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Periodic Solutions Arising in Ecology of Mangroves.

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ABSTRACT - Maximal monotone operators techniques and a fixed point argument are employed to prove the existence of periodic solutions for a coupled evolution system of partial differential equations modelling the movement of water and salt in a porous medium where the extraction of water takes place by the roots of mangroves.

1. Introduction.

The aim of this paper is to study the existence of periodic solutions for the evolution coupled system formed by a convection-diffusion equation and an ordinary differential equation

$$u_t - u_{xx} + (uq)_x = 0 \text{ in } Q := (0,d) \times P, \tag{1.1}$$

$$u(0,t) = \varphi(t) \text{ in } P, \tag{1.2}$$

$$u_x(d,t) = 0 \text{ in } P, \tag{1.3}$$

$$u(x,t+\omega) = u(x,t) \text{ in } Q,\ \omega > 0, \tag{1.4}$$

$$q_x + f(x,u) = 0 \text{ in } Q, \tag{1.5}$$

$$q(d,t) = 0 \text{ in } P, \tag{1.6}$$

$$q(x,t+\omega) = q(x,t) \text{ in } Q,\ \omega > 0, \tag{1.7}$$

describing the one-dimensional movement of water and salt under the surface of a porous medium with constant porosity, when a continuous extraction of water is determinated by the roots of mangroves. Mangroves

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are woody plants characterized by grow on saturated soils subject to regular inundations by tidal water with salt concentration c_w at the surface, close to that of sea water (see [3], [7]). In the flow water which carries salt from soil surface downward, the roots of mangroves take up fresh water from the saline soil leaving behind the salt. The mathematical formulation of model, is given in [8] or [9].

The unknown of model are the salt concentration u and the water flow q in the porous medium $(0, d)$. The function f, represents the volume of water taken up by the roots for unit volume of porous material per unit time. Conditions (1.3) and (1.6) mean that on the bottom of the porous media is assumed no flux boundary conditions for salt and water. To justify the periodicity of solutions, we assume in condition (1.2) that the salt concentration on the top of the porous medium varies periodically in the time for the mangroves model. Our approach to periodicity, shall be done looking for periodic solutions in suitable t-periodic functions space, rather than to take into consideration the fixed point of the Poincaré periodic map. Methodologically, the starting point relies on an abstract formulation of the problem to which to apply some techniques of the maximal monotone operators and to get solutions as fixed points of an operator equation. Both the stationary and the evolution model was considered in [8] where the existence and uniqueness of the solution, the stability of the steady state solution and the occurrence of dead core was proved. In [9], the authors studied a system with a dynamical boundary condition.

The plan of paper is the following: Section 2 is devoted to introduce the functional framework and give the definition of solution for our system. In Section 3, are applied some results of the maximal monotone mappings theory to study the evolution problem formulated as an abstract problem. Section 4, deals with an approximating problem and by means of a priori estimates, is obtained the convergence of the approximate solutions. Finally, in Section 5 a fixed point argument allows us to prove the existence of periodic solutions.

2. Preliminaries.

Throughout the paper we represent with $P := R/\omega Z$ the period interval $[0, \omega]$, so for the functions defined in Q we are automatically imposing the time ω-periodicity. Our system shall be studied within the functional framework represented by some useful spaces of t-periodic functions. Let

$$V := L^2(P; \mathcal{V})$$

be the Hilbert space endowed with the norm

$$\|v\|_V := \left(\int_Q |v(x,t)|^2 dxdt + \int_Q |v_x(x,t)|^2 dxdt \right)^{1/2} \tag{2.1}$$

where

$$\mathcal{V} := \{u \in L^2(0,d) : u(0) = 0\}$$

and let

$$L^\infty(P; W^{1,\infty}(0,d))$$

be the Banach space with the usual norm.

The space $V^* := L^2(P; \mathcal{V}^*)$ is the topological dual space of V and $\|.\|_*$ is its norm. The pairing of duality between V and V^* shall be denoted by $\langle ., . \rangle$.

