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Idempotent Subreducts of Semimodules over Commutative Semirings

DAVID STANOVSKÝ (*)

ABSTRACT - A short proof of the characterization of idempotent subreducts of semimodules over commutative semirings is presented. It says that an idempotent algebra embeds into a semimodule over a commutative semiring, if and only if it belongs to the variety of Szendrei modes.

1. Introduction.

Embedding one class of structures into a better understood one usually brings some new knowledge about the former class. We will focus on embeddings of algebras into reducts of semimodules over commutative semirings; hence we obtain *linear representations* for operations of the algebras.

Modes are idempotent algebras where every pair of operations commute with one another [10]. Indeed, idempotent subreducts of semimodules over commutative semirings are modes and it had been an open problem [10] whether the converse statement is true. Quite recently, N. Dojer observed that such modes satisfy the so-called *Szendrei identities* (they appeared in the paper [16] by Ágnes Szendrei) and Michał Stronkowski found a syntactical proof that these identities do not follow from the axioms of modes [14]. Thus there exist modes that are not idempotent subreducts of semimodules over commutative semirings; in fact, we present a simple example of such a mode in Example 2.

(*) Indirizzo dell'A.: Charles University in Prague, KA MFF UK, Sokolovská 83, 18675 Praha 8, Czech Republic.
E-mail: stanovsk@karlin.mff.cuni.cz
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Shortly after that, Stronkowski also proved that Szendrei modes are embeddable [15] and thus obtained the following characterization:

THEOREM 1 (M. Stronkowski [15]). *An idempotent algebra is a subreduct of a semimodule over a commutative semiring if and only if it is a Szendrei mode.*

The aim of the present paper is to provide a short proof of Theorem 1.

Actually, M. Stronkowski considered a more general situation: He proved the embedding theorem for (not necessarily idempotent) entropic algebras with onto operations. (Theorem 1 is an obvious corollary of this result.) The payoff for greater generality is much greater complexity of his proof; it does not simplify straightforwardly if idempotency is assumed. However, in the idempotent case, one can use several technical tricks developed by Á. Szendrei in [16], which make our proof rather short and transparent. Since modes have interested a number of mathematicians recently (see the monograph [10]), I think presenting a short proof is worthwhile.

The core of the proof of Theorem 1 is contained in Section 3. In Section 2, we present auxiliary results on free Szendrei modes, based mostly on the original Szendrei's paper [16]. Some partial results related to Theorem 1 can be found in [4][5][6][11][12][18]; a significant part of the survey [9] was devoted to the problem. Motivated by Example 2, the paper [7] is concerned with a broad class of modes that *do not* embed into semimodules. Related problems are discussed in the last section.

We quickly recall basic definitions. By a *commutative semiring* we mean an algebra $\boldsymbol{R} = (R, +, \cdot)$ such that both operations $+, \cdot$ are commutative and associative and distributive laws hold. A *semimodule* over a semiring $\boldsymbol{R}$ (or an $\boldsymbol{R}$*-semimodule*) is a "module without subtraction", it means an algebra $\boldsymbol{M} = (M, +, r\cdot : r \in R)$ such that $(M, +)$ is a commutative semigroup and $r\cdot$ are unary operations of multiplication by elements of $\boldsymbol{R}$ satisfying associative and distributive laws. Moreover, the semiring in our construction will be unitary, that is, it contains a unit element 1 which acts on semimodules as identity. Note that in $\boldsymbol{R}$-semimodules, a term t over variables $x_1, \ldots, x_n$ can always be written (uniquely) as

$$t = r_1 \cdot x_1 + \ldots + r_n \cdot x_n, \quad \text{for some } r_1, \ldots, r_n \in R.$$

An algebra $\boldsymbol{A}$ is called a *reduct* of an algebra $\boldsymbol{B}$, if all operations of $\boldsymbol{A}$ are term operations of $\boldsymbol{B}$. It is called a *subreduct*, if it is a subalgebra of a reduct of $\boldsymbol{B}$. (Sometimes we also say that $\boldsymbol{A}$ *embeds* into $\boldsymbol{B}$.)

