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(S_3, S_6)-Amalgams VI.

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Introduction.

In this, the penultimate part of [LPR1], we analyse Case 3 (as described in [Section 12; [LPR1]). Our main result, Theorem 14.1, asserts that Case 3 does not occur. Since $\mathrm{core}_{G_\alpha} V_\beta$ is so small, the core argument is, at times, especially deadly. Consequently we often end up in situations, where the subgroups we are interested in have trivial action on all the non-central G_δ-chief factors in Q_δ for many vertices $\delta \in O(S_3)$. This is especially unfortunate since, as Lemma 15.1 shows, U_δ (for $\delta \in O(S_3)$) possesses at least four non-central G_δ-chief factors. In fact the proof of Theorem 14.1 hinges upon overcoming just this type of situation. A significant step in dealing with this problem is made in Lemmas 16.3 and 16.4. It follows from these lemmas that there exists a critical pair (α, α') and $\rho \in \Delta(\alpha') \setminus \{\alpha' - 1\}$ for which $F_\alpha \cap Q_{\alpha'} \not\leq Q_\rho$. The group F_γ and its accomplice H_δ (where $\gamma \in O(S_3)$ and $\delta \in O(S_6)$) are defined in Section 14. These groups are "small" enough so as they fix many vertices yet are "large" from the point of view of the non-central chief factors they contain. Before the groups F_γ and H_δ can be successfully deployed we need to restrict the structure of the critical pairs in Γ. This we do in Section 15. Section 14, apart from defining F_γ and H_β, is concerned with eliminating the possibility $b = 3$.

Finally, as before we continue the section numbering in [LPR1] and note that this paper only refers to results and notation contained in sections 1, 2 and 12.

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14. $b = 3$ and the groups $T_{\delta\gamma}$.

Throughout this paper the following hypothesis is assumed to hold.

HYPOTHESIS 14.0. (i) *For each* $(\alpha, \alpha') \in \mathcal{C}$ *we have* $\alpha \in O(S_3)$ *and* $[Z_\alpha, Z_{\alpha'}] = 1$*; and*

(ii) $\mathrm{core}_{G_\alpha} V_\beta = V_\beta \cap V_{\alpha-1} = Z_\alpha$.

Our objective is to show that

THEOREM 14.1. *Hypothesis* 14.0 *cannot hold.*

Before tackling the case $b = 3$ we give some notation and define the groups $T_{\delta\gamma}$.

For $\delta \in O(S_6)$ we set

$$L_\delta := O^2(G_\delta),$$

$$Y_\delta := C_{V_\delta}(L_\delta) \quad \text{and}$$

$$C_\delta := C_{Q_\delta}(V_\delta).$$

We recall from Theorem 12.1 that for $\delta \in O(S_6)$, $\eta(G_\delta, V_\delta) = 1$ and V_δ/Z_δ is a quotient of $\binom{4}{1} \oplus 1$. Let $\gamma \in \Delta(\delta)$. If $V_\delta/Z_\delta \cong 4$, then we define

$$T_{\delta\gamma} := [V_\delta, Q_\gamma, Q_\gamma];$$

otherwise we define $T_{\delta\gamma}$ to be a normal fours subgroup of $G_{\delta\gamma}$ with

$$Z_\delta < T_{\delta\gamma} \leq Y_\delta.$$

Next we describe two groups which play a crucial role later in this section and in Section 16.

$$F_\gamma := \langle T_{\delta\gamma}{}^{G_\gamma} \rangle$$

$$H_\delta := \langle F_\gamma{}^{G_\delta} \rangle$$

The groups F_γ and H_δ are similar to the F_α and H_β defined in Section 12. (Indeed if $|Y_\delta| = 2^2$, then they are the same.) We will need a result analogous to Lemma 12.5.

LEMMA 14.2. *Let $(\alpha, \alpha') \in \mathcal{C}$. Then*

(i) $\eta(G_\alpha, F_\alpha) = 2$ *with* $Z_\alpha = [F_\alpha, Q_\alpha]$ *and* $|F_\alpha/Z_\alpha| \in \{2^2, 2^3\}$; *and*
(ii) $V_\beta \lneqq V_\beta F_\alpha \lneqq H_\beta$ *and* $\eta(G_\beta, H_\beta/V_\beta) \geq 1$.

PROOF. First we observe that if $V_\beta/Z_\beta \cong 4$, then $Z_\alpha \leq T_{\beta\alpha}$. While, if $V_\beta/Z_\beta \ncong 4$, then $Z_\alpha \cap T_{\beta\alpha} = Z_\beta$ by Lemma 1.1(ii). Hence $T_{\beta\alpha}Z_\alpha \cong E(2^3)$ in either case. Since $\Omega_1(Z(Q_\alpha)) = Z_\alpha$ by Lemma 12.2(i) and $V_\beta \leq Q_\alpha$, $1 \neq [T_{\beta\alpha}, Q_\alpha] \leq Z_\alpha$ and hence $[F_\alpha, Q_\alpha] = Z_\alpha$. Because $Z_\alpha = \mathrm{core}_{G_\alpha} V_\beta$ we clearly have $F_\alpha \gneqq T_{\beta\alpha}Z_\alpha$ and now (i) follows.

Since $[V_\beta, Q_\beta] = Z_\beta$, (i) implies that $V_\beta \lneqq V_\beta F_\alpha \leq H_\beta$. Also, from (i) and Lemma 12.2(ii), $2^2 \leq |[F_\alpha, Q_\beta]| \leq 2^3$. Now suppose that $\eta(G_\beta, H_\beta/V_\beta) = 0$. Then $H_\beta = V_\beta F_\alpha$. Hence $K_\beta := [H_\beta, Q_\beta] = [V_\beta, Q_\beta][F_\alpha, Q_\beta] = [F_\alpha, Q_\beta] \trianglelefteq G_\beta$. Since $|K_\beta| \leq 2^3$, $[K_\beta, L_\beta] = 1$ and so $Z_\alpha \cap K_\beta = Z_\beta$. Therefore $[F_\alpha, Q_\alpha \cap Q_\beta] \leq$ $\leq [F_\alpha, Q_\alpha] \cap K_\beta = Z_\alpha \cap K_\beta = Z_\beta$. Hence

$$[H_\beta, Q_\alpha \cap Q_\beta] = [V_\beta, Q_\alpha \cap Q_\beta][F_\alpha, Q_\alpha \cap Q_\beta] \leq Z_\beta$$

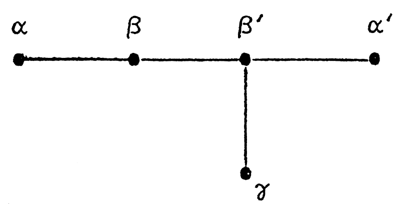
and so $Q_\alpha \cap Q_\beta \leq C_{Q_\beta}(H_\beta/Z_\beta)$. Since $|K_\beta| \geq 2^2$, we deduce that $Q_\alpha \cap Q_\beta =$ $= C_{Q_\beta}(H_\beta/Z_\beta) \trianglelefteq G_\beta$, contradicting Lemma 12.4(i). This proves (ii).

And now to the main business of this section.

THEOREM 14.3. $b \geq 5$.

We suppose the theorem is false and, in the following lemmas, seek a contradiction. So, by Lemma 11.1(iii), $b = 3$.

For $(\alpha, \alpha') \in \mathcal{C}$ we label vertices as indicated.



So $\Delta(\beta') = \{\beta, \gamma, \alpha'\}$.

LEMMA 14.4. *Let $(\alpha, \alpha') \in \mathcal{C}$.*

(i) $V_\beta/Z_\beta \cong 4$ *or* $\binom{4}{1}$.
(ii) $R := [V_\beta, V_{\alpha'}] = Z_\gamma$.
(iii) *There exists* $\rho \in \Delta(\alpha') \setminus \{\beta'\}$ *such that* $(\rho, \beta) \in \mathcal{C}$.

(iv) $Z_\alpha Q_{\alpha'}/Q_{\alpha'} = V_\beta Q_{\alpha'}/Q_{\alpha'}$ *is the central transvection of* $G_{\beta'\alpha'}/Q_{\alpha'}$ *on* $V_{\alpha'}/Z_{\alpha'}$ *and* $Z_\rho Q_\beta/Q_\beta = V_{\alpha'}Q_\beta/Q_\beta$ *is the central transvection of* $G_{\beta\beta'}/Q_\beta$ *on* V_β/Z_β.

(v) $C_{V_\beta}(V_{\alpha'}) = [V_\beta, Q_{\beta'}] = V_\beta \cap Q_{\alpha'} \underset{2}{\lneq} V_\beta$ *and*

$C_{V_{\alpha'}}(V_\beta) = [V_{\alpha'}, Q_{\beta'}] = V_{\alpha'} \cap Q_\beta \underset{2}{\lneq} V_{\alpha'}$.

(vi) *There exists* $\delta \in \Delta(\beta)$ *such that* $(\delta, \alpha') \in \mathcal{C}$ *and*

$$\langle G_{\delta\beta}, V_{\alpha'} \rangle = G_\beta.$$

Proof. First we note that

$$1 \neq [Z_\alpha, V_{\alpha'}] \leq R := [V_\beta, V_{\alpha'}] \leq V_\beta \cap V_{\alpha'} = Z_{\beta'}.$$

Since R is $G_{\beta'\gamma}$-invariant, $Z_\gamma \leq R$ whence $V_{\alpha'} \not\leq Q_\beta$ and so there exists $\rho \in \Delta(\alpha') \setminus \{\beta'\}$ such that $(\rho, \beta) \in \mathcal{C}$. Also we have $Z_{\beta'} = Z_\beta R \leq [V_\beta, G_\beta]$ and therefore $V_\beta = [V_\beta, G_\beta]$. So (i) holds. Further, $RY_{\alpha'}/Y_{\alpha'} = Z_{\beta'}Y_{\alpha'}/Y_{\alpha'} = C_{V_{\alpha'}/Y_{\alpha'}}(G_{\beta'\alpha'})$ together with Proposition 2.5(iii) yields that $Z_\alpha Q_{\alpha'}/Q_{\alpha'} = V_\beta Q_{\alpha'}/Q_{\alpha'}$ is the central transvection of $G_{\beta'\alpha'}/Q_{\alpha'}$ on $V_{\alpha'}/Z_{\alpha'}$ proving the first part of (iv). The second part of (iv) follows similarly.

Because $Q_{\beta'}Q_\beta = G_{\beta\beta'}$ and $V_\beta \cap Q_{\alpha'}$ is $Q_{\beta'}$-invariant with, by (iv), $Z_\beta \leq V_\beta \cap Q_{\alpha'} \underset{2}{\lneq} V_\beta$ we infer that $V_\beta \cap Q_{\alpha'} = [V_\beta, G_{\beta\beta'}]$. Then, by (iv), $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] \leq Z_\beta \cap Z_{\alpha'} = 1$ and thus $C_{V_\beta}(V_{\alpha'}) = V_\beta \cap Q_{\alpha'} = [V_\beta, Q_{\beta'}]$ with a symmetric statement with β and α' interchanged, so we have (v). From (v) $V_{\alpha'}$ acts upon V_β as an involution and a transvection and so $|R| = 2$. Thus $R = Z_\gamma$.

Since $V_{\alpha'} \not\leq Q_\beta$, by Proposition 2.8(viii) we may choose $\delta \in \Delta(\beta)$ such that $\langle G_{\delta\beta}, V_{\alpha'} \rangle = G_\beta$. If $Z_\delta \leq Q_{\alpha'}$, then $Z_\delta \leq V_\beta \cap Q_{\alpha'}$ and so $[Z_\delta, V_{\alpha'}] = 1$ by (iv). But then $Z_\delta \trianglelefteq G_\beta$, a contradiction. Hence $(\delta, \alpha') \in \mathcal{C}$, as required.

In view of Lemma 14.4(vi) we may, and shall, assume that (α, α') is a fixed critical pair for which $\langle G_{\alpha\beta}, V_{\alpha'} \rangle = G_\beta$. Also we set $J_\beta = \langle [U_\alpha, Q_\alpha]^{G_\beta} \rangle$.