We shall make the following assumptions on the data involved

H$_1$) $\varphi : P \to (0,1)$ such that $\varphi \in W^{1,2}(P)$;

H$_2$) $\left\{ \begin{array}{c} f : (0,d) \times [0,1] \to R, f \in L^\infty((0,d); C([0,1])),\ |f| \leqslant 1 \\ f(x,.) \text{ is nonincreasing in } [0,1] \text{ and } f(x,1) = 0, \text{ for a.e. } x \in (0,d). \end{array} \right\}$

REMARK. Condition H$_2$) implies that $f \geqslant 0$ in $(0,d) \times [0,1]$.

Next, we give the definition of weak-strong periodic solution to (1.1)-(1.7)

DEFINITION 2.1. A solution of (1.1)-(1.7) is a pair of functions (u, q) such that

$$u - \varphi(t) \in L^2(P; \mathcal{V}),\ u_t \in L^2(P; \mathcal{V}^*)$$

with

$$\int_Q u_t \xi dxdt + \int_Q u_x \xi_x dxdt - \int_Q uq\xi dxdt = 0,\ \forall \xi \in V \tag{2.2}$$

and

$$q \in L^\infty(P; W^{1,\infty}(0,d))$$

satisfying (1.5)-(1.7) almost everywhere.

3. The evolution problem.

We begin solving the evolution problem (1.1)-(1.4). Defined $h(x,t) := u(x,t) - \varphi(t)$ and fixed $w \in L^2(P;\mathcal{V})$, we consider the new problem

$$h_t - h_{xx} + ((w+\varphi)q)_x + \varphi'(t) = 0 \text{ in } Q, \tag{3.1}$$

$$h(0,t) = 0 \text{ in } P, \tag{3.2}$$

$$h_x(d,t) = 0 \text{ in } P, \tag{3.3}$$

$$h(x,t+\omega) = h(x,t) \text{ in } Q,\ \omega > 0, \tag{3.4}$$

$$q_x + f(x,w) = 0 \text{ in } Q, \tag{3.5}$$

$$q(d,t) = 0 \text{ in } P, \tag{3.6}$$

$$q(x,t+\omega) = q(x,t) \text{ in } Q,\ \omega > 0. \tag{3.7}$$

Problem (3.5)-(3.7) has a unique solution $q \in L^\infty(P; W^{1,\infty}(0,d))$, (see [5]).

DEFINITION 3.1. A weak periodic solution to (3.1)-(3.4), associated to w, is a function

$$h \in L^2(P;\mathcal{V}),\ h_t \in L^2(P;\mathcal{V}^*)$$

such that

$$\int_Q h_t\xi dxdt + \int_Q h_x\xi_x dxdt - \int_Q (w+\varphi(t))q\xi_x dxdt - \int_Q \varphi'(t)\xi dxdt = 0,\ \forall \xi \in V. \tag{3.8}$$

The existence of weak periodic solutions to (3.1)-(3.4) shall be established using the maximal monotone mappings theory, according to the following result.

THEOREM 3.1 ([1], [2], [6]). Let L be a linear closed densely defined operator from the reflexive Banach space $L^2(P;\mathcal{V})$ to $L^2(P;\mathcal{V}^*)$, L maximal monotone and let A be a bounded hemicontinuous monotone mapping $L^2(P;\mathcal{V})$ into $L^2(P;\mathcal{V}^*)$. Then, $L+A$ is maximal monotone in $L^2(P;\mathcal{V}) \times$ $\times L^2(P;\mathcal{V}^*)$. Moreover, if $L+A$ is coercive, then the $Range(L+A) =$ $= L^2(P;\mathcal{V}^*)$.

In order to utilize the results of the above theorem, we must formulate (3.8) as an abstract problem. Let us define the operators

$$L : \mathcal{D} \to L^2(P; \mathcal{V}^*)$$

as follows

$$\langle Lh,\, \xi\rangle := \int_Q h_t \xi \, dx dt$$

on the set

$$\mathcal{D} := \{h \in V : h_t \in V^*\}$$

and

$$A : L^2(P; \mathcal{V}) \to L^2(P; \mathcal{V}^*)$$

by setting

$$\langle Ah,\, \xi\rangle := \int_Q h_x \xi_x \, dx dt.$$

The linear operator L is closed, skew-adjoint (i.e. $L = -L^*$), maximal monotone (see [6]) and densely defined because $C(\overline{Q}) \subset \mathcal{D}$ is dense in V.

