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Testing for Cotorsionness over Domains.

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ABSTRACT - We are looking for reduced modules over an integral domain to test the cotorsionness of modules. Our focus is on the cotorsion theories due to Matlis, Enochs and Warfield, respectively. If κ is a cardinal such that $\kappa^{\aleph_0} = 2^\kappa$, then for domains R for which Q/R (Q is the quotient field of R) is not self-small there is a reduced strongly flat R-module U_κ of rank 2^κ that can be used to test whether or not an R-module of cardinality $\leq 2^\kappa$ is Matlis cotorsion; however, no such module exists if the cardinality restriction is removed. We also establish the existence of reduced modules which test Matlis cotorsion modules for Enochs or Warfield cotorsionness.

For a given torsion-free or flat R-module A of cardinality κ, we construct an Enochs, resp. Matlis cotorsion R-module M of cardinality $\leq 2^\kappa$ with $\mathrm{Ext}^1_R(A, M) = 0$ that is not Warfield, resp. not Enochs cotorsion.

1. Introduction.

We consider modules over integral domains R. The field of quotients of R will be denoted by Q, and we write $K = Q/R$. Throughout it is assumed that $Q \neq R$.

The classical cotorsion theory of abelian groups (due to Harrison [13]) has been extended to modules over arbitrary integral domains in three inequivalent ways (see e.g. Bazzoni-Salce [1] or Göbel-Trlifaj [12]). An R-module C is called *Matlis cotorsion* if $\mathrm{Ext}^1_R(Q, C) = 0$, *Enochs cotorsion* if $\mathrm{Ext}^1_R(F, C) = 0$ for all flat R-modules F, and *Warfield cotorsion* if $\mathrm{Ext}^1_R(G, C) = 0$ for all torsion-free R-modules G. Evidently,

$$\text{Warfield cotorsion} \Rightarrow \text{Enochs cotorsion} \Rightarrow \text{Matlis cotorsion},$$

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and examples show that in general the reverse implications fail. (However, all these cotorsion theories are equivalent for Dedekind domains.) Note that for reduced torsion-free modules, Matlis cotorsion means R-completeness, pure-injective modules are Enochs cotorsion, while Warfield cotorsion is equivalent to RD-injectivity (for RD-injectivity see e.g. Fuchs-Salce [7]).

An R-module A satisfying $\mathrm{Ext}^1_R(A, C) = 0$ for all Matlis cotorsion modules C is said to be *strongly flat* (see Bazzoni-Salce [2]), while the modules A that satisfy $\mathrm{Ext}^1_R(A, C) = 0$ for all Enochs, resp. Warfield cotorsion modules C are precisely the flat, resp. the torsion-free R-modules. In particular, the class of strongly flat R-modules and the class of Matlis cotorsion modules form a cotorsion theory in the sense of Salce [18], and so do the other two pairs of corresponding classes.

In this note we wish to discuss two problems. It is well known (see Enochs-Jenda [6]) that for all the three cotorsion theories mentioned above there exist test modules for cotorsionness, but none of them is reduced: they all contain a copy of Q. We are wondering if there exist *reduced* modules to test cotorsionness; more precisely, we would like to find

a) a reduced strongly flat module U such that $\mathrm{Ext}^1_R(U, C) = 0$ implies that C is Matlis cotorsion,

b) a reduced flat (but not strongly flat) module U such that the vanishing of the same Ext for a Matlis cotorsion C implies that C is Enochs cotorsion, and

c) a reduced torsion-free non-flat U for which $\mathrm{Ext}^1_R(U, C) = 0$ for an Enochs cotorsion C implies C is Warfield cotorsion.

While questions b) and c) have easy affirmative answers for all domains (see Theorems 4.1 and 4.2), the answer is in the negative for a) (cf. Corollary 3.4), though in cases in which K is not self-small cardinality restrictions yield more satisfactory results (Theorem 2.4).

Another problem we address is concerned with vanishing Ext. If A is a flat, but not strongly flat (torsion-free, but not flat) R-module, then in view of the cotorsion theories mentioned above we know that there must exist Matlis (Enochs) cotorsion modules M that are not Enochs (Warfield) cotorsion satisfying $\mathrm{Ext}^1_R(A, M) = 0$. We intend to show how to find in both cases such an M whose cardinality does not exceed $2^{|A|}$ (Theorems 5.1 and 5.2). The construction of vanishing Ext was pioneered by Eklof-Trlifaj [5]; here a different method will be adapted that will be used repeatedly.