Lemma 14.5.

(i) $[U_\alpha, Q_\alpha] \leq Z(U_\alpha)$.

(ii) $V_\beta \leq H_\beta \leq J_\beta \leq C_\beta \leq G_{\alpha'}$ *and* $H_{\alpha'} \leq C_{\alpha'} \leq G_\beta$.

(iii) $[F_\alpha, V_{\alpha'} \cap Q_\beta] = Z_\beta$.

Proof. From Lemma 14.4(v) $V_\beta \cap Q_{\alpha'} = [V_\beta, Q_{\beta'}] = C_{V_\beta}(V_{\alpha'})$, and $Q_\beta \not\leq Q_{\beta'}$ by Lemma 12.2(ii). Therefore, since $[V_\beta, Q_{\beta'}]$ is Q_β-invariant we get $V_\beta \cap Q_{\alpha'} = C_{V_\beta}(V_\gamma)$ and thus $V_\beta \cap Q_{\alpha'} \leq Z(U_{\beta'})$. This in turn implies that

$$Z(U_{\beta'}) \geq \left\langle \left[V_\beta, Q_{\beta'}\right]^{G_{\beta'}} \right\rangle = \left[U_{\beta'}, Q_{\beta'}\right],$$

from which (i) follows.

Appealing to Lemma 14.4(i) and Proposition 2.5(i) gives $T_{\beta\alpha} \leq [V_\beta, Q_\alpha]$ and hence $F_\alpha \leq [U_\alpha, Q_\alpha]$. Consequently $H_\beta = \langle {F_\alpha}^{G_\beta} \rangle \leq \langle [U_\alpha, Q_\alpha]^{G_\beta} \rangle = J_\beta$. By part (i) $[U_\alpha, Q_\alpha] \leq C_\beta$ whence $J_\beta \leq C_\beta$. Evidently $C_\beta \leq Q_{\beta'} \leq G_{\alpha'}$ and it is also clear that we have $H_{\alpha'} \leq C_{\alpha'} \leq G_\beta$.

Now we prove (iii). By Lemma 14.4(v) $V_{\alpha'} \cap Q_\beta$ centralizes V_β and so $V_{\alpha'} \cap Q_\beta \leq Q_\alpha$. Hence $[F_\alpha, V_{\alpha'} \cap Q_\beta] \leq [F_\alpha, Q_\alpha] = Z_\alpha$ by Lemma 14.2(i). Since $F_\alpha \leq G_{\alpha'}$ by (ii), we then get $[F_\alpha, V_{\alpha'} \cap Q_\beta] \leq Z_\alpha \cap V_{\alpha'} = Z_\beta$. If $[F_\alpha, V_{\alpha'} \cap Q_\beta] = 1$, then F_α centralizes a hyperplane of $V_{\alpha'}$ and so $F_\alpha Q_{\alpha'} = Z_\alpha Q_{\alpha'} = V_\beta Q_{\alpha'}$. Hence, using Lemma 14.4(ii),

$$\left[F_\alpha, V_{\alpha'}\right] \leq Z_{\alpha'} \left[V_\beta, V_{\alpha'}\right] = Z_{\alpha'} Z_\gamma = Z_{\beta'} \leq V_\beta.$$

But then $V_\beta F_\alpha$ is normalized by $\langle G_{\alpha\beta}, V_{\alpha'} \rangle = G_\beta$ by the choice of the critical pair, against Lemma 14.2(ii). This shows that $[F_\alpha, V_{\alpha'} \cap Q_\beta] = Z_\beta$.

LEMMA 14.6. *$V_\beta / Z_\beta \cong 4$. In particular, $T_{\beta\alpha} = [V_\beta, Q_\alpha, Q_\alpha]$ and $F_\alpha = [U_\alpha, Q_\alpha, Q_\alpha]$.*

PROOF. Assume that $V_\beta / Z_\beta \cong \binom{4}{1}$, and set $\Delta(\alpha) = \{\lambda, \mu, \beta\}$. Then by definition $T_{\beta\alpha} = Y_\beta \cong E(2^2)$ and $F_\alpha = \langle Y_\lambda, Y_\mu, Y_\beta \rangle$. Put $X = V_{\alpha'} \cap Q_\beta$. By Lemma 14.4(v) X centralizes V_β and so $X \leq Q_\alpha \leq G_\lambda$. Therefore $[X, Y_\lambda] \leq Z_\lambda$. Since $[X, F_\alpha] = Z_\beta$ by Lemma 14.5(iii), $[X, Y_\lambda] \leq Z_\lambda \cap Z_\beta = 1$. Likewise $[X, Y_\mu] = 1$ and thus $[X, F_\alpha] = 1$, contrary to $[X, F_\alpha] = Z_\beta$. Hence $V_\beta / Z_\beta \not\cong \binom{4}{1}$ and now the lemma follows from Lemma 14.4(i) and the definition of $T_{\beta\alpha}$ and F_α.

LEMMA 14.7. *Put $\bar{G}_{\alpha'} = G_{\alpha'} / Q_{\alpha'}$. Then*

(i) *$[J_\beta, J_\beta] \leq Z_\beta$; and*
(ii) *$\bar{J}_\beta = \bar{C}_\beta$ is the non-quadratic $E(2^3)$-subgroup of $\bar{G}_{\beta'\alpha'}$ on $V_{\alpha'} / Z_{\alpha'}$.*

PROOF. Put $\Delta(\alpha) = \{\lambda, \mu, \beta\}$, $\bar{G}_{\alpha'} = G_{\alpha'} / Q_{\alpha'}$ and $X = V_{\alpha'} \cap Q_\beta$. From Lemma 14.5(iii) $[F_\alpha, X] = Z_\beta$. Now $T_{\beta\alpha} \leq V_\beta$ and $[V_\beta, X] = 1$ means we can assume, without loss of generality, that

(14.7.1) $\left[T_{\lambda\alpha}, X\right] = Z_\beta \neq Z_\lambda.$

From this and Lemma 14.6 we immediately deduce

(14.7.2) $X \not\leq Q_\lambda$ and X does not centralize $T_{\lambda\alpha}/Z_\lambda = [V_\lambda, Q_\alpha, Q_\alpha]/Z_\lambda$; in particular, XQ_λ/Q_λ is not contained in the $E(2^3)$-subgroup of $G_{\lambda\alpha}/Q_\lambda$ acting quadratically on V_λ/Z_λ.

By Lemmas 14.4(v) and 14.5(ii) we have $J_\beta \leq C_\beta \leq Q_\alpha \leq G_\lambda$ and $J_\beta \geq [U_{\beta'}, Q_{\beta'}] \geq [V_{\alpha'}, Q_{\beta'}] = X$. Combining these observations with (14.7.2) and Lemma 11.1(vii) yields

(14.7.3) $\bar{J}_\beta$ is not contained in the $E(2^3)$-subgroup of $\bar{G}_{\beta'\alpha'}$ acting quadratically on $V_{\alpha'}/Z_{\alpha'}$.

Since $G_{\beta'\alpha'}$ is a 2-group, $\overline{\Phi(C_\beta)} = \Phi(\bar{C}_\beta) \leq \Phi(\bar{G}_{\beta'\alpha'})$. By Lemma 14.4(v) $\Phi(\bar{G}_{\beta'\alpha'}) \cap \bar{V}_\beta = 1$ and so $\Phi(C_\beta) \cap V_\beta \leq Q_{\alpha'}$. Because V_β/Z_β is a G_β-chief factor we deduce that

(14.7.4) $\Phi(C_\beta) \cap V_\beta \leq Z_\beta$

Clearly $\Phi(C_\beta) \cap \Phi(C_{\alpha'})$ is normalized by $G_{\beta'\gamma}$. If $\Phi(C_\beta) \cap \Phi(C_{\alpha'}) \neq 1$, then $\Phi(C_\beta) \cap \Phi(C_{\alpha'}) \geq \Omega_1(Z(G_{\beta'\gamma})) = Z_\gamma$ and hence $Z_\gamma \leq \Phi(C_\beta) \cap V_\beta$ which contradicts (14.7.4). Therefore we have

(14.7.5) $[\Phi(C_\beta), \Phi(C_{\alpha'})] = \Phi(C_\beta) \cap \Phi(C_{\alpha'}) = 1.$

Next, we assume that $\eta(G_\beta, \Phi(C_\beta)) \neq 0$. Then $\eta(G_{\alpha'}, \Phi(C_{\alpha'})) \geq 1$ and thus $\Phi(C_\beta) \leq Q_{\alpha'}$ by (14.7.5). Hence $[\Phi(C_\beta), V_{\alpha'}] \leq Z_{\alpha'}$. Since $V_{\alpha'} \not\leq Q_\beta$ by Lemma 14.4(iii), we then get $[\Phi(C_\beta), V_{\alpha'}] = Z_{\alpha'}$ whence $Z_{\alpha'} \leq \Phi(C_\beta) \cap V_\beta$, again contradicting (14.7.4). So we have shown that

(14.7.6) $\eta(G_\beta, \Phi(C_\beta)) = 0.$

Now we suppose that $\Phi(C_\beta) \not\leq Q_{\alpha'}$. Then $\overline{\Phi(C_\beta)} = Z(\bar{G}_{\beta'\alpha'}) \cap O^2(\bar{G}_{\alpha'}) \cong \cong E(2)$. Thus

$$Z_{\alpha'}[\Phi(C_\beta), V_{\alpha'}] = [V_{\alpha'}, Q_{\beta'}, Q_{\beta'}].$$

Clearly, $K_\beta := \Phi(C_\beta)V_\beta \trianglelefteq G_\beta$ with, by (14.7.6), $\eta(G_\beta, K_\beta/V_\beta) = 0$. Therefore $V_\beta[\Phi(C_\beta), V_{\alpha'}] = V_\beta[V_{\alpha'}, Q_{\beta'}, Q_{\beta'}]$ is normalized by L_β and hence by $Q := [L_\beta, Q_\beta]$. Appealing to Lemma 12.4(i) we conclude that

$$\begin{aligned} V_\beta[V_{\alpha'}, Q_{\beta'}, Q_{\beta'}] &= V_\beta[V_\gamma, Q_{\beta'}, Q_{\beta'}] = V_\beta[U_{\beta'}, Q_{\beta'}, Q_{\beta'}] \\ &= V_\beta F_{\beta'} \trianglelefteq \langle L_\beta, G_{\beta\beta'} \rangle = G_\beta. \end{aligned}$$

Thus $H_\beta = V_\beta F_{\beta'} = V_\beta F_\alpha$ which is impossible by Lemma 14.2(ii), and so

(14.7.7) $\Phi(C_\beta) \leq Q_{\alpha'}.$

Now part (ii) is a direct consequence of Lemma 14.5(ii), (14.7.3), (14.7.7) and the fact that $\bar{J}_\beta$ and $\bar{C}_\beta$ are normal subgroups of $\bar{Q}_{\beta'} = \bar{G}_{\beta'\alpha'}$ containing the central transvection $\bar{Z}_\alpha$.

From part (ii) we infer that $[J_\beta, [V_{\alpha'}, Q_{\beta'}]] \leq Z_{\beta'}$ and hence $[J_\beta, [U_{\beta'}, Q_{\beta'}]] \leq Z_{\beta'}$. Since $J_\beta = \langle [U_{\beta'}, Q_{\beta'}]^{G_\beta} \rangle$, we then obtain

$$[J_\beta, J_\beta] = \left\langle \left[J_\beta, \left[U_{\beta'} Q_{\beta'}\right]\right]^{G_\beta} \right\rangle \leq \left\langle {Z_{\beta'}}^{G_\beta} \right\rangle = V_\beta.$$

Noting that $[J_\beta, J_\beta] \leq [C_\beta, C_\beta] \leq \Phi(C_\beta)$ an application of (14.7.4) yields that $[J_\beta, J_\beta] \leq Z_\beta$, so completing the proof of the lemma.