In this paper, we consider algebras over an arbitrary signature Σ without constant symbols. An algebra is called *idempotent*, if each element forms a one-element subalgebra. Equivalently, if the identity

$$f(x, x, \dots, x) \approx x$$

holds for every operation f. An algebra is called *entropic*, if every pair of operations commute with one another. Equivalently, if the identity

$$f(g(x_{11}, \dots, x_{1n}), \dots, g(x_{m1}, \dots, x_{mn})) \approx g(f(x_{11}, \dots, x_{m1}), \dots, f(x_{1n}, \dots, x_{mn}))$$

holds for all operations f, g. Idempotent entropic algebras are called *modes*. The article [9] and the monograph [10] are good surveys of what is known in the theory of modes.

We say that an n-ary operation f satisfies *Szendrei identities* [14][16], if

$$f(f(x_{11}, \dots, x_{1n}), \dots, f(x_{n1}, \dots, x_{nn})) \\ \approx f(f(x_{\pi(11)}, \dots, x_{\pi(1n)}), \dots, f(x_{\pi(n1)}, \dots, x_{\pi(nn)}))$$

holds for every π, which is the permutation of the n^2 indices which fixes all indices except ij and ji, and switches these two, for some $1 \leq i,j \leq n$. (So we obtain $\binom{n}{2}$ identities.) Modes satisfying all Szendrei identities for every operation are called *Szendrei modes*. Note that Szendrei identities for an operation f imply that f commutes with itself, hence Szendrei algebras with a single operation are entropic. For a binary operation, there is just one Szendrei identity, and it is equivalent to the entropic identity; many authors call this identity *mediality* [4].

EXAMPLE 2. We define a ternary operation f on the set $\{0, 1, 2\}$ by

$$f(x, y, z) = \begin{cases} 2 - z & \text{if } x = y = 1, \\ z & \text{otherwise.} \end{cases}$$

So $f(f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3))$ is equal either $2 - z_3$, if $x_3 = y_3 = 1$ and $(z_1, z_2) \neq (1, 1)$, or $z_1 = z_2 = 1$ and $(x_3, y_3) \neq (1, 1)$; or it is equal z_3 otherwise. Consequently, the algebra $\boldsymbol{A} = (\{0, 1, 2\}, f)$ is a mode. However,

$$f(f(0,0,1), f(0,0,0), f(0,1,z)) = z \neq 2 - z = f(f(0,0,0), f(0,0,0), f(1,1,z))$$

for $z \neq 1$, so $\boldsymbol{A}$ is not a Szendrei mode.

The notation and terminology of universal algebra we use is rather standard and follows the book [8]. We assume the standard representation of free algebras in a variety $\mathcal{V}$ by terms modulo the identities of $\mathcal{V}$. Terms are considered as labeled rooted trees. Inner nodes are labeled by operation symbols, leaves by variables. *Depth* of a symbol/variable is defined as the distance from the root.

2. Free Szendrei modes.

Throughout the paper, we fix a signature Σ without constants (arity of a symbol σ will be denoted $\operatorname{ar}\sigma$) and let Ω denote the set of abstract symbols $\alpha_{\sigma,i}$ for every $\sigma \in \Sigma$ and $i = 1, \dots, \operatorname{ar}\sigma$, i.e.

$$\Omega = \{\alpha_{\sigma,i} : \sigma \in \Sigma,\ i = 1, \dots, \operatorname{ar}\sigma\}.$$

Let $\boldsymbol{R}_\Sigma$ denote the semiring with unit $\mathbb{N}[\Omega]/\theta$ of polynomials with (commutative) variables from Ω and coefficients from the set of natural numbers $\mathbb{N}$, modulo the congruence θ generated by all pairs

$$(\alpha_{\sigma,1} + \dots + \alpha_{\sigma,n},\ 1)$$

for every n-ary $\sigma \in \Sigma$. On every $\boldsymbol{R}_\Sigma$-semimodule $\boldsymbol{M}$, consider the operations g_σ defined by

$$g_\sigma(a_1, \dots, a_n) = \alpha_{\sigma,1} \cdot a_1 + \dots + \alpha_{\sigma,n} \cdot a_n$$

for every n-ary $\sigma \in \Sigma$. Since $\boldsymbol{R}_\Sigma$ is a commutative semiring, the algebra $(M, g_\sigma : \sigma \in \Sigma)$ is a Szendrei mode.