In the final section our focus is on modules over Matlis domains. We prove that there exist arbitrarily large cardinals κ such that there are torsion-free modules of cardinality 2^{2^κ} over such domains that are not Matlis cotorsion, but all of their torsion-free epic images of cardinalities $\leq 2^\kappa$ are Matlis cotorsion (Theorem 6.1). We would like to thank the referee for very useful suggestions incorporated in this article.

2. Reduced test modules for Matlis cotorsion.

We will use the notation $|R| = \rho$ for the cardinality of the domain. $\widetilde{M}$ will stand for the Matlis cotorsion hull of the R-module M. If M is h-reduced (i.e. has no h-divisible submodule $\neq 0$), then M is a submodule of $\widetilde{M} = \operatorname{Ext}^1_R(Q/R, M)$, and if M is in addition torsion-free, then this is just the completion of M in its R-topology.

Lemma 2.1. *If M denotes an h-reduced R-module of cardinality κ, then the cardinality of $\widetilde{M}/M = \operatorname{Ext}^1_R(Q, M)$ is at most κ^ρ.*

Proof. Let $0 \to H \to F \to Q \to 0$ be an exact sequence with F a free R-module of rank ρ. The induced exact sequence

$$\operatorname{Hom}_R(F, M) \to \operatorname{Hom}_R(H, M) \to \operatorname{Ext}^1_R(Q, M) \to \operatorname{Ext}^1_R(F, M) = 0$$

shows that $|\operatorname{Ext}^1_R(Q, M)| \leq |\operatorname{Hom}_R(H, M)|$. The latter cardinal is certainly not larger than $|M|^{|H|} \leq \kappa^\rho$. □

It turns out that among the reduced R-modules there exists no universal test module for Matlis cotorsionness, but there might exist some that test modules up to certain cardinalities. In our search for such reduced test modules, it is reasonable to concentrate on strongly flat modules.

Extensions of free R-modules by torsion-free divisible modules are immediately seen to be strongly flat, and of course so are their summands. The converse is also true (Bazzoni-Salce [1]): all strongly flat modules are summands of modules U that fit into an exact sequence of the form

$$0 \to F \to U \to D \to 0, \tag{1}$$

where F is a free R-module and D is a torsion-free divisible R-module (thus a direct sum of copies of Q). If U is reduced, then F is a dense submodule of U in the R-topology, and on the other hand, U is a pure submodule in the R-completion $\widetilde{F}$ of F.

Let us introduce some *ad hoc* terminology to simplify our statements. An infinite cardinal κ will be called *'good'* if it satisfies $\kappa^{\aleph_0} = 2^\kappa$.

E.g. $\aleph_0$ is a 'good' cardinal in this sense, but $2^{\aleph_0}$ is not. Note that arbitrarily large cardinals κ with this property exist, e.g. choose κ as a strong limit cardinal of cofinality ω; cf. Jech [15, p. 49]. Furthermore, a reduced strongly flat R-module U_κ will be said to be *'good'* if κ is a 'good' cardinal $> |R|$ and there is an exact sequence (1) where U_κ is the middle term and

(i) F is a free R-module of rank κ,
(ii) D is a torsion-free divisible R-module of rank $\kappa^{\aleph_0} = 2^\kappa$.

Observe that if U in (1) is a 'good' module, then so is $\widetilde{F}$, because by Lemma 2.1 we have $\kappa^{\aleph_0} \leq \kappa^\rho \leq \kappa^{\aleph_0 \kappa} = (2^\kappa)^\kappa = 2^\kappa$, thus $|\widetilde{F}| = 2^\kappa$. On the other hand, if $\widetilde{F}$ is 'good', then there is a large supply of 'good' modules between F and $\widetilde{F}$.

Let us point out that for certain domains R 'good' reduced strongly flat modules U_κ may not exist at all.

EXAMPLE 1. If R is an R-complete valuation domain with uncountably generated Q, then all free R-modules are R-complete (see e.g. Fuchs-Salce [7, p. 281]). Thus $F = U_\kappa$ in this situation, and so no 'good' U_κ may exist for any κ.

EXAMPLE 2. Let $R = \mathbb{Z}[X]$ be the polynomial ring where X stands for uncountably many indeterminates. It is shown in Göbel-May [10, Example 3.8] that R is complete in the S-topology, where S is the set of monomials in the indeterminates. The R-topology on R is finer than the S-topology, thus R is R-complete as well. It is readily seen that the same holds for all free R-modules, so in this case no 'good' U_κ may exist.