Lemma 14.8. *Put* ${V_\beta}^* = V_\beta[H_\beta, Q_\beta]$. *Then the following statements hold:*

(i) $\eta(G_\beta, C_\beta/V_\beta) = \eta(G_\beta, H_\beta/V_\beta) = 1$ *and the only non-central G_β-chief factor within C_β/V_β is not isomorphic to V_β/Z_β, as a G_β/Q_β-module;*

(ii) $J_\beta = H_\beta = [H_\beta, G_\beta]$ *and* $H_\beta/{V_\beta}^* \cong 4$ *or* $\begin{pmatrix}4\\1\end{pmatrix}$; *and*

(iii) ${V_\beta}^* = V_\beta[F_{\beta'}, Q_\beta]$ *and* ${V_\beta}^* \cap C_{\alpha'} = [V_\beta, Q_{\beta'}][F_{\beta'}, Q_\beta]$ *with* $[{V_\beta}^* : V_\beta] \leq 2$ *and* $[{V_\beta}^* : {V_\beta}^* \cap C_{\alpha'}] = 2$.

Proof. Because $C_{\alpha'} \leq G_\beta$

$$\left[C_{\alpha'}, J_\beta, J_\beta\right] \leq \left[J_\beta, J_\beta\right] \leq Z_\beta \leq V_{\alpha'},$$

by Lemma 14.7(i). Therefore

(14.8.1) J_β acts quadratically on $C_{\alpha'}/V_{\alpha'}$ and so (by Lemma 14.7(ii)) the non-central $G_{\alpha'}$-chief factors within $C_{\alpha'}/V_{\alpha'}$ are not isomorphic to $V_{\alpha'}/Z_{\alpha'}$ as $G_{\alpha'}/Q_{\alpha'}$-modules.

Using Lemma 14.7(ii) we see that

$$E\left(2^3\right) \lesssim \left[C_\beta, V_{\alpha'}\right] \leq \left[V_{\alpha'}, Q_{\beta'}\right] = V_{\alpha'} \cap Q_\beta \cong E\left(2^4\right).$$

Now $E(2^2) \cong Z_{\beta'} \leq V_{\alpha'} \cap Q_\beta$ and so $|[C_\beta/V_\beta, V_{\alpha'}]| \leq 2^2$. Then Lemmas 14.2(ii), 14.4(iv) and (14.8.1) force

(14.8.2) $\eta\left(G_\beta, C_\beta/V_\beta\right) = 1$.

Now set ${H_\beta}^* = [H_\beta, G_\beta]$, and note that $\eta(G_\beta, {H_\beta}^*) = 2$ and ${H_\beta}^* \geq V_\beta$. Also, by (14.8.1), $|[{H_\beta}^*/V_\beta, V_{\alpha'}]| = 2^2$ and thus $[V_{\alpha'}, Q_{\beta'}] = V_{\alpha'} \cap Q_\beta \leq {H_\beta}^*$. Employing Lemma 11.1(vii) gives ${H_\beta}^* \geq [U_{\beta'}, Q_{\beta'}]$. Since $J_\beta = \langle [U_{\beta'}, Q_{\beta'}]^{G_\beta} \rangle \geq$ $\geq H_\beta \geq {H_\beta}^*$ we obtain

(14.8.3) $J_\beta = H_\beta = \left[H_\beta, G_\beta\right]$.

If $\eta(G_\beta, V_\beta{}^*/V_\beta) \neq 0$, then $\eta(G_\beta, H_\beta/V_\beta{}^*) = 0$ and so $H_\beta = V_\beta[H_\beta, Q_\beta]F_\alpha$ which, commutating sufficiently often with Q_β, forces the untenable $H_\beta = V_\beta F_\alpha$. Thus $\eta(G_\beta, V_\beta{}^*) = \eta(G_\beta, V_\beta) = \eta(G_\beta, H_\beta/V_\beta{}^*) = 1$. Now $[F_\alpha : (F_\alpha \cap V_\beta)[F_\alpha, Q_\beta]] \leq 2$ means that we have $|F_\alpha V_\beta{}^*/V_\beta{}^*| = 2$ with $H_\beta/V_\beta{}^* = \langle (F_\alpha V_\beta{}^*/V_\beta{}^*)^{G_\beta} \rangle$. Since $H_\beta = [H_\beta, G_\beta]$ and $\eta(G_\beta, H_\beta/V_\beta{}^*) = 1$ we easily see that $H_\beta/V_\beta{}^* \cong 4$ or $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

From $\eta(G_\beta, V_\beta{}^*/V_\beta) = 0$, we clearly have $V_\beta{}^* = V_\beta[F_\alpha, Q_\beta] = V_\beta[F_{\beta'}, Q_\beta]$ with $[V_\beta{}^* : V_\beta] \leq 2$,

$$V_\beta{}^* \cap C_{\alpha'} = \left(V_\beta \cap C_{\alpha'}\right)\left[F_{\beta'}, Q_\beta\right] = \left[V_\beta, Q_{\beta'}\right]\left[F_{\beta'}, Q_\beta\right]$$

and $[V_\beta{}^* : V_\beta{}^* \cap C_{\alpha'}] = 2$. This completes the proof of the lemma.

We are now in a position to complete the proof of Theorem 14.3. Using Lemma 14.6 we see that

(14.3.1) $\quad Q_\beta/C_\beta \cong \left(V_\beta/Z_\beta\right)^* \cong V_\beta/Z_\beta.$

So we have $(Q_{\beta'} \cap Q_{\alpha'})/C_{\alpha'} \cong E(2^3)$. Since, by Lemmas 14.7(ii) and 14.8(ii), $H_\beta Q_{\alpha'}/Q_{\alpha'}$ is the non-quadratic $E(2^3)$-subgroup of $G_{\beta'\alpha'}/Q_{\alpha'}$ on $V_{\alpha'}/Z_{\alpha'}$, (14.3.1) implies

(14.3.2) $[(Q_{\beta'} \cap Q_{\alpha'})/C_{\alpha'}, H_\beta] \cong E(2)$; in particular $[H_\beta \cap Q_{\alpha'} : H_\beta \cap C_{\alpha'}] \geq 2$ and therefore $[H_\beta : H_\beta \cap C_{\alpha'}] \geq 2^4$.

We now observe from Lemma 14.8(iii) that $V_\beta{}^* \cap H_{\alpha'} = V_\beta{}^* \cap C_{\alpha'}$ with $[V_\beta{}^* : V_\beta{}^* \cap H_{\alpha'}] = 2$. Combining Lemmas 14.4(iii), (vi), 14.7(ii) and 14.8(ii) with Proposition 2.5(ii) gives

$$\left[H_\beta/V_\beta{}^* : \left[H_\beta/V_\beta{}^*, H_{\alpha'}\right]\right] = 2^2,$$

and consequently

$$\left[H_\beta : H_\beta \cap H_{\alpha'}\right] \leq \left[H_\beta : \left[H_\beta, H_{\alpha'}\right]\left(V_\beta{}^* \cap H_{\alpha'}\right)\right] \leq 2^3.$$

Since $H_\beta \cap H_{\alpha'} \leq H_\beta \cap C_{\alpha'}$, this clearly contradicts (14.3.2), so proving Theorem 14.3.

15. The structure of critical pairs.

The main result in this section is Theorem 15.7.

From Theorem 14.3 we have $b \geq 5$, and so U_α is elementary abelian. Our first result shows that there is an abundance of non-central chief factors in U_α.

LEMMA 15.1. *Let* $(\alpha, \alpha') \in \mathcal{C}$.

(i) $\eta(G_\alpha, U_\alpha) \geq 4$.

(ii) If $\eta(G_\alpha, U_\alpha) = 4$, then $[U_\alpha, Q_\alpha; 3] = [V_\beta, Q_\alpha; 3] = Z_\alpha$ and $V_\beta/Z_\beta \cong 4$ or $\binom{4}{1}$.

PROOF. Put ${V_\beta}^{(i)} = [V_\beta, Q_\alpha; i]$ and $U_\alpha^{(i)} = [U_\alpha, Q_\alpha; i]$ for $i \in \mathbb{N} \cup \{0\}$; let $n \in \mathbb{N}$ be such that ${V_\beta}^{(n)} \neq 1$ and ${V_\beta}^{(n+1)} = 1$. By Lemma 12.2(i) $\boldsymbol{\Omega}_1(Z(Q_\alpha)) = Z_\alpha$ and since ${V_\beta}^{(n)}$ is a $G_{\alpha\beta}$-invariant subgroup of $\boldsymbol{\Omega}_1(Z(Q_\alpha))$, Theorem 12.1 and Proposition 2.5(i) imply

(15.1.1) $n \geq 3$ and $Z_\beta \leq {V_\beta}^{(n)} \leq Z_\alpha = {U_\alpha}^{(n)}$.

Now suppose that $\eta(G_\alpha, {U_\alpha}^{(j)}/{U_\alpha}^{(j+1)}) = 0$ for some $j \in \{0, 1, \ldots, n-1\}$. Then ${U_\alpha}^{(j)} = {U_\alpha}^{(j+1)}{V_\beta}^{(j)}$ and hence ${U_\alpha}^{(j+1)} = {U_\alpha}^{(j+2)}{V_\beta}^{(j+1)}$. So ${U_\alpha}^{(j)} = {U_\alpha}^{(j+2)}{V_\beta}^{(j)}$ and consequently ${U_\alpha}^{(j)} = {V_\beta}^{(j)} \leq \mathrm{core}_{G_\alpha} V_\beta = Z_\alpha$. This implies ${U_\alpha}^{(n-1)} \leq Z_\alpha$ whence ${U_\alpha}^{(n)} = 1$, a contradiction. Thus we have

(15.1.2) $\eta\big(G_\alpha, {U_\alpha}^{(j)}/{U_\alpha}^{(j+1)}\big) \geq 1$ for $j \in \{0, 1, \ldots, n-1\}$.

Clearly (15.1.1) and (15.1.2) give (i). Now assume that $\eta(G_\alpha, U_\alpha) = 4$. Then, by (15.1.1) and (15.1.2), $Z_\beta \leq {V_\beta}^{(3)} \leq {U_\alpha}^{(3)} = Z_\alpha$. Since ${V_\beta}^{(3)} \not\leq Z_\beta$, ${V_\beta}^{(3)} = {U_\alpha}^{(3)} = Z_\alpha$ which establishes (ii).

LEMMA 15.2. *For* $(\alpha, \alpha') \in \mathcal{C}$, $V_{\alpha'} \not\leq G_\alpha$*; in particular* $V_{\alpha'} \not\leq Q_\beta$.

PROOF. Let $(\alpha, \alpha') \in \mathcal{C}$, put $\Delta(\alpha) = \{\lambda, \mu, \beta\}$ and assume by way of contradiction that $V_{\alpha'} \leq G_\alpha$. Since $Z_\alpha \leq G_{\alpha'}$ and $Z_\alpha \not\leq Q_{\alpha'}$, we have

(15.2.1) $V_{\alpha'} \leq G_{\alpha\beta}$ with $[Z_\alpha, V_{\alpha'}] = Z_\beta \neq Z_{\alpha'}$.

Note that there exists $\rho \in \Delta(\alpha')$ such that $Z_\rho \not\leq Q_\alpha$; moreover Z_ρ acts as an involution on U_α.

(15.2.2) $U_\alpha \not\leq G_{\alpha'}$; in particular $U_\alpha \not\leq Q_{\alpha'-1}$.

Suppose that $U_\alpha \leq G_{\alpha'}$ holds. Then, since U_α is abelian and $V_{\alpha'} \leq G_\alpha$, U_α acts quadratically on $V_{\alpha'}$. Since $\eta(G_\alpha, U_\alpha) \geq 4$ by Lemma 15.1(i) and $[U_\alpha \cap Q_{\alpha'} : U_\alpha \cap Q_{\alpha'} \cap Q_\rho] \leq 2$, we see that $U_\alpha Q_{\alpha'}/Q_{\alpha'} \cong E(2^3)$ with $\eta(G_\alpha, U_\alpha) = 4$. Also, from $|[U_\alpha, V_{\alpha'}]| \geq 2^4$, Lemma 15.1(ii) and Proposition 2.5(ii) force $V_{\alpha'}/Z_{\alpha'} \cong \binom{4}{1}$ with $[U_\alpha, V_{\alpha'}] \cong E(2^4)$.