For a set A, we will denote

- $\boldsymbol{F}(A) = (F(A), +, r\cdot : r \in R_\Sigma)$ the free $\boldsymbol{R}_\Sigma$-semimodule over A;
- $\boldsymbol{G}(A) = (G(A), g_\sigma : \sigma \in \Sigma)$ the subalgebra of $(F(A), g_\sigma : \sigma \in \Sigma)$ generated by the set A.

Clearly, for $u \in F(A)$, we have $u \in G(A)$, iff there is a Σ-term t such that $u = t(a_1, \dots, a_n)$ for some $a_1, \dots, a_n \in A$.

THEOREM 3. *The algebra $\boldsymbol{G}(A)$ is a free Szendrei mode over the set A.*

The theorem is an easy consequence of results of Á. Szendrei [16]. We outline its proof in the rest of the section.

A term is called *completely expanded*, if all variables have equal depth. A completely expanded term is called *isosceles*, if at each particular depth level, all the nodes at that depth are labeled with the same operation symbol, except possibly the variables at the deepest level. E.g., the term $f(g(x, y), g(y, z))$ is isosceles, while $f(g(x, x), h(x, x))$ is not.

The *address* of an occurence of a symbol/variable σ of depth k in a term t is the sequence $(b_0, \dots, b_{k-1})$ of natural numbers such that the (shortest) path from the root to σ uses b_i-th branch of the tree on i-th depth level. The *trace* of an occurence of a symbol/variable σ of depth k in a term t is the sequence $(\sigma_0, \dots, \sigma_{k-1})$ of operation symbols such that the i-th node on the path from the root to σ is labeled by σ_i.

Thus, isosceles terms are precisely those terms, where all occurences of variables have the same trace; it is called the trace of an isosceles term. An identity $t \approx s$ is called *isosceles*, if both t, s are isosceles terms with the same trace.

LEMMA 4 ([16], Lemma 2.2). *For every pair of terms t_1, t_2, there are isosceles terms s_1, s_2 with the same trace such that $t_1 \approx s_1$ and $t_2 \approx s_2$ are provable from the idempotent identities. Consequently, every identity is equivalent to an isosceles identity (called an* isosceles expansion*) relative to the idempotent identities.*

Let Ω^* denote the set of all monomials

$$\prod_{\sigma\in\Sigma} \alpha_{\sigma,1}^{k_{\sigma,1}} \cdots \alpha_{\sigma,\mathrm{ar}\,\sigma}^{k_{\sigma,\mathrm{ar}\,\sigma}}$$

(it means that all but finitely many $k_{\sigma,i}$'s are zeros). For a given trace τ, let Ω_τ denote the set of all $\omega \in \Omega^*$ such that for each $\sigma \in \Sigma$, the sum $k_{\sigma,1} + \ldots + k_{\sigma,\mathrm{ar}\,\sigma}$ is equal to the number of occurences of σ in the trace τ. Thus Ω_τ consists of monomials that may appear in an interpretation of an isosceles term of trace τ in $\boldsymbol{G}(A)$.