An example for a domain where 'good' modules U_κ exist in a large supply is provided by any Matlis domain, see Section 6 *infra*.

A necessary and sufficient condition for a domain R to admit arbitrarily large 'good' modules U_κ is given in the next theorem.

THEOREM 2.2. *A domain R admits arbitrarily large 'good' reduced strongly flat modules U_κ if and only if the R-module $K = Q/R$ is not self-small.*

PROOF. We are making use of the Matlis category equivalence (see Matlis [17] or Fuchs-Salce [7, p. 280]) when we use the isomorphism $\widetilde{F} \cong \mathrm{Hom}_R(K, K \otimes_R F) = \mathrm{Hom}_R(K, \oplus_\kappa K)$ to obtain the Matlis cotorsion hull (R-completion) $\widetilde{F}$ of the free R-module F of rank κ.

First suppose that K is self-small. Then $\widetilde{F} \cong \mathrm{Hom}_R(K, \oplus_\kappa K) \cong \oplus_\kappa \mathrm{Hom}_R(K,K) \cong \oplus_\kappa \widetilde{R}$. Therefore, if $\kappa > |\widetilde{R}|$, then $|\widetilde{F}| = \kappa$, so we cannot find a 'good' module U_κ whenever $\kappa > |\widetilde{R}|$.

Next assume K is not self-small. This means that there exists a homomorphism $\eta : K \to \oplus_{\aleph_0} K$ such that all the projections of $\mathrm{Im}\,\eta$ on the summands are different from 0. Now let κ be any 'good' cardinal $> |\widetilde{R}|$. For every countable subset I of κ, we can map K via η into $\oplus_I K$, and these maps are different for different subsets I of κ. Since there are $\kappa^{\aleph_0}$ ways to select I, we conclude that $|\widetilde{F}| \geq \kappa^{\aleph_0}$. Consequently, $|\widetilde{F}/F| \geq \kappa^{\aleph_0}$, and so such an $\widetilde{F}$ serves as a 'good' U_κ. □

Note that K can not be self-small if it decomposes into a direct sum of infinitely many non-zero summands. This is the case e.g. if R is an h-local domain with infinitely many maximal primes. A useful sufficient criterion for K to be not self-small is contained in the following simple lemma.

LEMMA 2.3. *If the module K admits a countably generated direct summand $\neq 0$, then it is not self-small.*

PROOF. It is enough to show that a countably generated direct summand T of K is not self-small. T is divisible, so it is of the form $T = \cup_{n<\omega} r_n^{-1}R/R$ with suitable $r_n \in R$ with $r_n^{-1}R \subset r_{n+1}^{-1}R$ for all n. We get an embedding of T in a countable direct sum of copies of T by embedding T in the nth copy of T after applying multiplication by r_n. Thus T is not self-small, so in this case by Theorem 2.2 there exist arbitrarily large 'good' modules. □

Next we prove that a 'good' strongly flat module U_κ can serve as a test module for modules of cardinality not exceeding 2^κ. (We adapt cardinality arguments similar to those used by Hunter [14].)

THEOREM 2.4. *Let $U = U_\kappa$ be a 'good' strongly flat module for a 'good' cardinal κ, and suppose $|R| \leq \kappa$. If C is any R-module of cardinality $\leq 2^\kappa$, then $\mathrm{Ext}^1_R(U, C) = 0$ implies that C is Matlis cotorsion.*

PROOF. The exact sequence (1) induces the exact sequence

$$\mathrm{Hom}_R(F, C) \to \mathrm{Ext}^1_R(D, C) \to \mathrm{Ext}^1_R(U, C) \to \mathrm{Ext}^1_R(F, C) = 0. \tag{2}$$

Working toward contradiction, assume C is not Matlis cotorsion, i.e. $\mathrm{Ext}^1_R(Q, C) \neq 0$. Evidently, $|\mathrm{Hom}_R(F, C)| = |C|^\kappa \leq (2^\kappa)^\kappa = 2^\kappa$, while we have

$$|\operatorname{Ext}^1_R(D,C)| = |\operatorname{Ext}^1_R(\oplus_{2^\kappa} Q, C)| = \prod_{2^\kappa} |\operatorname{Ext}^1_R(Q,C)| \geq 2^{2^\kappa}.$$

The inequality $2^\kappa < 2^{2^\kappa}$ implies that in the exact sequence (2) it is impossible to have $\operatorname{Ext}^1_R(U,C) = 0$. □

In case $R = \mathbb{Z}$, we have $\rho = \aleph_0$. Choosing $\kappa = \aleph_0$, it is clear that a large variety of abelian groups can be chosen as test modules for (Matlis) cotorsionness of abelian groups up to the power of the continuum. Göbel-Prelle [11] point out that the collection of, and hence the direct sum of, the slender subgroups of the Baer-Specker group $\mathbb{Z}^{\aleph_0}$ is an adequate test group. Our result asserts that even a single adequate slender subgroup $U_{\aleph_0}$ will suffice.