Now, we investigate the properties of A.

PROPOSITION 3.2. Under assumptions H_1)-H_2), the operator A is:

i) hemicontinuous;
ii) monotone;
iii) coercive.

PROOF. By the Hölder inequality one has

$$|\langle Ah,\, \xi\rangle| \leqslant \left(\int_Q |h_x|^2 dx dt\right)^{1/2} \left(\int_Q |\xi|^2 dx dt\right)^{1/2} \leqslant \|h\|_V \|\xi\|_V$$

and consequently

$$\|Ah\|_* \leqslant \|h\|_V.$$

The hemicontinuity descends from [4, Theorems 2.1 and 2.3].

ii)
$$\langle Ah_1 - Ah_2,\, h_1 - h_2\rangle = \int_Q |(h_1 - h_2)_x|^2 dxdt \geqslant 0.$$

iii) The Poincaré inequality gives us

$$\langle Ah,\, h\rangle = \int_Q |h_x|^2 dxdt \geqslant c\|h\|_V^2. \tag{3.9}$$

From (3.9) one has

$$\frac{\langle Ah,\, h\rangle}{\|h\|_V} \geqslant c\|h\|_V \to +\infty \text{ as } \|h\|_V \to +\infty.$$

This completes the proof. □

In conclusion, defined the linear functional $G \in V^*$ as

$$\langle G,\, \xi\rangle = -\int_Q (w + \varphi(t))q\xi_x dxdt - \int_Q \varphi'(t)\xi dxdt$$

we can rewrite problem (3.8) in the following abstract form

$$Lh + Ah = G \tag{3.10}$$

to which to apply Theorem 3.1.

Finally, closing the section we state its main result

THEOREM 3.3. If H_1)-H_2) are fulfilled, then problem (3.10) has a unique weak periodic solution.

PROOF. The existence of weak periodic solutions derive from Theorem 3.1, while the uniqueness is a consequence of the strict monotonicity. □

4. Convergences.

In the previous section, has been show the existence and uniqueness of the weak-strong periodic solution (h, q) corresponding to $w \in L^2(Q)$. Let w_n be a sequence in $L^2(Q)$ such that $w_n \to w$ strongly in $L^2(Q)$, we denote with $h_n \in V$, $q_n \in L^\infty(P; W^{1,\infty}(0, d))$ respectively, the periodic weak-strong solution of

$$\int_Q h_{nt}\xi dxdt + \int_Q h_{nx}\xi_x dxdt - \int_Q (w_n + \varphi(t))q_n\xi_x dxdt - \int_Q \varphi'(t)\xi dxdt = 0, \tag{4.1}$$

for any $\xi \in V$, respectively

$$q_{nx} + f(x, w_n) = 0 \text{ in } Q, \tag{4.2}$$

$$q_n(d, t) = 0 \text{ in } P, \tag{4.3}$$

$$q_n(x, t+\omega) = q_n(x, t) \text{ in } Q,\ \omega > 0. \tag{4.4}$$

Integrating (4.2) over Q and taking into account H_2), we get

$$\|q_n\|_{L^\infty(P;W^{1,\infty}(0,d))} \leqslant M \tag{4.5}$$

where M denotes various positive constants independent of n.