LEMMA 5. *Let τ be a trace and $p = \sum_{\omega\in\Omega_\tau} c_\omega \omega$, $q = \sum_{\omega\in\Omega_\tau} d_\omega \omega$ two polynomials from $\mathbb{N}[\Omega]$. If they are θ-equivalent, then they are equal.*

PROOF. We pass the situation into the polynomial ring $\mathbb{Z}[\Omega]$: If p, q are θ-equivalent in $\mathbb{N}[\Omega]$, then they are equivalent also in the congruence generated by all pairs $(\alpha_{\sigma,1} + \cdots + \alpha_{\sigma,\mathrm{ar}\,\sigma}, 1)$, $\sigma \in \Sigma$, in $\mathbb{Z}[\Omega]$, and thus $p - q$ belongs to the ideal $\boldsymbol{I}$ of $\mathbb{Z}[\Omega]$ generated by all polynomials $g_\sigma = \alpha_{\sigma,1} + \cdots + \alpha_{\sigma,\mathrm{ar}\,\sigma} - 1$, $\sigma \in \Sigma$. We prove that this implies $p = q$ by showing that

(✠) ***I*** *does not contain a non-zero polynomial $f = \sum_{\omega\in\Omega_\tau} b_\omega \omega$ with $b_\omega \in \mathbb{Z}$.*

Note that the following conditions are equivalent:

(1) $f \in I$;

(2) there exist polynomials $f_\sigma \in \mathbb{Z}[\Omega]$ such that $f = \sum_{\sigma\in\Sigma} f_\sigma g_\sigma$;

(3) in f, substituting $1 - \alpha_{\sigma,2} - \ldots - \alpha_{\sigma,\mathrm{ar}\,\sigma}$ for every $\alpha_{\sigma,1}$, yields zero polynomial.

It follows from (3) that $\boldsymbol{I}$ is a prime ideal. We prove (⌘) by induction on $k = \sum_{\sigma\in\tau} \mathrm{ar}\,\sigma$. If all symbols in τ have arity 1, then $f = b\omega$ and it fails condition (3) unless $b = 0$. Otherwise, consider a counterexample $f \in \mathbb{Z}[\Omega]$ of minimal degree. Choose an arbitrary σ of arity > 1. The variable $\alpha_{\sigma,n} \in \Omega$ does not divide f: if it did, we had $f = \alpha_{\sigma,n} \cdot g$, and thus, by primeness, $g \in I$ would be a smaller counterexample. So, substituting 0 for $\alpha_{\sigma,n}$ in f yields a non-zero polynomial, which, as follows from (2), belongs to the respective ideal $\boldsymbol{I}$ in $\mathbb{Z}[\Omega \setminus \{\alpha_{\sigma,n}\}]$, hence we reduced k by one. □

Let t be an isosceles term with trace τ. We say that an occurence of a variable in t has the property $\delta(\omega)$ for an $\omega \in \Omega_\tau$, if it can be reached by $k_{\sigma,i}$ choices of i-th branch in the nodes labeled by σ. Finally, for every $\omega \in \Omega_\tau$, let $\Delta(\omega, x, t)$ denote the number of occurences of the variable x in t with the property $\delta(\omega)$. E.g., if $t = f(g(x, y), g(y, z))$, then $\Delta(\alpha_{f,1}\alpha_{g,1}, x, t) = \Delta(\alpha_{f,1}\alpha_{g,2}, y, t) = \Delta(\alpha_{f,2}\alpha_{g,1}, y, t) = \Delta(\alpha_{f,2}\alpha_{g,2}, z, t) = 1$. If $t = f(f(x, y), f(y, z))$, then $\Delta(\alpha_{f,1}\alpha_{f,2}, y, t) = 2$.

Lemma 6 ([16], Theorem 2.8). *The following statements are equivalent for an isosceles identity $t \approx s$.*

(1) *$t \approx s$ is provable from entropy and Szendrei identities.*

(2) *$\Delta(\omega, x, t) = \Delta(\omega, x, s)$ for every variable x that occurs in t or s and every $\omega \in \Omega_\tau$.*

Proposition 7. *The following statements are equivalent for terms t, s over variables $x_1, \ldots, x_m$.*

(1) *$t \approx s$ holds in all Szendrei modes.*

(2) *There is an isosceles expansion $t^* \approx s^*$ of the identity $t \approx s$ that is provable from entropy and Szendrei identities.*

(3) *Any isosceles expansion $t^* \approx s^*$ of the identity $t \approx s$ is provable from entropy and Szendrei identities.*

(4) *$t(a_1, \ldots, a_m) = s(a_1, \ldots, a_m)$ holds in the algebra $\boldsymbol{G}(a_1, \ldots, a_m)$.*

Proof. (3) ⇒ (2) ⇒ (1) ⇒ (4) are trivial. We prove (4) ⇒ (3).