The preceding theorem is a sharp result in the sense that for modules of cardinalities $\geq 2^{2^\kappa}$ a 'good' U_κ cannot be a test module for Matlis cotorsionness. In fact, this follows from the next result (choose $A = U_\lambda$ with $\lambda = 2^\kappa$).

Proposition 2.5. *For every reduced strongly flat R-module A of cardinality $\lambda > |R|$ there exists an R-module M of cardinality $\leq 2^\lambda$ such that*

(i) $\operatorname{Ext}^1_R(A,M) = 0$, and
(ii) M is not Matlis cotorsion.

Proof. Starting with the list of all R-homomorphisms $\phi_i : A \to$ $\to Q$ $(i \in I)$, define the module $B = \bigoplus_{i\in I} A$ along with the map $\psi = \oplus_{i\in I}\phi_i : B \to Q$; thus every ϕ_i lifts to a map $\sigma_i : A \to B$ such that $\phi_i = \psi\sigma_i$. Next extend ψ to $\widetilde{\psi} : \widetilde{B} \to Q$ from the Matlis cotorsion hull $\widetilde{B}$ of B to Q. Then $M = \operatorname{Ker}\widetilde{\psi}$ is as desired. In fact, $\widetilde{B}/M \cong Q$ and $\widetilde{M} = \widetilde{B}$, so M is not Matlis cotorsion. It is straightforward to verify that $|M| \leq 2^\lambda$. Finally, the vanishing of Ext in (i) follows from the fact that in the exact sequence $\operatorname{Hom}_R(A,\widetilde{B}) \to \operatorname{Hom}_R(A,Q) \to \operatorname{Ext}^1_R(A,M) \to \operatorname{Ext}^1_R(A,\widetilde{B}) = 0$ the map between the Homs is by construction surjective. □

3. A construction.

That no reduced R-module of cardinality $\leq \kappa$ can exist which could test modules of cardinalities $\geq 2^\kappa$ for Matlis cotorsionness will follow from the corollary of the next theorem. The theorem that follows is essentially Theorem 2 in Eklof-Trlifaj's paper [5] for which we give a different proof, using injective resolutions rather than free presentations. (This method will be used in the proofs of Theorems 3.2, 5.1 and 5.2.)

THEOREM 3.1. *Suppose that κ and λ are infinite cardinals such that $|R| \leq \kappa$ and $\lambda^\kappa = \lambda$. For every pair L, A of R-modules of cardinality $\leq \kappa$, there exists an R-module M such that*

(i) $|M| \leq \lambda$;
(ii) $\mathrm{Ext}^1_R(A, M) = 0$;
(iii) *M is the union of a well-ordered ascending chain of submodules M_σ $(\sigma < \lambda)$ with $M_0 = L$ and $M_{\sigma+1}/M_\sigma$ a direct sum of copies of A.*

PROOF. We may assume that L is not injective. Our starting point is the exact sequence $0 \to L \to E(L) \to D_0 \to 0$, where $E(L)$ denotes the injective hull of L. Thus $D_0 \neq 0$ is a divisible torsion module. Observe that $|E(L)| \leq \kappa^\kappa \leq \lambda$. We follow two steps to obtain another short exact sequence.

First, we replace $E(L)$ by its direct sum with a direct sum of copies of A and extend the map $\phi : E(L) \to D_0$ accordingly so as to obtain an exact sequence in the second row of the following commutative diagram such that every homomorphism $A \to D_0$ lifts to a homomorphism $A \to C_0$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_0 = L & \longrightarrow & E(L) & \xrightarrow{\phi} & D_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M_1 & \longrightarrow & C_0 = \oplus A \oplus E(L) & \xrightarrow{\psi} & D_0 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & C_1 = E(C_0) & \xrightarrow{\phi_1} & D_1 & \longrightarrow & 0.
\end{array}$$

The next step is to replace the middle term C_0 by its injective hull $C_1 = E(C_0)$ and modify the quotient D_0 accordingly (keeping the kernel unchanged); D_1 is likewise torsion divisible. The arising diagram is commutative, with all the vertical arrows representing inclusions.