Chosen $\xi = h_n$ as a test function in (4.1), one has

$$\int_Q |h_{nx}|^2 dxdt = \int_Q (w_n + \varphi(t)) q_n h_{nx} dxdt + \int_Q \varphi'(t) h_n dxdt$$

by which

$$\int_Q |h_{nx}|^2 dxdt = \int_Q w_n q_n h_{nx} dxdt + \int_Q \varphi(t) q_n h_{nx} dxdt + \int_Q \varphi'(t) h_n dxdt.$$

The Hölder and Poincaré inequalities yield

$$\int_Q |h_{nx}|^2 dxdt \leqslant M \Big(\int_Q |h_{nx}|^2 dxdt \Big)^{1/2} \Bigg(\Big(\int_Q |w_n|^2 dxdt \Big)^{1/2} + \Big(\int_Q |\varphi(t)|^2 dxdt \Big)^{1/2} + \Big(\int_Q |\varphi'(t)|^2 dxdt \Big)^{1/2} \Bigg)$$

which implies

$$\int_Q |h_{nx}|^2 dxdt \leqslant M^2 \Bigg(\Big(\int_Q |w_n|^2 dxdt \Big)^{1/2} + \Big(\int_Q |\varphi(t)|^2 dxdt \Big)^{1/2} + \Big(\int_Q |\varphi'(t)|^2 dxdt \Big)^{1/2} \Bigg)^2 \leqslant M'$$

with the positive constant M' independent of n.

The Poincaré inequality, permits to derive the classical energy estimate

$$\int_Q |h_n(x,t)|^2 dxdt + \int_Q |h_{nx}(x,t)|^2 dxdt \leqslant C. \tag{4.6}$$

From (4.1) and (4.6), we infer that h_{nt} is bounded with respect to the V^* norm. Hence, we conclude that h_n lies in a bounded set of $\mathcal{D}$ that is

$$\|h_n\|_{\mathcal{D}} \leqslant M.$$

We are ready to take the limit $n \to +\infty$. Passing to subsequence if necessary again denoted by h_n, one obtains

$$h_n \rightharpoonup h \text{ in } \mathcal{D}.$$

By a result of [6, Theorem 5.1], the sequence h_n is precompact in $L^2(Q)$, therefore

$$h_n \to h \text{ in } L^2(Q) \text{ and a.e. in } Q.$$

Because of the boundedness of the previously sequences, we can infer that

$$h_{nt} \rightharpoonup h_t \text{ in } L^2(P; \mathcal{V}^*)$$

$$h_{nx} \rightharpoonup h_x \text{ in } L^2(P; \mathcal{V})$$

$$q_n \rightharpoonup q \text{ in } L^\infty(P; W^{1,\infty}(0,d)) \text{ weak-}*$$

$$f(x, w_n) \to f(x, w) \text{ in } L^2(Q).$$

5. A fixed point argument.

To prove the existence of periodic solutions, we shall utilize a fixed point argument which makes possible to invoke the Schauder fixed point theorem.

Let

$$\Phi : L^2(Q) \to L^2(Q)$$

be the mapping so defined

$$\Phi(w) = h$$

where h is the unique weak periodic solution of (3.8) corresponding to $w \in L^2(Q)$.

The mapping Φ is well-defined and its properties are expressed below.

LEMMA 5.1. The mapping Φ is continuous.

PROOF. The above convergences show the continuity of Φ because $\Phi(w_n) = h_n$ converges strongly to $\Phi(w) = h$ in $L^2(Q)$ when $n \to +\infty$. □

LEMMA 5.2. There exists a constant $R > 0$ such that

$$\|\Phi(w)\|_{L^2(Q)} \leqslant R, \ \forall w \in L^2(Q).$$

PROOF. Passing to the limit in (4.6) leads to conclusion. □

Since $\Phi(L^2(Q)) \subset \mathcal{D}$ and the embedding $\mathcal{D} \subset L^2(Q)$ is compact, the operator Φ is compact from $L^2(Q)$ into itself.

In conclusion, our main result is the next

THEOREM 5.3. If H_1)-H_2) hold, there exists at least a weak-strong periodic solution to (1.1)-(1.7).

PROOF. Lemmas 5.1 and 5.2, tell us that the operator Φ is both compact and continuous from $L^2(Q)$ into $L^2(Q)$. From the Schauder fixed point theorem, one has the existence of periodic solutions for the mangroves model. □

REMARK. Using as test functions in (2.2), $\xi = \min\{0, \ u\}$ and $\xi = \max\{0, u - 1\}$ it is easy to see that

$$0 \leqslant u(x,t) \leqslant 1, \ \text{ a.e. in } Q.$$

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