Assume the equality $t(a_1, \ldots, a_m) = s(a_1, \ldots, a_m)$ in $\boldsymbol{G}(a_1, \ldots, a_m)$. Then also $t^*(a_1,\dots,a_m) = s^*(a_1,\dots,a_m)$ for any isosceles expansion $t^* \approx s^*$ of the identity $t \approx s$. Let τ be its trace. Then

$$t^*(a_1,\dots,a_m) = \sum_{i=1}^{m}\left(\sum_{\omega\in\Omega_\tau} c_{i,\omega}\omega\right)\cdot a_i \text{ and } s^*(a_1,\dots,a_m) = \sum_{i=1}^{m}\left(\sum_{\omega\in\Omega_\tau} d_{i,\omega}\omega\right)\cdot a_i,$$

where $c_{i\omega} = \Delta(\omega, x_i, t^*)$ and $d_{i\omega} = \Delta(\omega, x_i, s^*)$. Hence

$$\sum_{i=1}^{m}\left(\sum_{\omega\in\Omega_\tau} c_{i,\omega}\omega\right)\cdot a_i = \sum_{i=1}^{m}\left(\sum_{\omega\in\Omega_\tau} d_{i,\omega}\omega\right)\cdot a_i$$

holds in the free $\boldsymbol{R}_\Sigma$-semimodule over $a_1,\dots,a_m$ and, consequently, the polynomials $\sum_{\omega\in\Omega_\tau} c_{i,\omega}\omega$ and $\sum_{\omega\in\Omega_\tau} d_{i,\omega}\omega$ are θ-equivalent for every i, and so, by Lemma 5, are equal. Particularly, $\Delta(\omega, x_i, t^*) = c_{i,\omega} = d_{i,\omega} = \Delta(\omega, x_i, s^*)$ for every ω and i and we can use Lemma 6. □

Theorem 3 follows immediately from Proposition 7.

3. Proof of Theorem 1.

Since subreducts of semimodules over commutative semirings satisfy both entropy and Szendrei identities, one implication of Theorem 1 is clear. In the rest of the section, we prove the converse.

Let $\boldsymbol{A} = (A, f_\sigma : \sigma \in \Sigma)$ be an arbitrary Szendrei mode and let's denote π the projection of the free Szendrei mode $\boldsymbol{G}(A) = (G(A), g_\sigma : \sigma \in \Sigma)$ onto the algebra $\boldsymbol{A}$, extending the identity mapping on generators. We define a relation ρ on $F(A)$ consisting of all pairs

$$(w + \omega\cdot b,\ w + \omega\alpha_{\sigma,1}\cdot a_1 + \ldots + \omega\alpha_{\sigma,n}\cdot a_n),$$

where $\sigma \in \Sigma$ is an n-ary symbol, $w \in F(A)$, $\omega \in \Omega^*$ and $a_1,\dots,a_n, b \in A$ such that $b = f_\sigma(a_1,\dots,a_n)$.