We repeat the step of obtaining the third row from the first one transfinitely, continuing from the last row of the preceding diagram, and taking direct limits of exact sequences at limit ordinals. The λth step yields an exact sequence

$$(4) \qquad 0 \to M = \bigcup_{\sigma<\lambda} M_\sigma \to C = \bigcup_{\sigma<\lambda} E(C_\sigma) \stackrel{\phi^*}{\longrightarrow} D = \bigcup_{\sigma<\lambda} D_\sigma \to 0.$$

Hence we obtain the induced exact sequence

$$(5) \qquad \mathrm{Hom}_R(A, C) \to \mathrm{Hom}_R(A, D) \to \mathrm{Ext}^1_R(A, M) \to \mathrm{Ext}^1_R(A, C).$$

The map between the Homs is surjective, since $|A| \le \kappa < \mathrm{cf}\,\lambda$ implies that the image of any homomorphism $A \to D$ is contained already in some D_σ with $\sigma < \lambda$, so it lifts to a map $A \to E(C_{\sigma+1}) < C$. Furthermore, the last term vanishes (though C need not be injective), because the factor set of any extension of C by A already belongs to $E(C_\sigma)$ for some $\sigma < \lambda$, and therefore it has to be a transformation set, indicating splitting. This establishes the equality $\mathrm{Ext}^1_R(A, M) = 0$.

If $|D_\sigma| \le \lambda$, then because of $|A| \le \kappa$ the cardinality of the set of homomorphisms $A \to D_\sigma$ is at most $\lambda^\kappa = \lambda$. Therefore, we have $|C_{\sigma+1}| \le \lambda$ and $|D_{\sigma+1}| \le \lambda$, whence the inequality $|C| \le \lambda$, and so also $|M| \le \lambda$ follows.

The step of passing from M_σ to $M_{\sigma+1}$ is described by the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_\sigma & \longrightarrow & C_\sigma & \longrightarrow & D_\sigma & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M_{\sigma+1} & \longrightarrow & \oplus A \oplus C_\sigma & \longrightarrow & D_\sigma & \longrightarrow & 0 \end{array}$$

which shows that $M_{\sigma+1}/M_\sigma \cong \oplus A$. Thus M has been constructed in the way as described in the theorem. □

In the following proof we will use the notation $\Lambda_L(A)$ for a module M in Theorem 3.1 constructed by using the given modules L and A.

THEOREM 3.2. *Let the domain R and the cardinals κ, λ be as in the preceding theorem. If there is a torsion-free R-module L of cardinality $< \lambda$ that is not Matlis cotorsion, then to every reduced torsion-free R-module A of cardinality $\le \kappa$ there exists an R-module M of cardinality $\le \lambda$ that contains L, is not Matlis cotorsion, and satisfies* $\mathrm{Ext}^1_R(A, M) = 0$.

PROOF. Let L be a torsion-free, not Matlis cotorsion R-module. We are going to use the same kind of construction as above starting from the exact sequence $0 \to L \to \widetilde{L} \to D_0 \to 0$; here $D_0 \neq 0$ is a direct sum of copies of Q.

This time we change the middle term $\widetilde{L}$ by adding a summand $\Lambda_{\oplus A}(A)$ for sufficiently many copies of A and extend the map $\phi : \widetilde{L} \to D_0$ to a map ψ such that every map $A \to D_0$ should lift to $A \to C_0$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & \widetilde{L} & \xrightarrow{\phi} & D_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M_1 & \longrightarrow & C_0 = \Lambda_{\oplus A}(A) \oplus \widetilde{L} & \xrightarrow{\psi} & D_0 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & C_1 = \widetilde{C}_0 & \xrightarrow{\phi_1} & D_1 & \longrightarrow & 0.
\end{array}$$

The second step consists of the formation of the Matlis cotorsion hull of the middle term. Here again, $D_1 = \oplus Q$. Taking direct limits we get an exact sequence like (4), and hence (5). As in the proof of Theorem 3.1 we can conclude that $\mathrm{Ext}^1_R(A, C) = 0$ which implies $\mathrm{Ext}^1_R(A, M) = 0$. It is obvious that C can not contain any copy of Q, so the sequence (4) does not split. We conclude that M has a non-split extension by a direct sum of copies of Q, i.e. M is not Matlis cotorsion. □

The idea of the proof above applies to verify the following theorem.