Since Z_ρ interchanges λ and μ, Z_ρ normalizes $V := V_\lambda V_\mu$. Since $V_\lambda \cap V_\mu = Z_\alpha$ and $V_{\alpha'} \cong E(2^6)$, $\bar{V} := V/Z_\alpha = \bar{V}_\lambda \times \bar{V}_\mu$ with $\bar{V}_\lambda \cong E(2^4) \cong \bar{V}_\mu$

and hence $[\bar{V}, Z_\rho] \cong E(2^4)$. Now $[Z_\alpha, Z_\rho] = Z_\beta$ and $V \leq U_\alpha$ yields

$$E(2^5) \underset{\sim}{\leq} [V, Z_\rho] \leq [U_\alpha, V_{\alpha'}],$$

contrary to $[U_\alpha, V_{\alpha'}] \cong E(2^4)$. So we have (15.2.2).

Because $b \geq 5$ and $V_\lambda \underset{Z_\rho}{\sim} V_\mu$, $[V_\lambda, V_{\alpha'-2}] = [V_\mu, V_{\alpha'-2}] \leq V_\lambda \cap V_\mu = Z_\alpha$ and so $[U_\alpha, V_{\alpha'-2}] \leq Z_\alpha$. From (15.2.2) and $[U_\alpha, V_{\alpha'-2}] \leq C_{Z_\alpha}(V_{\alpha'}) = Z_\beta$ we obtain

(15.2.3) $[U_\alpha, V_{\alpha'-2}] = Z_\beta$.

(15.2.4) $U_\alpha \not\leq Q_{\alpha'-2}$

Assume $U_\alpha \leq Q_{\alpha'-2}$ holds. Since $U_\alpha \not\leq Q_{\alpha'-1}$ by (15.2.2) and $Z_{\alpha'-1} = Z_{\alpha'-2}Z_{\alpha'}$, $[U_\alpha, Z_{\alpha'}] \neq 1$. Hence $[U_\alpha \cap Q_{\alpha'-1} \cap Q_{\alpha'}, V_{\alpha'}] = 1$ and so $U_\alpha \cap Q_{\alpha'-1} \cap Q_{\alpha'} \leq C_{U_\alpha}(Z_\rho)$. Now $[U_\alpha : U_\alpha \cap Q_{\alpha'-1}] \leq 2$ and $\eta(G_\alpha, U_\alpha) \geq 4$ forces $(U_\alpha \cap Q_{\alpha'-1})Q_{\alpha'}/Q_{\alpha'}$ to be the quadratic $E(2^3)$-subgroup of $G_{\alpha'-1\alpha'}/Q_{\alpha'}$ on $V_{\alpha'}/Z_{\alpha'}$ and $[U_\alpha : U_\alpha \cap Q_{\alpha'-1}] = 2$. Also we note that $[U_\alpha \cap Q_{\alpha'-1}, V_{\alpha'}]Z_{\alpha'}$ has index 2^2 in $V_{\alpha'}$ and that $[U_\alpha \cap Q_{\alpha'-1}, V_{\alpha'}] \leq U_\alpha$. So U_α centralizes $[U_\alpha \cap Q_{\alpha'-1}, V_{\alpha'}]$ and $U_\alpha \not\leq Q_{\alpha'-1}$ whence, using the core argument, $[U_\alpha \cap Q_{\alpha'-1}, V_{\alpha'}] \leq \mathrm{core}_{G_{\alpha'-1}} V_{\alpha'} = Z_{\alpha'-1}$. But this is impossible since $(U_\alpha \cap Q_{\alpha'-1})Q_{\alpha'}/Q_{\alpha'}$ in the quadratic $E(2^3)$-subgroup of $G_{\alpha'-1\alpha'}/Q_{\alpha'}$ on $V_{\alpha'}/Z_{\alpha'}$. Therefore $U_\alpha \not\leq Q_{\alpha'-2}$.

(15.2.5) $[Z_\alpha, V_{\alpha'}] = [U_\alpha, V_{\alpha'-2}] = Z_\beta \leq V_{\alpha'-2} \cap V_{\alpha'} = Z_{\alpha'-1}$ and $Z_{\alpha'-1} = Z_{\alpha'-2}Z_\beta$.

By (15.2.3) and (15.2.4) $Z_\beta \neq Z_{\alpha'-2}$. Now (15.2.5) follows from (15.2.1) and (15.2.3).

Put $\widetilde{V}_{\alpha'-2} = V_{\alpha'-2}/Y_{\alpha'-2}$ and $P_{\alpha'-2} = \langle U_\alpha, G_{\alpha'-2\alpha'-1}\rangle$. Since $Z_{\alpha'-1} = Z_{\alpha'-2}Z_\beta$, $P_{\alpha'-2}$ centralizes $Z_{\alpha'-1}$ and, as $U_\alpha \not\leq G_{\alpha'-2\alpha'-1}$ by (15.2.2), $\bar{P}_{\alpha'-2} := P_{\alpha'-2}/Q_{\alpha'-2} \cong S_4 \times \mathbb{Z}_2$ with $\widetilde{V}_{\alpha'-2}|_{\bar{P}_{\alpha'-2}} \cong \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. However, by (15.2.5), $[U_\alpha, \widetilde{V}_{\alpha'-2}] = \widetilde{Z}_\beta = \widetilde{Z}_{\alpha'-1} = C_{\widetilde{V}_{\alpha'-2}}(P_{\alpha'-2})$ which implies $U_\alpha \leq O_2(P_{\alpha'-2}) \leq G_{\alpha'-2\alpha'-1}$. This contradiction completes the proof of Lemma 15.2.

LEMMA 15.3. *Let* $(\alpha, \alpha') \in \mathcal{C}$ *and* $R = [V_\beta, V_{\alpha'}]$. *If* $\eta(G_\beta, W_\beta) = 2$ *or* $\eta(G_\beta, [W_\beta, Q_\beta]V_\beta/V_\beta) = 0$, *then the following hold:*

(i) $R \leq Z_{\alpha+2} \cap Z_{\alpha'-1}$ *with* $RZ_\beta = Z_{\alpha+2}$ *and* $RZ_{\alpha'} = Z_{\alpha'-1}$. *If, moreover,* $b = 5$, *then* $R = Z_{\alpha+3} = Z_{\alpha'-2}$;

(ii) $V_\beta = [V_\beta, G_\beta]$ *and so* $V_\beta/Z_\beta \cong 4$ *or* $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$;

(iii) $V_\beta Q_{\alpha'}/Q_{\alpha'}$ *is the central transvection of* $G_{\alpha'-1\alpha'}/Q_{\alpha'}$ *acting on*

$V_{\alpha'}/Z_{\alpha'}$ with a symmetric statement holding for $V_{\alpha'}$ with the roles of β and α' interchanged; and

(iv) *either $W_\beta \leq Q_{\alpha'-2}$ or $W_\beta Q_{\alpha'-2}/Q_{\alpha'-2}$ is the central transvection of $G_{\alpha'-3\alpha'-2}/Q_{\alpha'-2}$ on $V_{\alpha'-2}/Z_{\alpha'-2}$, and so $[W_\beta : W_\beta \cap Q_{\alpha'-2}] \leq 2$.*

PROOF. Put $W_\beta{}^{(1)} = [W_\beta, Q_\beta]$ and $W_\beta{}^* = W_\beta{}^{(1)} V_\beta$. Note that $\eta(G_\beta, W_\beta/W_\beta{}^*) \neq 0$ else $W_\beta = W_\beta{}^{(1)} U_\alpha$ which produces the absurd $W_\beta = U_\alpha$. So we have $\eta(G_\beta, W_\beta{}^*/V_\beta) = 0$. Appealing to Lemma 12.7 gives $R \leq V_{\alpha'-2} \cap V_{\alpha'} = Z_{\alpha'-1}$. Since $V_{\alpha'} \not\leq Q_\beta$ by Lemma 15.2, there exists $\rho \in \Delta(\alpha')$ such that $(\rho, \beta) \in \mathcal{C}$, and likewise we deduce $R \leq V_\beta \cap V_{\alpha+3} = Z_{\alpha+2}$. This proves (i)

Since $Z_{\alpha+2} = RZ_\beta \leq [V_\beta, G_\beta]$, the statements in (ii) and (iii) follow readily while (iv) is an easy consequence of (iii).

LEMMA 15.4. *Let $(\alpha, \alpha') \in \mathcal{C}$. If $Z_{\alpha'} \leq V_\beta$, then $|V_{\alpha'} Q_\beta/Q_\beta| = 2$.*

PROOF. Let $(\alpha, \alpha') \in \mathcal{C}$ be such that $Z_{\alpha'} \leq V_\beta$ and assume that $|V_{\alpha'} Q_\beta/Q_\beta| \neq 2$. So, by Lemma 15.2, $|V_{\alpha'} Q_\beta/Q_\beta| \geq 2^2$. Also Lemma 15.3 implies that

(15.4.1) $\eta\big(G_\beta, W_\beta\big) \geq 3$.

Clearly we have $W_\beta \leq G_{\alpha'-2}$. Since $b \geq 5$, $Z_{\alpha'} \leq V_\beta \leq Z(W_\beta)$ and so W_β centralizes $Z_{\alpha'-1} = Z_{\alpha'-2} Z_{\alpha'}$. Thus $W_\beta \leq C_{G_{\alpha'-2}}(Z_{\alpha'-1} Y_{\alpha'-2}/Y_{\alpha'-2})$ and therefore $X := W_\beta \cap G_{\alpha'-2\alpha'-1}$ has index at most 2 in W_β by the parabolic argument (Lemma 3.10). Note that $X \leq C_{G_{\alpha'-1}}(Z_{\alpha'-1}) = Q_{\alpha'-1} \leq G_{\alpha'}$, and set $X^* = (X \cap Q_{\alpha'})V_\beta$. Then $[X : X^*] \leq 2^3$ and

$$\big[X^*, V_{\alpha'}\big] = \big[X \cap Q_{\alpha'}, V_{\alpha'}\big] R \leq Z_{\alpha'} R \leq V_\beta.$$

Hence, putting $\widetilde{W}_\beta = W_\beta/V_\beta$, we have $\widetilde{X}^* \leq C_{\widetilde{W}_\beta}(V_{\alpha'})$. Using (15.4.1) we then get

$$2^4 \leq \big[\widetilde{W}_\beta : C_{\widetilde{W}_\beta}(V_{\alpha'})\big] \leq \big[\widetilde{W}_\beta : \widetilde{X}^*\big] = \big[W_\beta : X^*\big] = \big[W_\beta : X\big]\big[X : X^*\big] \leq 2^4$$

and consequently $[W_\beta : X] = 2$ and $[X : X^*] = 2^3$. In particular

(15.4.2) $G_{\alpha'-1\alpha'} = Q_{\alpha'} X$.

Since $[X, V_{\alpha'}] \leq X \cap V_{\alpha'} \leq Q_\beta \cap V_{\alpha'}$ and $[V_{\alpha'} : Q_\beta \cap V_{\alpha'}] \geq 2^2$, (15.4.2) together with Proposition 2.5(i) implies that

(15.4.3) $\big[X, V_{\alpha'}\big] = X \cap V_{\alpha'} = Q_\beta \cap V_{\alpha'} \underset{4}{<} V_{\alpha'}$ with $V_{\alpha'}/Z_{\alpha'} \cong 1 \oplus 4$ or $1 \oplus \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

It now follows from (15.4.3) that $Z_{\alpha'-1} = Z_{\alpha'-2}Z_{\alpha'} \le Q_\beta \cap V_{\alpha'} = [X, V_{\alpha'}] \le [V_{\alpha'}, G_{\alpha'}]$ whence $V_{\alpha'} = [V_{\alpha'}, G_{\alpha'}]$, which contradicts the structure of $V_{\alpha'}$ given in (15.4.3). Thus we infer that $|V_{\alpha'}Q_\beta/Q_\beta| = 2$ must hold.