Lemma 8. *Let $(u, v) \in \rho$. Then $u \in G(A)$ iff $v \in G(A)$. Moreover, if $u, v \in G(A)$, then $\pi(u) = \pi(v)$.*

Proof. Let $u \in G(A)$. Then $u = t(a_1,\dots,a_k)$ for some k-ary Σ-term t and some $a_1,\dots,a_k \in A$. Since $(u, v) \in \rho$, we have

$$u = w + \omega\cdot a_i$$

for certain $1 \le i \le k$ and

$$v = w + \omega\alpha_{\sigma,1} \cdot b_1 + \ldots + \omega\alpha_{\sigma,n} \cdot b_n$$

for some n-ary $\sigma \in \Sigma$, $w \in F(A)$, $\omega \in \Omega^*$ and $b_1, \ldots, b_n \in A$ such that $a_i = f_\sigma(b_1, \ldots, b_n)$ in $\boldsymbol{A}$. Let s be the term resulting from t by replacing one of the occurence of a_i with $\delta(\omega)$ property with the term $\sigma(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are new variables. Then $v = s(a_1, \ldots, a_k, b_1, \ldots, b_n)$ and thus $v \in G(A)$.

[Example: Let $\boldsymbol{A} = (A, *)$ (thus $\Omega = \{\alpha_{*,1}, \alpha_{*,2}\}$), $t = x * y$, $u = t(a, b) = \alpha_{*,1}a + \alpha_{*,2}b$, and $b = c * d$. Then $s(x, y, u, v) = x * (u * v)$, and so $v = s(a, b, c, d) = \alpha_{*,1}a + \alpha_{*,2}\alpha_{*,1}c + \alpha_{*,2}\alpha_{*,2}d$].

Now, let $v \in G(A)$. Then $v = s(a_1, \ldots, a_k)$ for some k-ary Σ-term s and some $a_1, \ldots, a_k \in A$. Since $(u, v) \in \rho$, we have

$$u = w + \omega \cdot b$$

and

$$v = w + \omega\alpha_{\sigma,1} \cdot a_{j_1} + \ldots + \omega\alpha_{\sigma,n} \cdot a_{j_n}$$

for some n-ary $\sigma \in \Sigma$, $w \in F(A)$, $\omega \in \Omega^*$, certain $1 \le j_1, \ldots, j_n \le k$ and $b \in A$ such that $b = f_\sigma(a_{j_1}, \ldots, a_{j_n})$. According to Lemma 4, $v = s'(a_1, \ldots, a_k)$ for an isosceles term s', and thus, according to Lemma 6, $v = s''(a_1, \ldots, a_k)$ for an isosceles term s'', in which the involved occurences of $a_{j_1}, \ldots, a_{j_n}$ are next each other, i.e., they form a subterm $\sigma(a_{j_1}, \ldots, a_{j_n})$. (Recall that Lemma 6 allows to switch any two occurences with the same $\delta(\omega)$ property.) Now, let t be the term that results from s'' by replacing the subterm $\sigma(a_{j_1}, \ldots, a_{j_n})$ by a single new variable. Then $u = t(a_1, \ldots, a_k, b)$ and thus $u \in G(A)$.

[Example: Let $\boldsymbol{A} = (A, *)$, $s = (x * y) * (u * v)$, $v = s(a, b, c, d) = \alpha_{*,1}\alpha_{*,1}a + \alpha_{*,1}\alpha_{*,2}b + \alpha_{*,2}\alpha_{*,1}c + \alpha_{*,2}\alpha_{*,2}d$, and $e = b * d$ — this perfectly fine constellation, since $\alpha_{*,1}\alpha_{*,2} = \alpha_{*,2}\alpha_{*,1}$. Then $s' = s$ and $s'' = (x * u) * (y * v)$, so that $v = s''(a, b, c, d)$, and we may define $t(x, y, z, u, w) = (x * u) * w$. Then $u = t(a, b, c, d, e) = \alpha_{*,1}\alpha_{*,1}a + \alpha_{*,1}\alpha_{*,2}c + \alpha_{*,2}e$.]

