\mathcal{E})=0$. Hence we get the assertion. □

REMARK 4.1. (1) If $(X,\mathcal{E})$ is the type (2.2) in Theorem 4.2, then a classification of $(X,\mathcal{E})$ has been obtained by [20].

(2) If $v(X,\mathcal{E})\leq 1$, then we see that $v(X,\mathcal{E})\geq q(X)$.

PROPOSITION 4.1. *Let* $(X,\mathcal{E})$ *be a generalized polarized manifold of dimension* $n\geq 3$. *Assume that* $r\geq n-1$ *and* $\mathcal{E}$ *is spanned. Then* $v(X,\mathcal{E})\geq q(X)$.

PROOF. Set $(P,H):=(\mathbb{P}_X(\mathcal{E}),H(\mathcal{E}))$. Then H is ample and spanned. So by Proposition 2.1 (2.1) we obtain

$$b_{2n-3}(P,H)\geq b_{2n-3}(P).$$

On the other hand

$$b_{2n-3}(P,H)-b_{2n-3}(P)$$

$$=2v(X,\mathcal{E})-2q(X).$$

Hence we get the assertion. □

REMARK 4.2. (1) By Theorem 4.2 we can determine the type of $(X,\mathcal{E})$ such that $v(X,\mathcal{E})=q(X)$ and $q(X)\leq 1$.

(2) For every integer $q \geq 2$, there exists an example of $(X, \mathcal{E})$ with $v(X, \mathcal{E}) = q = q(X)$ (This was given by the referee.): Let C be a smooth projective curve with $g(C) = q \geq 2$. We note that there exist vector bundles $\mathcal{F}$ and $\mathcal{G}$ on C with rank $\mathcal{F} = n$ and rank $\mathcal{G} = n-1$ such that $\mathcal{E} := H(\mathcal{F}) \otimes \pi^*(\mathcal{G})$ is an ample vector bundle of rank $n-1$ on X, where $X := \mathbb{P}_C(\mathcal{F})$, $H(\mathcal{F})$ is the tautological line bundle of $\mathcal{F}$ and $\pi : X \to C$ is the bundle projection. Then we can easily check that $v(X, \mathcal{E}) = cg(X, \mathcal{E}) = q \geq 2$.

(3) We see that if $\mathcal{E}$ is ample and spanned, rank $\mathcal{E} = n-1$ and $v(X, \mathcal{E}) = q(X) \geq 2$, then $(X, \mathcal{E})$ is isomorphic to the type in (2) above. This has been proved by [17, Theorem].

Here we give the following conjecture which was pointed out by the referee:

CONJECTURE 4.2. *Let X be a smooth projective variety of dimension $n \geq 3$ and let $\mathcal{E}$ be an ample vector bundle on X with* rank $\mathcal{E} = r$. *Assume that $r \geq n-1$ and $q(X) \geq 2$. If $\mathcal{E}$ is spanned and $v(X, \mathcal{E}) = q(X)$, then $r = n-1$.*

REMARK 4.3. We consider the case where $r \geq n \geq 3$. (The following (a) and (b) were also pointed out by the referee.)

(a) If $K_X + c_1(\mathcal{E})$ is not ample, then by [4, Main Theorem] $(X, \mathcal{E})$ is one of the following 6 types and we can calculate $v(X, \mathcal{E})$ and $q(X)$:

(a.1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1})$. In this case $(v(X, \mathcal{E}), q(X)) = (1 + (1/2) \cdot (n-2)(n+1), 0)$.

(a.2) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$. In this case $(v(X, \mathcal{E}), q(X)) = (0, 0)$.

(a.3) $X \cong \mathbb{P}_C(\mathcal{F})$ for a vector bundle $\mathcal{F}$ of rank n on a smooth curve C, and $\mathcal{E} \cong H(\mathcal{F}) \otimes \pi^*(\mathcal{G})$ for a vector bundle $\mathcal{G}$ on C with rank $\mathcal{G} = n$, where $\pi : X \to C$ is the bundle projection. In this case $(v(X, \mathcal{E}), q(X)) = (g(C) + (n-1)(g(C) - 1 + c_1(\mathcal{F}) + c_1(\mathcal{G})), g(C))$.

(a.4) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^n}(2))$. In this case $(v(X, \mathcal{E}), q(X)) = = (n-1, 0)$.

(a.5) $(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n})$. In this case $(v(X, \mathcal{E}), q(X)) = (1 + (1/2)(n-2) \cdot (n+1), 0)$.

(a.6) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus n})$. In this case $(v(X, \mathcal{E}), q(X)) = (n-1, 0)$.

Here we consider the case where $(X, \mathcal{E})$ is the type (a.3) above. Then $1 \leq c_n(\mathcal{E}) = c_1(\mathcal{F}) + c_1(\mathcal{G})$ because $\mathcal{E}$ is ample. Hence $g(C) - 1 + c_1(\mathcal{F}) +$

$+c_1(\mathcal{G}) \geq 0$ and we get $v(X,\mathcal{E}) \geq g(C) = q(X)$. Moreover if $g(C) \geq 1$, then we see that $v(X,\mathcal{E}) \geq q(X) + (n-1) > q(X)$.

We also note that if $\mathcal{E}$ is spanned by global sections, then $c_n(\mathcal{E}) \geq 2$ by [21, (3.4) Theorem] and we obtain $v(X,\mathcal{E}) \geq q(X) + (n-1) > q(X)$.

(b) Next we assume that $K_X + c_1(\mathcal{E})$ is ample. Then we can prove the following:

PROPOSITION 4.2. (1) *If* $K_X + c_1(\mathcal{E})$ *is ample, then* $v(X,\mathcal{E}) \geq 1 + \frac{1}{2}(n-1)$.

(2) *If* $K_X + c_1(\mathcal{E})$ *is ample and* $\mathcal{E}$ *is spanned, then* $v(X,\mathcal{E}) \geq n$.

PROOF. (1) By assumption, we have $c_n(\mathcal{E}) \geq 1$ and $(K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}) \geq 1$. Hence we get the assertion of (1).

(2) Since $\mathcal{E}$ is spanned and $K_X + c_1(\mathcal{E})$ is ample, we see that $c_n(\mathcal{E}) \geq 2$ by [21, (3.4) Theorem]. Hence $(n-2)c_n(\mathcal{E}) + (K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}) \geq 2(n-2) + 2$ because the term on the left is even. Therefore we get the assertion of (2). □

The above results suggest that for the case where $r \geq n$ and $v(X,\mathcal{E}) \neq q(X)$ there are some gaps for the value of $v(X,\mathcal{E}) - q(X)$, depending on n. We will investigate this in a future paper.

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