COROLLARY 15.5. *If* $(\alpha,\alpha')\in\mathcal{C}$ *and* $|V_\beta Q_{\alpha'}/Q_{\alpha'}|\ge 2^2$, *then* $|V_{\alpha'}Q_\beta/Q_\beta|\ge 2^2$.

PROOF. Suppose we have $(\alpha, \alpha') \in \mathcal{C}$ with $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \ge 2^2$. Then $Z_\beta \not\le V_{\alpha'}$ by Lemmas 15.2 and 15.4. Hence $[V_{\alpha'} \cap Q_\beta, V_\beta] = 1$. Since V_β cannot centralize a hyperplane of $V_{\alpha'}$, $[V_{\alpha'} : V_{\alpha'} \cap Q_\beta] \ge 2^2$ which proves the result.

LEMMA 15.6. *Let* $(\alpha, \alpha') \in \mathcal{C}$ *and suppose that* $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \ge 2^2$, *and hence by Corollary* 15.5 $|V_{\alpha'}Q_\beta/Q_\beta| \ge 2^2$. *Then the following statements hold.*

(i) $V_\beta Q_{\alpha'}/Q_{\alpha'} \cong V_{\alpha'}Q_\beta/Q_\beta \cong E(2^2)$;

(ii) $W_\beta \cap G_{\alpha'-1} \le G_{\alpha'}$;

(iii) $P_{\alpha'-2} := \langle W_\beta, Q_{\alpha'-1}\rangle \le G_{\alpha'-2} = Q_{\alpha'-2}P_{\alpha'-2}$ *and* $P_{\alpha'-2} \ge L_{\alpha'-2}$;

(iv) $b = 5$; *and*

(v) *if* $\widehat{G}_{\alpha'} = G_{\alpha'}/Q_{\alpha'}$, *then* $\widehat{W}_{\alpha'-2} \cong E(2^3)$ *and* $W_{\alpha'-2}$ *acts quadratically on* $V_{\alpha'}/Z_{\alpha'}$ *with*

$$\binom{4}{1} \lesssim V_{\alpha'}/Z_{\alpha'} \lesssim \binom{4}{1} \oplus 1.$$

PROOF. Put $R = [V_\beta, V_{\alpha'}]$ and $\widehat{G}_{\alpha'} = G_{\alpha'}/Q_{\alpha'}$. By Lemmas 15.2 and 15.4 $Z_{\alpha'} \not\le V_\beta$ and $Z_\beta \not\le V_{\alpha'}$. Hence $[V_\beta, Y_{\alpha'}] = [V_{\alpha'}, Y_\beta] = 1$. Since $[V_{\alpha'} : RY_{\alpha'}] = [V_\beta : RY_\beta] = 2^2$ we obtain (i). Noting that $R \not\le Z_{\alpha'-1}$ and $[W_\beta, V_\beta] = 1$, the core argument yields (ii).

Lemma 15.3 (iii) gives

(15.6.1) $\eta(G_\beta, W_\beta) \ge 3$.

(15.6.2) $V_\beta Q_{\alpha'}/Q_{\alpha'} \le (W_\beta \cap G_{\alpha'})Q_{\alpha'}/Q_{\alpha'} \lesssim E(2^3)$.

Since $W_\beta \cap G_{\alpha'}$ centralizes $RY_{\alpha'}/Y_{\alpha'} \cong E(2^2)$, Proposition 2.5(vi) implies that $(W_\beta \cap G_{\alpha'})Q_{\alpha'}/Q_{\alpha'}$ is elementary abelian, so giving (15.6.2).

We now establish part (iii). Suppose that $Q_{\alpha'-2}P_{\alpha'-2} \ne G_{\alpha'-2}$. Then $\langle W_\beta, G_{\alpha'-2\alpha'-1}\rangle \ne G_{\alpha'-2}$ and so, by the parabolic argument, $[W_\beta : W] \le 2$ where $W := W_\beta \cap G_{\alpha'-2\alpha'-1}$. By (ii) $W \le G_{\alpha'}$, and by (15.6.2) $[W : V_\beta(W \cap Q_{\alpha'})] \le 2$. Clearly $[V_\beta(W \cap Q_{\alpha'}), V_{\alpha'}] \le RZ_{\alpha'}$. Putting $\bar{W}_\beta = W_\beta/V_\beta$ we have $[\bar{W}_\beta : \overline{V_\beta(W \cap Q_{\alpha'})}] \le 2^2$ and $|[\overline{V_\beta(W \cap Q_{\alpha'})}, V_{\alpha'}]| \le 2$.

Since $V_{\alpha'}Q_\beta/Q_\beta \cong E(2^2)$, this gives $\eta(G_\beta, \bar{W}_\beta) \leq 1$, contradicting (15.6.1). This proves part (iii).

As a direct consequence of (iii), Proposition 2.5(ii) and Lemmas 12.7 and 15.2 we have

(15.6.3) (i) $Z_{\alpha'}R$ is not normal in $G_{\alpha'-1\alpha'}$ and $Z_\beta R$ is not normal in $G_{\beta\alpha+2}$.

(ii) Either

(a) $4 \underset{\sim}{\leq} V_{\alpha'}/Z_{\alpha'} \underset{\sim}{\leq} 4 \oplus 1$ and $\widehat{V}_\beta \sim \langle s_1, t\rangle$ or

(b) $\begin{pmatrix}4\\1\end{pmatrix} \underset{\sim}{\leq} V_{\alpha'}/Z_{\alpha'} \underset{\sim}{\leq} \begin{pmatrix}4\\1\end{pmatrix} \oplus 1$ and $\widehat{V}_\beta \sim \langle s_1, t\rangle$ or $\langle s_2, t\rangle$ (using the notation of Proposition 2.5).

Moving onto part (iv), we now assume $b > 5$ and derive a contradiction. Set $W_0 = W_\beta \cap Q_{\alpha'-2}$. By (ii) $W_0 \leq G_{\alpha'}$. Since W_β is abelian we observe that $[W_\beta : W_0] \leq 2^3 \geq [W_0 : W_0 \cap Q_{\alpha'}]$. Now W_β being abelian and (iii) imply that $Z_{\alpha'} \not\leq W_\beta$. Hence $[W_0 \cap Q_{\alpha'}, V_{\alpha'}] = 1$, and so $[W_\beta : C_{W_\beta}(V_{\alpha'})] \leq 2^6$. Therefore, by (15.6.1), $\eta(G_\beta, W_\beta) = 3$, $[W_\beta : C_{W_\beta}(V_{\alpha'})] = 2^6$ and $W_0Q_{\alpha'}/Q_{\alpha'} \cong E(2^3)$.

(15.6.4) The non central G_β chief factors in W_β are isomorphic natural S_6-modules.

Observe that $V_{\alpha'}$ acts quadratically upon W_β since $[W_\beta, V_{\alpha'}] \leq W_{\alpha'-2}$. By (15.6.3)(i) $V_{\alpha'}Q_\beta/Q_\beta \not\leq Z(G_{\beta\alpha+2}/Q_\beta)$ from which (15.6.4) follows.

Since $W_0Q_{\alpha'}/Q_{\alpha'} \cong E(2^3)$ with W_0 acting quadratically on $V_{\alpha'}$, we may choose $t \in W_0$ so that $V_\beta\langle t\rangle Q_{\alpha'} = W_0Q_{\alpha'}$ with t acting as a transvection on $V_{\alpha'}/Y_{\alpha'}$. Hence t acts as a transvection on $V_{\alpha'}/Z_{\alpha'}$ and, recalling that $Z_{\alpha'} \not\leq W_\beta$, we see that $C := C_{V_{\alpha'}}(t)$ has index 2 in $V_{\alpha'}$. Therefore $C \not\leq Q_\beta$. Now

$$[W_0, C] = \big[V_\beta\langle t\rangle\big(W_0 \cap Q_{\alpha'}\big), C\big] = \big[V_\beta, C\big] \leq V_\beta$$

and consequently $[W_\beta/V_\beta : C_{W_\beta/V_\beta}(C)] \leq 2^3$. Since $\eta(G_\beta, W_\beta/V_\beta) = 2$, C must induce a transvection on at least one non-central G_β-chief factor within W_β/V_β. Appealing to (15.6.4) gives that C induces a transvection on V_β/Y_β and hence on V_β/Z_β. Because $Z_\beta \not\leq V_{\alpha'}$, C induces a transvection on V_β. But then $[W_0, C] = [V_\beta, C] \cong \mathbb{Z}_2$ whereas $[W_0, C/Y_{\alpha'}] \cong E(2^2)$. From this untenable situation we deduce that $b = 5$.

Finally we consider part (v). By (15.6.3) $\widehat{V}_\beta \ntrianglelefteq \widehat{G}_{\alpha'-1\alpha'} = \widehat{Q}_{\alpha'-1}$. So, since $V_\beta \leq W_{\alpha'-2}$ (as $b = 5$) and $\widehat{W}_{\alpha'-2} \trianglelefteq \widehat{Q}_{\alpha'-1}$, $\widehat{V}_\beta \lneqq \widehat{W}_{\alpha'-2}$. Evidently we have $[V_\beta, W_{\alpha'-2}] \leq V_\beta$ and consulting (15.6.3)(ii) we see that $\widehat{V}_\beta \cap (\widehat{G}_{\alpha'-1\alpha'})' = 1$.

So $\widehat{W}_{\alpha'-2} \leq C_{\widehat{G}_{\alpha'-1\alpha'}}(\widehat{V}_\beta)$. Hence $\widehat{W}_{\alpha'-2} \cong E(2^3)$ by (15.6.3)(ii). Now if (v) is false, then (15.6.3)(i) must hold and therefore $\widehat{W}_{\alpha'-2}$ is the non-quadratic $E(2^3)$-subgroup of $\widehat{G}_{\alpha'-1\alpha'}$ (acting on $V_{\alpha'}/Z_{\alpha'}$). Thus $Y_{\alpha'}[V_{\alpha'}, W_{\alpha'-2}] = Y_{\alpha'}[V_{\alpha'}, G_{\alpha'-1\alpha'}] \underset{2}{\lneq} V_{\alpha'}$. Next we observe that $[W_{\alpha'-2}, W_{\alpha'-2}] \leq Q_{\alpha'}$. So, since $G_{\alpha'-2}$ is transitive on vertices distance 2 away from $\alpha'-2$ and $b=5$, $[W_{\alpha'-2}, W_{\alpha'-2}] \leq Q_\beta$ also. This then gives, since $Y_{\alpha'} \leq Q_\beta$, that

$$Y_{\alpha'}\big[V_{\alpha'}, W_{\alpha'-2}\big] \leq Y_{\alpha'}\big[W_{\alpha'-2}, W_{\alpha'-2}\big] \leq Q_\beta$$

Therefore $[V_{\alpha'} : V_{\alpha'} \cap Q_\beta] = 2$, against $[V_{\alpha'} : V_{\alpha'} \cap Q_\beta] = 2^2$. This establishes (v) and completes the proof of the lemma.