So, as we have seen, if $u, v \in G(A)$ and $(u, v) \in \rho$, then we can write $u = t(a_1, \ldots, a_k)$ and $v = s(a_1, \ldots, a_k)$ for terms t, s such that s results from t by replacing an occurence of a variable b by the subterm $\sigma(b_1, \ldots, b_n)$, for some $b, b_1, \ldots, b_n \in \{a_1, \ldots, a_k\}$ with $b = f_\sigma(b_1, \ldots, b_n)$ in $\boldsymbol{A}$. Hence, because π is a homomorphism identical on A, we have $\pi(u) = \pi(v)$. □

Let $\bar{\rho}$ be the symmetric transitive closure of ρ. Then $\bar{\rho}$ is a congruence of the $\boldsymbol{R}_\Sigma$-semimodule $\boldsymbol{F}(A)$, so $\boldsymbol{F}(A)/\bar{\rho}$ is again an $\boldsymbol{R}_\Sigma$-semimodule.

LEMMA 9. *The Szendrei mode* $\boldsymbol{A}$ *embeds into the reduct* $(F(A)/\bar{\rho},\ g_\sigma : \sigma \in \Sigma)$ *of the* $\boldsymbol{R}_\Sigma$*-semimodule* $\boldsymbol{F}(A)/\bar{\rho}$.

PROOF. The embedding is $a \mapsto [a]_{\bar{\rho}}$. This is a homomorphism, because

$$g_\sigma([a_1]_{\bar{\rho}}, \dots, [a_n]_{\bar{\rho}}) = \alpha_{\sigma,1} \cdot [a_1]_{\bar{\rho}} + \dots + \alpha_{\sigma,n} \cdot [a_n]_{\bar{\rho}}$$

$$= [\alpha_{\sigma,1} \cdot a_1 + \dots + \alpha_{\sigma,n} \cdot a_n]_{\bar{\rho}} = [f_\sigma(a_1, \dots, a_n)]_{\bar{\rho}}.$$

(The first equality is the definition of g_σ, the last follows from the definition of ρ.) So it remains to prove that the mapping is injective. Assume $[a]_{\bar{\rho}} = [b]_{\bar{\rho}}$ for some $a, b \in A$, it means $(a, b) \in \bar{\rho}$. Hence there is a chain $a = u_0, u_1, \dots, u_{n-1}, u_n = b$ such that $(u_i, u_{i+1}) \in \rho \cup \rho^{-1}$. It follows from Lemma 8 that $u_0, \dots, u_n \in G(A)$ and thus that $\pi(u_0) = \pi(u_1) = \dots = \pi(u_n)$. However, $\pi(a) = \pi(b)$ iff $a = b$, because π is the identity on A. □

This ultimately proves Theorem 1.

4. Concluding remarks.

Two similar types of representation appear in the literature:

- *Quasi-(semi)linear algebras* are subreducts of (semi)modules; their operations can be expressed as (semi)module terms, i.e. $r_1 \cdot x_1 + \dots + r_n \cdot x_n$.
- *Quasi-(semi)affine algebras* are subreducts of (semi)modules with additional constants pointing to every element; their operations can be expressed as (semi)module polynomials, i.e. $c + r_1 \cdot x_1 + \dots + r_n \cdot x_n$ with a constant c.

In this terminology, what we did, is characterizing *idempotent quasi-semilinear algebras over commutative semirings*.

We wish to discuss a couple of related questions. First, why do we consider *idempotent* subreducts only? One reason is that my original intention was to answer the open problem posed in [10], to characterize *modes* embeddable into semimodules over commutative semirings. Even when Stronkowski's result appeared, it was still desirable to find a short and transparent proof for the idempotent case. A characterization of not necessarily idempotent subreducts is an open problem.

Regarding semilinear representations over *general semirings*, the problem is ultimately solved. J. Ježek [3] proved that actually *every* algebra (without constants) is a subreduct of a semimodule over a semiring.

And what about *semiaffine* representations? Since idempotent quasi-semiaffine algebras over commutative semirings are also Szendrei modes, we obtain "quasi-semiaffine over c.s. $\Leftrightarrow$ quasi-semilinear over c.s." for idempotent algebras. However, according to Ježek and Kepka [4], there is a (non-idempotent) algebra which is quasi-semiaffine over c.s. but not quasi-semilinear over c.s.