THEOREM 3.3. *Suppose that κ is an infinite cardinal such that $|R| \leq \kappa$. There exists an R-module M of cardinality 2^κ that is not Matlis cotorsion, but satisfies*

$$\mathrm{Ext}^1_R(A, M) = 0$$

for all reduced torsion-free R-modules A of cardinality $\leq \kappa$.

PROOF. We argue as above in the proof of Theorem 3.2, just in the 2^κ steps we have to use every R-module A of cardinality $\leq \kappa$ exactly 2^κ times (rather than a fixed A all the times) with the proviso that each A is used cofinally (i.e. we never stop using any A: for any $\sigma < 2^\kappa$, every A has to be used after the σth step). □

From Theorem 3.2 we derive the conclusions pointed out at the beginning of this section.

COROLLARY 3.4. *There is no reduced R-module of cardinality $\leq \kappa$ to test modules of cardinalities $\geq 2^\kappa$ for Matlis cotorsionness.* □

4. Reduced test modules for Enochs and Warfield cotorsion.

Since there is no universal reduced test module for Matlis cotorsion, there can not exist any for Enochs or for Warfield cotorsion either. But if we wish to test a module M for Enochs or for Warfield cotorsionness that is already known to be Matlis cotorsion, then we can manage with reduced test modules, since Q is needed only to ensure the Matlis cotorsion property.

Thus we are now in search for reduced flat, resp. reduced torsion-free R-modules U such that the equation $\operatorname{Ext}^1_R(U, M) = 0$ will force a Matlis cotorsion module M to be Enochs, resp. Warfield cotorsion. Such modules U are provided by the following two theorems (the proofs use an idea from Bican-El Bashir-Enochs [3]).

THEOREM 4.1. *Assuming $|R| \leq \kappa$, let U be the direct sum of all pairwise non-isomorphic reduced flat R-modules of cardinality $\leq \kappa$. If M is a Matlis cotorsion module that satisfies* $\operatorname{Ext}^1_R(U, M) = 0$, *then M is Enochs cotorsion.*

PROOF. It is well known that if $|R| \leq \kappa$, then every R-module $A \neq 0$ contains a pure submodule $B \neq 0$ of cardinality $\leq \kappa$. Now if A is flat, then both B and A/B are flat, and by induction it follows that, for some ordinal τ, A contains a continuous well-ordered ascending chain $0 = A_0 <$ $< A_1 < \ldots < A_\alpha <$ $(\alpha < \tau)$ of pure submodules such that $A = \cup_{\alpha<\tau} A_\alpha$ and the factor modules $A_{\alpha+1}/A_\alpha$ are flat modules of cardinality $\leq \kappa$. If $\operatorname{Ext}^1_R(U, M) = 0$, then evidently also $\operatorname{Ext}^1_R(A_{\alpha+1}/A_\alpha, M) = 0$ for all $\alpha < \tau$. In view of Lemma 1 in [5], we then have also $\operatorname{Ext}^1_R(A, M) = 0$. This equation holds for every flat R-module A, and so M must be Enochs cotorsion. □

The same proof applies in a simpler form to verify the analogous result (arguing with relative divisibility rather than purity):

THEOREM 4.2. *Assume again that $|R| \leq \kappa$, and let U be the direct sum of all pairwise non-isomorphic ideals of the domain R. If M is a Matlis (or Enochs) cotorsion module satisfying* $\operatorname{Ext}^1_R(U, M) = 0$, *then M is Warfield cotorsion.*

PROOF. It is shown in [7, Corollary 8.4, p. 458] that an R-module M is Warfield cotorsion if and only if it satisfies $\operatorname{Ext}^1_R(A, M) = 0$ for all ideals A of R as well as for $A = Q$. □

5. More vanishing Exts.

In this section we wish to prove that for every reduced torsion-free module A of cardinality $\leq \kappa$ that is not flat (for every flat module A that is not strongly flat) there exists a Matlis cotorsion (Enochs cotorsion) module M of cardinality $\leq 2^\kappa$ that is not Enochs cotorsion (not Warfield cotorsion) and still it satisfies $\mathrm{Ext}^1_R(A, M) = 0$. The method employed in the proof of Theorem 3.1 will be used *mutatis mutandis*.