THEOREM 15.7. *Let $(\alpha, \alpha') \in \mathcal{C}$. Then*

(i) $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = |V_{\alpha'} Q_\beta/Q_\beta| = 2$;
(ii) $|[V_\beta, V_{\alpha'}]| = 2$ *and* $[V_\beta, V_{\alpha'} \cap Q_\beta] = 1 = [V_{\alpha'}, V_\beta \cap Q_{\alpha'}]$; *and*
(iii) *there exists $\delta \in \Delta(\beta)$ such that $(\delta, \alpha') \in \mathcal{C}$ and $\langle G_{\delta\beta}, V_{\alpha'}\rangle = G_\beta$.*

PROOF. (i) Assume by way of contradiction that the assertion is false. Then from Lemma 15.6 we have

(15.7.1) $b = 5$;

(15.7.2) if $\widehat{G}_{\alpha'} = G_{\alpha'}/Q_{\alpha'}$, then $\widehat{W}_{\alpha'-2} \cong E(2^3)$ and $W_{\alpha'-2}$ acts quadratically on $V_{\alpha'}/Z_{\alpha'}$ where

$$\binom{4}{1} \lessapprox V_{\alpha'}/Z_{\alpha'} \lessapprox \binom{4}{1} \oplus 1; \text{ and}$$

(15.7.3) $P_{\alpha'-2} := \langle W_\beta, Q_{\alpha'-1}\rangle \leq G_{\alpha'-2} = Q_{\alpha'-2}P_{\alpha'-2}$.

Put ${W_\beta}^* = [W_\beta, W_\beta]V_\beta$ and $R = [V_\beta, V_{\alpha'}]$. Bringing together (15.7.1), (15.7.2) and Lemmas 11.1(vii), 15.6(ii) we obtain

(15.7.4) $W_\beta Q_{\alpha'-2}/Q_{\alpha'-2} \cong W_{\alpha'} Q_{\alpha'-2}/Q_{\alpha'-2} \cong E(2^3)$ with both groups acting quadratically on $V_{\alpha'-2}/Z_{\alpha'-2}$ and ${W_\beta}^* \leq Q_{\alpha'-2} \cap W_\beta \leq G_{\alpha'}$.

By (15.7.4) we have

$$Y_{\alpha+3} = Y_{\alpha'-2} \leq \big[W_\beta, V_{\alpha'-2}\big]Z_{\alpha+2} \leq \big[W_\beta, W_\beta\big]V_\beta = {W_\beta}^*.$$

Since G_β is transitive on vertices distance 2 from β, $\langle Y_\beta, Y_\gamma, Y_{\alpha+3}\rangle \leq {W_\beta}^*$, where $\Delta(\alpha+2) = \{\beta, \gamma, \alpha+3\}$. Now (15.7.2) and the definition of $F_{\alpha+2}$ implies that $F_{\alpha+2} \leq {W_\beta}^*$, whence $H_\beta \leq {W_\beta}^*$. Therefore $\eta(G_\beta, {W_\beta}^*) \geq 2$ by Lemma 14.2(ii). Since ${W_\beta}^*$ centralizes $RY_{\alpha'}/Y_{\alpha'} \cong E(2^2)$, ${W_\beta}^*$ acts quadratically on $V_{\alpha'}/Y_{\alpha'}$. Then $V_{\alpha'}Q_\beta/Q_\beta \cong E(2^2)$ forces

(15.7.5) $|W_\beta{}^* Q_{\alpha'}/Q_{\alpha'}| = 2^3$ with $Z_{\alpha'} \leq [W_\beta{}^*, V_{\alpha'}] \cong E(2^4)$; hence $Z_{\alpha'-1} = = Z_{\alpha'-2} Z_{\alpha'} \leq W_\beta{}^*$.

From (15.7.4) we have $[[V_{\alpha'-2}, W_\beta], W_\beta] \leq Z_{\alpha'-2}$ and thus $[[W_\beta, W_\beta], W_\beta] \leq \leq V_\beta$. So $[W_\beta, W_\beta{}^*] \leq V_\beta$. Hence, as $Z_{\alpha'-1} \leq W_\beta{}^*$ by (15.7.5),

$$\big[Z_{\alpha'-1}, W_\beta\big] \leq \big[W_\beta{}^*, W_\beta\big] \cap V_{\alpha'-2} \leq V_\beta \cap V_{\alpha'-2} = Z_{\alpha+2}.$$

Therefore W_β normalizes the group $X := Y_{\alpha'-2} Z_{\alpha+2} Z_{\alpha'-1}$. Symmetrically we deduce that $W_{\alpha'}$ also normalizes X. By (15.7.3) $W_\beta Q_{\alpha'-2} \neq W_{\alpha'} Q_{\alpha'-2}$ and so Proposition 2.8(vii) yields that

(15.7.6) $P := \langle W_\beta, Q_{\alpha'-2}, W_{\alpha'} \rangle = N_{G_{\alpha'-2}}(X)$ with $|X/Y_{\alpha'-2}| \in \{2, 2^2\}$ and $P/Q_{\alpha'-2} \cong S_4 \times \mathbb{Z}_2$.

If $|X/Y_{\alpha'-2}| = 2$, then $X = Y_{\alpha'-2} Z_{\alpha'-1}$ is normalized by $P_{\alpha'-2}$ which is impossible by (15.7.3). So $|X/Y_{\alpha'-2}| = 2^2$. But then $O_2(P/Q_{\alpha'-2})$ is the only $E(2^3)$-subgroup of $P/Q_{\alpha'-2}$ which acts quadratically on $V_{\alpha'-2}/Z_{\alpha'-2}$, so (15.7.4) forces $W_\beta Q_{\alpha'-2} = W_{\alpha'} Q_{\alpha'-2}$, a contradiction. Therefore (i) holds.

(ii) In view of part (i) and Lemma 15.2 it is sufficient for us to show that for $(\alpha, \alpha') \in \mathcal{C}$ $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1$. So we assume $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] \neq 1$ and argue for a contradiction. Since $V_\beta \not\leq Q_{\alpha'}$, we have

(15.7.7) $Z_{\alpha'} = \big[V_\beta \cap Q_{\alpha'}, V_{\alpha'}\big] \leq V_\beta$ and $\big|\big[V_\beta, V_{\alpha'}\big]\big| \geq 2^2$.

Clearly there exists $\rho \in \Delta(\alpha')$ for which $[V_\beta \cap Q_{\alpha'}, Z_\rho] = Z_{\alpha'}$. Put $\Delta(\rho) = \{\alpha', \sigma, \tau\}$ and $V = V_\sigma V_\tau$. Since $V_\beta \cap Q_{\alpha'}$ interchanges σ and τ, $V_\sigma \cap V_\tau = Z_\rho$ and $Z_{\alpha'} \leq [V_\beta \cap Q_{\alpha'} V]$, we deduce that

(15.7.8) $\big|\big[V_\beta \cap Q_{\alpha'}, V\big]\big| \geq |V_\sigma|/2$.

(15.7.9) $U_\rho \not\leq G_\beta$.

For $U_\rho \leq G_\beta$ implies that V normalizes $V_\beta \cap Q_{\alpha'}$ and so

$$|[V_\beta \cap Q_{\alpha'}, V]| \leq |V_\beta|/4 = |V_\sigma|/4,$$

contradicting (15.7.8). Hence $U_\rho \not\leq G_\beta$.

(15.7.10) $U_\rho \not\leq Q_{\alpha+3}$

Assume $U_\rho \leq Q_{\alpha+3}$ holds. Then $U_\rho \leq G_{\alpha+2}$ and so $U_\rho \not\leq Q_{\alpha+2}$ by (15.7.8). Since $[V_\beta, V_{\alpha'}] \leq V_\beta \cap V_{\alpha'}$ and $[U_\rho, V_{\alpha'}] = 1$, the core argument forces

$$\big[V_\beta, V_{\alpha'}\big] \leq \mathrm{core}_{G_{\alpha+2}} V_\beta = Z_{\alpha+2}.$$

Then (15.7.7) gives $Z_{\alpha+2} = [V_\beta, V_{\alpha'}] \leq V_\alpha$, which yields $U_\rho \leq C_{G_{\alpha+2}}(Z_{\alpha+2}) = Q_{\alpha+2} \leq G_\beta$, against (15.7.9). Therefore $U_\rho \not\leq Q_{\alpha+3}$, as asserted.

Since $[V_{\alpha+3}, V_{\alpha'}] = 1 = [V_\beta, V_{\alpha+3}] = 1$, $V_\beta \cap Q_{\alpha'}$ centralizes $[V_{\alpha+3}, U_\rho] = [V_{\alpha+3}, V] = [V_{\alpha+3}, V_\sigma][V_{\alpha+3}, V_\tau]$. From $V_\beta \cap Q_{\alpha'}$ interchanging σ and τ we get $[V_{\alpha+3}, U_\rho] = [V_{\alpha+3}, V_\sigma] = [V_{\alpha+3}, V_\tau] \leq V_\sigma \cap V_\tau \cap C(V_\beta \cap Q_{\alpha'}) = C_{Z_\rho}(V_\beta \cap Q_{\alpha'}) = Z_{\alpha'}$. Now (15.7.10) implies that

(15.7.11) $\quad \left[V_{\alpha+3}, U_\rho\right] = Z_{\alpha'} \leq V_{\alpha+3}$ and $Z_{\alpha'} \neq Z_{\alpha+3}$.

Combining (15.7.7) and (15.7.11) gives $Z_{\alpha'} \leq V_\beta \cap V_{\alpha+3} = Z_{\alpha+2}$ and then $Z_{\alpha+2} = Z_{\alpha+3} Z_{\alpha'}$. So $U_\rho \leq P_{\alpha+3} := C_{G_{\alpha+3}}(Z_{\alpha+2})$ with $\widetilde{V}_{\alpha+3}|_{P_{\alpha+3}} \cong \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ (where $\widetilde{V}_{\alpha+3} = V_{\alpha+3}/Y_{\alpha+3}$). From (15.7.11) $U_\rho \leq O_2(P_{\alpha+3}) \leq G_{\alpha+2\alpha+3}$ which then gives $U_\rho \leq Q_{\alpha+2}$, contradicting (15.7.9) and concluding the proof of (ii).

(iii) Since $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1$ by (ii), we may argue as in Lemma 14.4(vi).

16. Case 3 bites the dust.

Employing Theorem 15.7 we begin this section by determining, for a critical pair (α, α'), which vertices H_β fixes and the location of the commutator $[F_\alpha, V_{\alpha'} \cap Q_\beta]$.

LEMMA 16.1. *Let* $(\alpha, \alpha') \in \mathcal{C}$. *Then*

(i) $H_\beta \leq Q_{\alpha'-2} \cap Q_{\alpha'-1}$;
(ii) $[F_\alpha, V_{\alpha'} \cap Q_\beta] \leq Z_\beta$; *and*
(iii) H_β *is abelian.*

PROOF. Assume first that $H_\beta \not\leq Q_{\alpha'-2}$. Then there exists $\beta - 1 \in \Delta(\beta)$ such that $F_{\beta-1} \not\leq Q_{\alpha'-2}$. Putting $\Delta(\beta-1) = \{\beta, \sigma, \tau\}$ we have $F_{\beta-1} = \langle T_{\beta\beta-1}, T_{\sigma\beta-1}, T_{\tau\beta-1}\rangle$. Since $T_{\beta\beta-1} \leq V_\beta \leq Q_{\alpha'-2}$, we may assume without loss of generality that $T_{\sigma\beta-1} \not\leq Q_{\alpha'-2}$ and hence $V_\sigma \not\leq Q_{\alpha'-2}$. So we may find $\gamma \in \Delta(\sigma)$ such that $(\gamma, \alpha'-2) \in \mathcal{C}$. By Theorem 15.7 $V_{\alpha'-2}Q_\sigma/Q_\sigma$ acts as a transvection upon V_σ/Z_σ with $C_{V_\sigma}(V_{\alpha'-2}) = V_\sigma \cap Q_{\alpha'-2} \lneqq_2 V_\sigma$. Hence $V_\sigma = (V_\sigma \cap Q_{\alpha'-2})T_{\sigma\beta-1}$ and consequently $[V_\sigma, V_{\alpha'-2}] = [T_{\sigma\beta-1}, V_{\alpha'-2}]$. If $V_\beta/Z_\beta \not\cong E(2^4)$, then $[T_{\sigma\beta-1}, V_{\alpha'-2}] \leq Z_\sigma$. While if $V_\beta/Z_\beta \cong E(2^4)$, then $T_{\sigma\beta-1}/Z_\sigma = [V_\sigma, Q_{\beta-1}; 2]/Z_\sigma$ is centralized by all transvections of $G_{\sigma\beta-1}/Q_\sigma$.