What about subreducts of *modules*? We don't know any general results about quasi-linear algebras, but there are several papers on quasi-affine algebras. Indeed, they are *abelian*, in the sense of commutator theory [1]. Not all abelian algebras are quasi-affine, though this is true under various additional assumptions, such as congruence modularity [2]. R. Quackenbush [13] proved that quasi-affine algebras form a quasivariety, axiomatized by a scheme of quasiidentities that could be considered as a "more restrictive abelianess". For more information, see the survey paper [17]. We don't know whether quasi-affine algebras without constants are quasi-linear.

Finally, let's look at representations over *commutative* rings. Particularly, which *modes* are embeddable into modules over commutative rings? Chapter 7 of the book [10] is devoted to this problem. For instance, cancellative modes are quasi-linear, and [15] contains a non-idempotent generalization of this statement. However, no characterization is known. Quasi-linear and quasi-affine algebras are abelian. It is not difficult to prove that abelian modes satisfy Szendrei identities. Is it true that all abelian modes are quasi-linear (or quasi-affine) over commutative rings?

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REFERENCES

[1] R. FREESE - R. MCKENZIE, *Commutator theory for congruence modular varieties*, London Mathematical Society Lecture Note Series, **125** (Cambridge University Press, 1987).

[2] C. HERRMANN, *Affine algebras in congruence modular varieties,* Acta. Sci. Math., **41** (1979), pp. 119–125.

[3] J. JEŽEK, *Terms and semiterms,* Comment. Math. Univ. Carolinae, **20** (1979), pp. 447–460.
[4] J. JEŽEK - T. KEPKA, *Medial groupoids,* Rozpravy ČSAV 93/2, 1983.
[5] J. JEŽEK - T. KEPKA, *Linear equational theories and semimodule representations,* Int. J. Algebra Comput., 8 (1998), pp. 599–615.
[6] K. KEARNES, *Semilattice modes I: the associated semiring,* Algebra Universalis, **34**, 2 (1995), pp. 220–272.
[7] A. KRAVCHENKO - A. PILITOWSKA - A. ROMANOWSKA - D. STANOVSKÝ, *Differential modes,* Internat. J. Algebra Comput., **18**, 3 (2008), pp. 567–588.
[8] R. MCKENZIE - G. MCNULTY - W. TAYLOR, *Algebras, Lattices, Varieties, Volume I.* Wadsworth & Brooks/Cole, 1987.
[9] A. ROMANOWSKA, *Semi-affine modes and modals,* Scientiae Mathematicae Japonicae, **61** (2005), pp. 159–194.
[10] A. ROMANOWSKA - JDH SMITH, *Modes.* World Scientific, 2002.
[11] A. ROMANOWSKA - A. ZAMOJSKA-DZIENIO, *Embedding semilattice sums of cancellative modes into semimodules,* Contributions to General Algebra, **13** (2001), pp. 295–303.
[12] A. ROMANOWSKA - A. ZAMOJSKA-DZIENIO, *Embedding sums of cancellative modes into semimodules,* Czech. Math. J., **55**, 4 (2005), pp. 975–991.
[13] R. QUACKENBUSH, *Quasi-affine algebras,* Algebra Universalis, **20** (1985), pp. 318–327.
[14] M. STRONKOWSKI, *On free modes,* Comment. Math. Univ. Carolinae, **47**, 4 (2006), pp. 561–568.
[15] M. STRONKOWSKI, *On embeddings of entropic algebras,* PhD Thesis, Warsaw University of Technology, 2006.
[16] Á. SZENDREI, *Identities satisfied by convex linear forms,* Algebra Universalis, **12** (1981), pp. 103–122.
[17] Á. SZENDREI, *Modules in general algebra,* in: Contributions to General Algebra, **10** (Proc. Klagenfurt Conf., 1997), Verlag Johannes Heyn, Klagenfurt, 1998; pp. 41–53.
[18] A. ZAMOJSKA-DZIENIO, *Medial modes and rectangular algebras,* Comment. Math. Univ. Carolinae, **47**, 1 (2006), pp. 21–34.

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