THEOREM 5.1. *Suppose that κ is an infinite cardinal satisfying $|R| \leq \kappa$. If A is a reduced flat R-module of cardinality κ that is not strongly flat, then there exists a Matlis cotorsion R-module M of cardinality $\leq 2^\kappa$ that is not Enochs cotorsion such that* $\mathrm{Ext}^1_R(A, M) = 0$.

PROOF. To begin, we refer to Lemma 4.1 in Fuchs-Salce-Trlifaj [8] and to its proof. This shows that if A is a flat, but not strongly flat R-module, then there is a non-splitting exact sequence $0 \to L \to N \to A \to 0$ where N is strongly flat and L is (torsion-free and) Matlis cotorsion. Since the sequence does not split, L cannot be Enochs cotorsion. An easy argument convinces us that $|L| \leq 2^\kappa$ holds for the module L in the proof of the mentioned lemma.

We start by replacing N by its Enochs cotorsion hull $EC(N)$ and modify the cokernel accordingly: $0 \to L \to EC(N) \to A_0 \to 0$ where all the modules are torsion-free and A_0 is flat containing A. We now proceed in the same way as above in the proof of Theorem 3.1, but we use Enochs cotorsion hulls rather than injective hulls all the way:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & EC(N) & \xrightarrow{\alpha} & A_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M_1 & \longrightarrow & C_0 = \oplus A \oplus EC(N) & \xrightarrow{\beta} & A_0 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & C_1 = EC(\oplus A \oplus EC(N)) & \longrightarrow & A_1' & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & L_1 = \widetilde{M_1} & \longrightarrow & C_1 & \xrightarrow{\alpha_1} & A_1 & \longrightarrow & 0,
\end{array}
$$

where in the second sequence we have added a direct sum of copies of A to insure that every homomorphism $A \to A_0$ lifts to C_0. In addition, in the fourth sequence M_1 is replaced by its Matlis cotorsion hull $\widetilde{M}_1$ (M_1 is relatively divisible in C_1, so its R-completion is contained in C_1) and A_1' by its reduced part A_1; this is a summand of A'_1 so that the composite map $A_0 \to A'_1 \to A_1$ is monic. Evidently, the module A'_1 is flat, and so is A_1. This triple step is repeated 2^κ times (taking unions and Enochs cotorsion hulls at limit ordinals) to obtain an exact sequence

$$(6) \quad 0 \to M = \bigcup_{\sigma<2^\kappa} M_\sigma = \bigcup_{\sigma<2^\kappa} L_\sigma \to C = \bigcup_{\sigma<2^\kappa} C_\sigma \xrightarrow{\beta} B = \bigcup_{\sigma<2^\kappa} A_\sigma \to 0.$$

This yields the exact sequence

$$\mathrm{Hom}_R(A, C) \to \mathrm{Hom}_R(A, B) \to \mathrm{Ext}^1_R(A, M) \to \mathrm{Ext}^1_R(A, C).$$

The arguments in the proof of Theorem 3.1 can be repeated to conclude that $\mathrm{Ext}^1_R(A, M) = 0$ and $|M| \leq 2^\kappa$. The module M is Matlis cotorsion, since in every extension of M by Q the factor sets are contained in L_σ for some σ.

It remains to verify that M is not Enochs cotorsion. Note that B is a flat module as the direct limit of the flat modules A_α, and $EC(N)$ is a summand of C, say with $\pi : C \to EC(N)$ as projection map. If the exact sequence (6) were splitting, and $\gamma : B \to C$ were a splitting map for β (i.e. $\beta\gamma = \mathbf{1}_B$), then $\pi\gamma$ would be a splitting map for α – an obvious contradiction. Consequently, M has a non-splitting extension C by the flat module B, so it cannot be Enochs cotorsion. □

The analogous result for Enochs cotorsion modules reads as follows.

THEOREM 5.2. *Suppose that κ is an infinite cardinal satisfying $|R| \leq \kappa$. If A is a reduced torsion-free R-module of cardinality κ that is not flat, then there exists an Enochs cotorsion R-module M of cardinality $\leq 2^\kappa$ that is not Warfield cotorsion such that* $\mathrm{Ext}^1_R(A, M) = 0$.