Hence in either case we have

$$[V_\sigma, V_{\alpha'-2}] = [T_{\sigma\beta-1}, V_{\alpha'-2}] \le Z_\sigma,$$

whence $V_{\alpha'-2} \le Q_\sigma$, a contradiction. So we have shown that $H_\beta \le Q_{\alpha'-2}$.

Assume next that $H_\beta \not\le Q_{\alpha'-1}$. So H_β is transitive on $\Delta(\alpha'-1) \setminus \{\alpha'-2\}$ and therefore, as $H_\beta \le Q_\beta$, $Q_\beta V_{\alpha'} = Q_\beta U_{\alpha'-1}$. Again, we may find $\beta - 1 \in \Delta(\beta) \setminus \{\alpha+2\}$ such that $F_{\beta-1} \not\le Q_{\alpha'-1}$ (note that $F_{\alpha+2} \le U_{\alpha+2} \le \le Q_{\alpha'-1}$). Since $\eta(G_{\alpha'-1}, U_{\alpha'-1}) \ge 4$ by Lemma 15.1 and $U := U_{\alpha'-1} \cap \cap Q_\beta \lneqq_2 U_{\alpha'-1}$, we infer that $|[F_{\beta-1}, U]| \ge 2^3$. On the other hand, by the $G_{\beta-1}/Q_{\beta-1}$ module structure of $F_{\beta-1}/Z_{\beta-1}$, obviously $|[F_{\beta-1}, U]| \le 2^3$, and so we have $Z_{\beta-1} \le [F_{\beta-1}, U] \cong E(2^3)$. Thence $Z_{\beta-1} \le U_{\alpha'-1}$ and so $[Z_{\beta-1}, U_{\alpha'-1}] = 1$. Consequently $U \le Q_{\beta-1}$ and thus, by Lemma 14.2(i),

$$[F_{\beta-1}, U] \le [F_{\beta-1}, Q_{\beta-1}] = Z_{\beta-1},$$

contrary to $|[F_{\beta-1}, U]| = 2^3$. Thus we conclude that $H_\beta \le Q_{\alpha'-1}$, and we have proven (i).

For (ii) observe that from Theorem 15.7 $V_{\alpha'} \cap Q_\beta \le Q_\alpha$ and hence $[F_\alpha, V_{\alpha'} \cap Q_\beta] \le [F_\alpha, Q_\alpha] = Z_\alpha$. From (i) we have $F_\alpha \le H_\beta \le G_{\alpha'}$ and therefore

$$[F_\alpha, V_{\alpha'} \cap Q_\beta] \le V_{\alpha'} \cap Z_\alpha \le Q_{\alpha'} \cap Z_\alpha = Z_\beta,$$

as required.

Because $H_\beta \le W_\beta \le Q_\alpha$ we have, using Lemma 14.2(i), $[F_\alpha, H_\beta] \le Z_\alpha$ and hence $[H_\beta, H_\beta] \le V_\beta$. Since, by Theorem 15.7, $V_\beta Q_{\alpha'}/Q_{\alpha'}$ acts as a transvection of $G_{\alpha'-1\alpha'}/Q_{\alpha'}$ on $V_{\alpha'}/Z_{\alpha'}$, it follows from part (i) that $[H_\beta, H_\beta] \le Q_{\alpha'}$. Recalling that $Y_\beta \le C_{V_\beta}(V_{\alpha'}) \le Q_{\alpha'}$ we see that $[H_\beta, H_\beta] \le Y_\beta$. Now $H_\beta \le Q_{\alpha+3}$ (using part (i) if $b = 5$). Let $\gamma \in \Delta(\beta)$ and $\delta \in \Delta(\gamma) \setminus \{\beta\}$. Since G_β is transitive on vertices distance 2 from β, $H_\beta \le Q_\delta$ and therefore

$$[T_{\delta\gamma}, H_\beta] \le [V_\delta, Q_\delta] \cap Y_\beta = Z_\delta \cap Y_\beta = 1.$$

Hence

$$[H_\beta, H_\beta] = \left\langle [T_{\delta\gamma}, H_\beta]^{G_\beta} \right\rangle = 1,$$

which proves (iii).

LEMMA 16.2. *Let* $(\alpha, \alpha') \in \mathcal{C}$ *and suppose that* $U_\alpha \le G_{\alpha'}$. *Then* $[U_\alpha, H_\beta] = 1$.

PROOF. Set $\Delta(\alpha) = \{\beta, \lambda, \mu\}$. Again we have $H_\beta \le Q_{\alpha+3}$ and hence $H_\beta \le Q_\lambda \cap Q_\mu$. Therefore

$$[H_\beta, U_\alpha] = [H_\beta, V_\lambda V_\mu] = [H_\beta, V_\lambda][H_\beta, V_\mu] \le Z_\lambda Z_\mu = Z_\alpha$$

By Lemma 16.1 $H_\beta \le G_{\alpha'}$ and thus, as $Z_\alpha Q_{\alpha'}/Q_{\alpha'} = V_\beta Q_{\alpha'}/Q_{\alpha'}$ acts as a transvection on $V_{\alpha'}/Z_{\alpha'}$, $[H_\beta, U_\alpha] \le Q_{\alpha'}$. Hence $[H_\beta, U_\alpha] \le Z_\alpha \cap Q_{\alpha'} = Z_\beta$. This in turn implies that $[H_\beta, V_\lambda] \le Z_\lambda \cap Z_\beta = 1$ as well as $[H_\beta, V_\mu] \le$ $\le Z_\mu \cap Z_\beta = 1$, so giving $[H_\beta, U_\alpha] = 1$.

LEMMA 16.3. *There exists a critical pair (α, α') which satisfies the following two conditions:*

(i) $\langle G_{\alpha\beta}, V_{\alpha'}\rangle = G_\beta$ *and*
(ii) $[F_\alpha, V_{\alpha'} \cap Q_\beta] = 1$.

PROOF. Suppose the lemma is false. So by Lemma 16.1(ii) $[F_\delta, V_{\delta'} \cap Q_{\delta+1}] = Z_{\delta+1}$ for all $(\delta, \delta') \in \mathcal{C}$ with $\langle G_{\delta\delta+1}, V_{\delta'}\rangle = G_{\delta+1}$. By Theorem 15.7(iii) we may select a critical pair (α, α') satisfying condition (i). So, since $F_\alpha \le H_\beta \le G_{\alpha'}$,

$$Z_\beta = [F_\alpha, V_{\alpha'} \cap Q_\beta] \le [F_\alpha, V_{\alpha'}] \le V_{\alpha'}.$$

Now $[T_{\beta\alpha}, V_{\alpha'} \cap Q_\beta] \le [V_\beta, V_{\alpha'} \cap Q_\beta] = 1$ and so we have $[T_{\lambda\alpha}, V_{\alpha'} \cap Q_\beta] = Z_\beta$ where $\lambda \in \Delta(\alpha) \setminus \{\beta\}$. Because $T_{\lambda\alpha} \trianglelefteq G_{\lambda\alpha}$ and $V_{\alpha'} \cap Q_\beta \le Q_\alpha \le G_\lambda$ we obtain $Z_\alpha = Z_\lambda Z_\beta \le T_{\lambda\alpha}$ and hence, by the definition of $T_{\lambda\alpha}$,

(16.3.1) $V_\beta/Z_\beta \cong 4$ and $T_{\lambda\alpha} = [V_\lambda, Q_\alpha; 2] \cong E(2^3)$.

From $[T_{\lambda\alpha}, V_{\alpha'} \cap Q_\beta] = Z_\beta \ne Z_\lambda$ we also deduce that

(16.3.2) $V_{\alpha'} \cap Q_\beta \not\le Q_\lambda$ and $V_{\alpha'} \cap Q_\beta$ does not centralize $T_{\lambda\alpha}/Z_\lambda$; in particular, $(V_{\alpha'} \cap Q_\beta)Q_\lambda/Q_\lambda$ and hence $(Q_\alpha \cap Q_\beta)Q_\lambda/Q_\lambda$ is not contained in the $E(2^3)$-subgroup of $G_{\lambda\alpha}/Q_\lambda$ acting quadratically on V_λ/Z_λ.

By Lemma 16.1(iii) H_β is abelian and hence H_β acts quadratically on $V_{\alpha'}$. Since $E(2^2) \cong Z_\beta[V_\beta, V_{\alpha'}] \le [H_\beta, V_{\alpha'}]$ and $\eta(G_\beta, H_\beta/V_\beta) \ge 1$, (16.3.1) implies that

(16.3.3) $E(2^3) \cong C_{V_{\alpha'}}(H_\beta) = [H_\beta, V_{\alpha'}] \ge Z_\beta [V_\beta, V_{\alpha'}] Z_{\alpha'-1} \gtrsim E(2^2)$.

Furthermore, $\eta(G_\beta, H_\beta) = 2$ and $V_{\alpha'}Q_\beta/Q_\beta$ acts as a transvection on each of the two non-central G_β chief factors within H_β which must be isomorphic natural G_β/Q_β-modules.

(16.3.4) $F_\alpha Q_{\alpha'} \gneq V_\beta Q_{\alpha'}$ and, in particular, $\left|(Q_{\alpha'-2} \cap Q_{\alpha'-1})Q_{\alpha'}/Q_{\alpha'}\right| \geq 2^3$

Suppose that $F_\alpha Q_{\alpha'} = V_\beta Q_{\alpha'}$ holds. Then

$$[F_\alpha, V_{\alpha'}] = [F_\alpha \cap Q_{\alpha'}, V_{\alpha'}][V_\beta, V_{\alpha'}] \leq Z_{\alpha'}[V_\beta, V_{\alpha'}] \cong E(2^2).$$

Since $[V_\beta, V_{\alpha'}] \neq Z_\beta = [F_\alpha, V_{\alpha'} \cap Q_\beta]$ we get

$$Z_{\alpha'} \leq [F_\alpha, V_{\alpha'}] = Z_\beta[V_\beta, V_{\alpha'}] \leq V_\beta.$$

But then $V_\beta F_\alpha \trianglelefteq \langle G_{\alpha\beta}, V_{\alpha'}\rangle = G_\beta$, against Lemma 14.2(i). Therefore we must have $F_\alpha Q_{\alpha'} \gneq Z_\alpha Q_{\alpha'} = V_\beta Q_{\alpha'}$. Hence $|(Q_{\alpha'-2} \cap Q_{\alpha'-1})Q_{\alpha'}/Q_{\alpha'}| \geq 2^2$. Combining (16.3.2) and Lemma 11.1(vii) with the fact that $\langle s_1, t\rangle$ is not a normal subgroup of T (see Proposition 2.5(ii)) yields that $|(Q_{\alpha'-2} \cap Q_{\alpha'-1})Q_{\alpha'}/Q_{\alpha'}| \geq 2^3$, as required.

(16.3.5) $[U_\alpha, Z_{\alpha'}] = 1$

From (16.3.3)

$$Z_{\alpha'} \leq [H_\beta, V_{\alpha'}] \leq H_\beta \leq W_\beta.$$

If $b > 5$, then (16.3.5) clearly holds, so we now assume $b = 5$. Since $Z_\beta \leq V_{\alpha'}$, we have

$$Z_\beta \leq V_\beta \cap V_{\alpha+3} \cap V_{\alpha'} = Z_{\alpha+2} \cap Z_{\alpha'-1}.$$

Because $Z_\beta \neq Z_{\alpha+3} \leq Z_{\alpha+2} \cap Z_{\alpha'-1}$ we deduce that $Z_{\alpha+2} = Z_{\alpha'-1}$. Consequently $Z_{\alpha'} \leq Z_{\alpha'-1} = Z_{\alpha+2} \leq V_\beta$. Now $[V_\beta, W_\beta] = 1$ implies that (16.3.5) holds when $b = 5$.