PROOF. Let A be a torsion-free R-module that is not flat. Then there is an ideal I of R such that $\mathrm{Tor}^R_1(A, R/I) \neq 0$. We are going to refer to the well-known natural isomorphism $\mathrm{Ext}^1_R(A, \mathrm{Hom}_\mathbb{Z}(R/I, \mathbb{Q}/\mathbb{Z})) \cong$ $\cong \mathrm{Hom}_\mathbb{Z}(\mathrm{Tor}^R_1(A, R/I), \mathbb{Q}/\mathbb{Z})$ to conclude that the pure-injective (and hence Enochs cotorsion) R-module $L = \mathrm{Hom}_\mathbb{Z}(R/I, \mathbb{Q}/\mathbb{Z})$ satisfies $\mathrm{Ext}^1_R(A, L) \neq 0$. As $|R/I| \leq \kappa$, we have certainly $|L| \leq 2^\kappa$. L is not Warfield cotorsion, because it has a non-splitting extension by the torsion-free A.

Thus there is a non-splitting exact sequence $0 \to L \to N \to A \to 0$ with L Enochs, but not Warfield cotorsion. We are going to use as a starting point of our argument the exact sequence $0 \to L \to \widehat{N} \to A_0 \to 0$ with A_0 torsion-free ($A \leq A_0$), where $\widehat{N}$ denotes the RD-injective hull of N. We proceed by imitating the proof of Theorem 5.1 almost *verbatim*, we just form RD-injective hulls rather than pure-injective hulls in each inductive step. □

Similar proofs apply to verify the analogues of Theorem 3.3 for Enochs and Warfield cotorsion modules, strengthening the last two theorems by replacing A by the collection of all reduced flat, resp. torsion-free R-modules of cardinality $\leq \kappa$.

6. Matlis domains.

We now specialize the domain, and assume that R is a Matlis domain, i.e. it satisfies $\mathrm{p.d.}_R Q = 1$. In this case more definite statements can be made.

To begin with, note that over a Matlis domain all 'good' modules have projective dimension 1; this is clear from the exact sequence (1).

Furthermore, by Lee [16] (see also Fuchs-Salce [7, p. 141]) for a Matlis domain R, the module K decomposes into a direct sum of countably generated submodules. Thus Lemma 2.3 implies that for a Matlis domain R the module K is not self-small, and so by Theorem 2.2 R admits arbitrarily large 'good' modules U_κ.

The following result generalizes a theorem on abelian groups due to Göbel [9].

THEOREM 6.1. *Let R be a Matlis domain. For every 'good' infinite cardinal $\kappa \geq |R|$, there exist torsion-free R-modules of cardinality 2^{2^κ} that are not Matlis cotorsion, but all torsion-free epic images of cardinalities $\leq 2^\kappa$ are Matlis cotorsion.*

PROOF. Apply Theorem 3.1 to $A = U_\kappa$ to obtain a module M of rank 2^{2^κ} that is not Matlis cotorsion and satisfies $\mathrm{Ext}^1_R(U_\kappa, M) = 0$. Let M/N be a torsion-free factor module of rank $\leq 2^\kappa$. In the exact sequence

$$\mathrm{Ext}^1_R(U_\kappa, M) \to \mathrm{Ext}^1_R(U_\kappa, M/N) \to \mathrm{Ext}^2_R(U_\kappa, N)$$

the last term vanishes, since $\mathrm{p.d.}\, U_\kappa = 1$ for Matlis domains. Hence $\mathrm{Ext}^1_R(U_\kappa, M/N) = 0$, so by Theorem 2.4 M/N is Matlis cotorsion. □

We can also ask what can be said if $\mathrm{Ext}^1_R(U_\kappa, M) = 0$ holds for a module M whose rank exceeds the cardinality 2^κ.

PROPOSITION 6.2. *Let again κ be a 'good' cardinal with $|R| \leq \kappa$, where R is a Matlis domain. A torsion-free R-module M of rank $> 2^\kappa$ satisfying $\mathrm{Ext}^1_R(U_\kappa, M) = 0$ is a subdirect product of Matlis cotorsion modules of ranks 2^κ.*

PROOF. It is clear that we can represent M as a subdirect product of torsion-free modules of ranks 2^κ. In the case of Matlis domains, p.d. $U_\kappa = 1$, so epic images N of M also satisfy $\mathrm{Ext}^1_R(U_\kappa, N) = 0$. By Theorem 2.4 these modules N are Matlis cotorsion modules. Hence the assertion is evident. □

We can add that in case all the modules are reduced, a subdirect product of Matlis cotorsion modules is Matlis cotorsion if and only if this subdirect product is a closed submodule in the cartesian product (in the R-topology).

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