We claim that $U_\alpha \leq Q_{\alpha'-2}$. For if $U_\alpha \not\leq Q_{\alpha'-2}$ then there exists $\alpha - 2 \in \Delta^{[2]}(\alpha)$ such that $(\alpha-2, \alpha'-2) \in \mathcal{C}$. Moreover, by Theorem 15.7(iii), we can also assume that $\langle G_{\alpha-2\alpha-1}, V_{\alpha'-2}\rangle = G_{\alpha-1}$ (where $\{\alpha-1\} = \Delta(\alpha) \cap \Delta(\alpha-2)$). Hence, by our supposition,

$$Z_{\alpha-1} = [F_{\alpha-2}, V_{\alpha'-2} \cap Q_{\alpha-1}] \leq V_{\alpha'-2} \leq Q_{\alpha'}$$

whence $Z_\alpha = Z_{\alpha-1}Z_\beta \leq Q_{\alpha'}$. With this contradiction we have established the claim. Now, using (16.3.5), we deduce that $U_\alpha \leq C_{G_{\alpha'-1}}(Z_{\alpha'-1}) = Q_{\alpha'-1} \leq G_{\alpha'}$. Applying Lemma 16.2 gives $[U_\alpha, H_\beta] = 1$. Therefore U_α centralizes $[H_\beta, V_{\alpha'}] \cong E(2^3)$ and so U_α acts quadratically on $V_{\alpha'}$. Since $F_\alpha \leq U_\alpha$, (16.3.4) implies that $|U_\alpha Q_{\alpha'}/Q_{\alpha'}| \geq 2^2$ so we see that

$$[U_\alpha, V_{\alpha'}] \leq C_{V_{\alpha'}}(U_\alpha) = [H_\beta, V_{\alpha'}].$$

Hence we obtain $[U_\alpha, V_{\alpha'}] \leq [H_\beta, V_{\alpha'}] \leq H_\beta$. Thus $H_\beta U_\alpha \trianglelefteq \langle G_{\alpha\beta}, V_{\alpha'}\rangle = G_\beta$ from which we conclude that $W_\beta = H_\beta U_\alpha$ and $\eta(G_\beta, W_\beta) = \eta(G_\beta, H_\beta) = 2$.

Since $\eta(G_\beta, W_\beta/[W_\beta, Q_\beta]V_\beta) \neq 0$, there exists a G_β-invariant subgroup E of W_β containing $[W_\beta, Q_\beta]V_\beta$ such that W_β/E is a natural or the dual of an orthogonal G_β/Q_β-module. Appealing to [Proposition 3; LPR2] yields $V_\lambda E/E \cong E(2^3)$ (where $\lambda \in \Delta(\alpha) \setminus \{\beta\}$). Since $[Q_\lambda, V_\lambda] = Z_\lambda \leq Z_\alpha \leq E$, $(Q_\lambda \cap Q_\alpha)Q_\beta/Q_\beta$ centralizes the group $V_\lambda E/E \cong E(2^3)$. Using (16.3.4) and Lemma 11.1(vii) $|(Q_\lambda \cap Q_\alpha)Q_\beta/Q_\beta| \geq 2^3$ and so it is the $E(2^3)$-quadratic subgroup of $G_{\alpha\beta}/Q_\beta$ on V_β/Z_β. This contradicts (16.3.2) and concludes the proof of the lemma.

From now on (α, α') will be a critical pair satisfying conditions (i) and (ii) of Lemma 16.3.

LEMMA 16.4. (i) $F_\alpha Q_{\alpha'} = Z_\alpha Q_\alpha = V_\beta Q_{\alpha'}$. *Moreover* $[F_\alpha, V_{\alpha'}] = Z_{\alpha'}[V_\beta, V_{\alpha'}] \cong E(2^2)$ *with* $[F_\alpha \cap Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'} \not\leq V_\beta F_\alpha$; *and*
(ii) $H_{\alpha+3} \leq Q_\beta \cap Q_\alpha$.

PROOF. Recall that $F_\alpha \leq H_\beta \leq G_{\alpha'}$. Since F_α centralizes a hyperplane in $V_{\alpha'}$ (by condition (ii) of Lemma 16.3) we obviously have $F_\alpha Q_{\alpha'} = Z_\alpha Q_{\alpha'} = V_\beta Q_{\alpha'}$. Now the other statements in (i) are easy consequences of the fact that $H_\beta \gneqq V_\beta F_\alpha$.

In proving (ii) we may assume $b = 5$. Suppose for the moment that $U_\alpha \leq Q_{\alpha'-2}$ and let $\lambda \in \Delta(\alpha) \setminus \{\beta\}$. By Lemma 15.2 we cannot have $(\tau, \lambda) \in \mathcal{C}$ for any $\tau \in \Delta(\alpha' - 2)$, and hence $V_{\alpha'-2} \leq Q_\lambda$. Because $Z_\lambda \not\leq V_{\alpha'-2}$, $[V_\lambda, V_{\alpha'-2}] = 1$ and therefore $[U_\alpha, V_{\alpha'-2}] = 1$. So $U_\alpha \leq G_{\alpha'}$ and hence, using Lemma 16.2, $[U_\alpha, H_\beta] = 1$. Thus $[H_{\alpha+3}, U_{\alpha+2}] = 1$. In particular $[H_{\alpha+3}, V_\beta] = 1$ which then yields $H_{\alpha+3} \leq Q_\alpha \cap Q_\beta$. Therefore (ii) holds when $U_\alpha \leq Q_{\alpha'-2}$. Now we consider the case $U_\alpha \not\leq Q_{\alpha'-2}$. Then there exists $\alpha - 2 \in \Delta^{[2]}(\alpha)$ such that $(\alpha - 2, \alpha' - 2) \in \mathcal{C}$. Set $\{\alpha - 1\} = \Delta(\alpha) \cap \Delta(\alpha - 2)$. By Lemma 16.2 there exists $\tau \in \Delta(\alpha' - 2)$ such that $(\tau, \alpha - 1) \in \mathcal{C}$. Applying Lemma 16.1(i) to this critical pair gives $H_{\alpha'-2} \leq Q_\alpha \cap Q_\beta$. This proves (ii) since $\alpha' - 2 = \alpha + 3$.

By Lemma 16.4(i) there exists $\rho \in \Delta(\alpha') \setminus \{\alpha' - 1\}$ such that $[F_\alpha \cap Q_{\alpha'}, Z_\rho] = Z_{\alpha'}$ and (hence) $F_\alpha \cap Q_{\alpha'} \not\leq Q_\rho$. Also we note that

LEMMA 16.5. $(\rho, \beta) \in \mathcal{C}$.

PROOF. If the lemma is false, then $Z_\rho \leq Q_\beta \leq G_\alpha$ whence $Z_{\alpha'} = [F_\alpha \cap Q_{\alpha'}, Z_\rho] \leq [F_\alpha, Z_\rho] \leq F_\alpha$, contrary to Lemma 16.4(i).

LEMMA 16.6. $\eta(G_\rho, U_\rho) \leq 3$.

PROOF. Put $\Delta(\rho) = \{\alpha', \sigma, \tau\}$.

Assume for the moment that $V_\sigma \not\leq Q_{\alpha+3}$. Then there exists $\sigma+1 \in \Delta(\sigma)$ such that $(\sigma+1, \alpha+3) \in \mathcal{C}$. By Lemma 15.2 there exists $\mu \in \Delta(\alpha+3)$ for which $(\mu, \sigma) \in \mathcal{C}$ and by Theorem 5.7(iii) we may choose μ so as $\langle G_{\mu\alpha+3}, V_\sigma\rangle = G_{\alpha+3}$. Additionally assume that $[F_\mu, V_\sigma \cap Q_{\alpha+3}] = 1$ (so condition (ii) of Lemma 16.3 holds for (μ, σ)). Applying Lemma 16.4(i) to (μ, σ) gives $[F_\mu \cap Q_\sigma, V_\sigma] = Z_\sigma$. Since $V_\sigma \leq G_{\alpha+3}$, we then obtain

$$Z_\sigma = \left[F_\mu \cap Q_\sigma, V_\sigma\right] \leq \left[H_{\alpha+3}, V_\sigma\right] \leq H_{\alpha+3}.$$

By Lemma 16.1(i) $H_{\alpha+3} \leq Q_\beta$ and thus $Z_\rho = Z_{\alpha'} Z_\sigma \leq Q_\beta$, contrary to $(\rho, \beta) \in \mathcal{C}$ (Lemma 16.5). Therefore we must have $[F_\mu, V_\sigma \cap Q_{\alpha+3}] \neq 1$ and recourse to Lemma 16.1(ii) gives $[F_\mu, V_\sigma \cap Q_{\alpha+3}] = Z_{\alpha+3}$. Since $F_\mu \leq G_\sigma$ by Lemma 16.1(i), this gives $Z_{\alpha+3} \leq V_\sigma$. Clearly, as $F_\alpha \leq U_\alpha$ and $b \geq 5$, $F_\alpha \cap Q_{\alpha'}$ centralizes $Z_{\alpha+3}$ and therefore $Z_{\alpha+3} \leq V_\sigma \cap V_\tau = Z_\rho$ by the core argument. Because $Z_\rho \not\leq Q_\beta$ and $Z_{\alpha'} \not\leq Q_\beta$ it follows that $Z_{\alpha+3} = Z_{\alpha'}$. But then $Z_{\alpha'} \leq V_\beta$ which is ruled out by Lemma 16.4(i). Hence we deduce that $V_\sigma \leq Q_{\alpha+3}$. As a consequence of this Lemma 15.2 implies that $(\alpha+2, \sigma) \notin \mathcal{C}$ and so $[Z_{\alpha+2}, V_\sigma] \leq Z_\sigma$. In view of $Z_\rho = Z_{\alpha'} Z_\sigma, Z_{\alpha'} \leq Q_\beta$ and $(\rho, \beta) \in \mathcal{C}$, we have $[Z_{\alpha+2}, V_\sigma] = 1$. Consequently $V_\sigma \leq C_{G_{\alpha+2}}(Z_{\alpha+2}) = Q_{\alpha+2}$. Similarly we obtain $V_\tau \leq Q_{\alpha+2}$ and thus

(16.6.1) $\quad U_\rho \leq G_\beta$.

Since $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1$ and $F_\alpha \cap Q_{\alpha'} \not\leq Q_\rho$,

$$\begin{aligned}\left[V_\beta \cap Q_{\alpha'}, U_\rho\right] &= \left[V_\beta \cap Q_{\alpha'}, V_\sigma\right]\left[V_\beta \cap Q_{\alpha'}, V_\tau\right] \\ &= \left[V_\beta \cap Q_{\alpha'}, V_\sigma\right] \leq V_\beta \cap V_\sigma.\end{aligned}$$

The core argument and Lemma 16.5 give

$$\left[V_\beta \cap Q_{\alpha'}, U_\rho\right] \leq V_\beta \cap Z_\rho = Z_{\alpha'}.$$

Now $Z_{\alpha'} \not\leq V_\beta$ by Lemma 16.4(i) forces $[V_\beta \cap Q_{\alpha'}, U_\rho] = 1$. Hence, by (16.6.1), $|U_\rho Q_\beta / Q_\beta| \leq 2$. So

(16.6.2) $\quad \left[U_\rho : U_\rho \cap Q_\alpha\right] \leq 2^2$.

Observe that

$$\left[F_\alpha \cap Q_{\alpha'}, U_\rho \cap Q_\alpha\right] \leq Z_\alpha \cap U_\rho \leq Z_\alpha \cap Q_{\alpha'} = Z_\beta.$$

Therefore for $f \in F_\alpha \cap Q_{\alpha'}$, $|[f, U_\rho]| \leq 2^3$ by (16.6.2). Since $F_\alpha \cap Q_{\alpha'} \not\leq Q_\rho$, we conclude that $\eta(G_\rho, U_\rho) \leq 3$.

Together Lemmas 15.1 and 16.6 are responsible for the demise of Case 3, and so Theorem 14.1 is proven.

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