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On the b-ary Expansion of an Algebraic Number.

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ABSTRACT - Let $b \geq 2$ be an integer and denote by $\omega(b)$ the number of its prime divisors. As a consequence of our main theorem, we establish that, for n large enough, there are at least $(\log n)^{1+1/(\omega(b)+4)} \cdot (\log \log n)^{-1/4}$ non-zero digits among the first n digits of the b-ary expansion of every algebraic, irrational, real number. Analogous results are given for the Hensel expansion of an algebraic, irrational p-adic number and for the β-expansion of 1, when β is either a Pisot, or a Salem number.

1. Introduction.

Let $b \geq 2$ be an integer. Despite some recent progress, the b-ary expansion of an irrational algebraic number is still very mysterious. It is commonly believed that $\sqrt{2}$, and every algebraic irrational number, should share most of the properties satisfied by almost all real numbers (here and below, 'almost all' always refers to the Lebesgue measure); in particular, every digit $0, 1, \ldots, b-1$ should appear in its b-ary expansion with the same frequency $1/b$. However, it is still unknown whether, for $b \geq 3$, three different digits occur infinitely often in the b-ary expansion of $\sqrt{2}$.

There are several ways to measure the complexity of a real number. A first one is the block complexity. For a real number θ and a positive integer n, let denote by $p(n, \theta, b)$ the total number of distinct blocks of n digits in the b-ary expansion of θ. It is shown in [2] with the help of a combinatorial transcendence criterion from [5] that the complexity function $n \mapsto p(n, \xi, b)$ of an irrational algebraic number ξ grows faster than any linear function. Since the complexity function of a rational number is uniformly bounded, this shows that the algebraic, irrational numbers are 'not too simple' in this sense.

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Another point of view is taken in [6, 18], where the authors obtained very nice lower bounds for the number of occurrences of the digit 1 among the first n digits of the binary expansion of an algebraic irrational number. Apparently, their approach does not extend to a base b with $b \geq 3$.

In the present paper, we consider the question of the complexity of an irrational algebraic number from a third point of view. Our aim is to estimate the asymptotic behaviour of the number of digit changes in its b-ary expansion. For a non-zero real number θ, write

$$\theta = \pm \sum_{k \geq -k_0} \frac{a_k}{b^k} = a_{-k_0} \ldots a_0 . a_1 a_2 \ldots ,$$

where $k_0 \geq 0$, $a_{-k_0} \neq 0$ if $k_0 > 0$, the a_k's are integers from $\{0, 1, \ldots, b-1\}$ and a_k is non-zero for infinitely many indices k. The sequence $(a_k)_{k \geq -k_0}$ is uniquely determined by θ: it is its b-ary expansion. We then define the function nbdc, 'number of digit changes', by

$$\mathrm{nbdc}(n, \theta, b) = \mathrm{Card}\{1 \leq k \leq n : a_k \neq a_{k+1}\},$$

for any positive integer n. Apparently, this function has not been studied up to now. If one believes that every irrational algebraic number ξ behaves in this respect like almost all numbers, then $n \mapsto \mathrm{nbdc}(n, \xi, b)$ should grow linearly in terms of n. Not surprisingly, much less can be proved. Indeed, it easily follows from Ridout's Theorem [16], recalled in Section 3, that we have

$$\frac{\mathrm{nbdc}(n, \xi, b)}{\log n} \underset{n \to +\infty}{\longrightarrow} +\infty, \tag{1.1}$$

a proof of this is given at the beginning of Section 4. The main object of this note is to improve (1.1) by using a quantitative version of Ridout's Theorem. As a corollary, we get lower and upper bounds for the sum of the first n digits of an algebraic irrational number.

Our paper is organized as follows. The main results are stated in Section 2 and proved in Section 4, with the help of an auxiliary theorem given in Section 3. We display in Sections 5 and 6 the analogues of our main result for the Hensel expansion of irrational algebraic p-adic numbers and for β-expansions, respectively. Furthermore, we discuss in Section 7 various results on the transcendence of the series $\sum_{j \geq 1} b^{-n_j}$, for an integer $b \geq 2$ and a rapidly growing sequence of positive integers $(n_j)_{j \geq 1}$. In the last section, we briefly consider an analogous problem for continued fraction expansions.

2. Results.

For any integer x at least equal to 2, we denote by $\omega(x)$ the number of its prime divisors.

THEOREM 1. *Let $b \geq 2$ be an integer. For every irrational, real algebraic number ξ, there exists an effectively computable constant $c(\xi, b)$, depending only on ξ and b, such that*

$$\text{nbdc}(n, \xi, b) \geq 3\,(\log n)^{1+1/(\omega(b)+4)} \cdot (\log\log n)^{-1/4}, \tag{2.1}$$

for every integer $n \geq c(\xi, b)$.

We stress that for every non-zero rational number p/q and for every integer $b \geq 2$, there exist integers ℓ_0 and C such that $\text{nbdc}(n, p/q, b^\ell) \leq C$ for $\ell \geq \ell_0$. The growth of the functions $n \mapsto \text{nbdc}(n, \theta, b^\ell)$ can be used to measure the complexity of the real number θ. In this respect, the 'simplest' numbers are the rational numbers and Theorem 1 shows that algebraic irrational numbers are 'not too simple'.

The proof of Theorem 1 given in Section 4 yields a (very) slightly better estimate than (2.1) and an explicit value for the constant $c(\xi, b)$.

We display two immediate corollaries of Theorem 1. A first one concerns the number of non-zero digits in the expansion of a non-zero algebraic number in an integer base, a question already investigated in [6, 18].

COROLLARY 1. *Let $b \geq 2$ be an integer. Let ξ be a non-zero algebraic number. Then, for n large enough, there are at least*

$$(\log n)^{1+1/(\omega(b)+4)} \cdot (\log\log n)^{-1/4}$$

non-zero digits among the first n digits of the b-ary expansion of ξ.

For $b = 2$, Corollary 1 gives a much weaker result than the one obtained by Bailey, Borwein, Crandall, and Pomerance [6], who proved that, among the first n digits of the binary expansion of a real irrational algebraic number ξ of degree d, there are at least $c(\xi)n^{1/d}$ occurrences of the digit 1, where $c(\xi)$ is a suitable positive constant (see also Rivoal [18]).

Besides the number of non-zero digits, we may as well study the sum of digits. To this end, for an integer $b \geq 2$ and a non-zero real number θ with b-ary expansion

$$\theta = \pm \sum_{k \geq -k_0} \frac{a_k}{b^k},$$

set

$$\mathcal{S}(n,\theta,b)=\sum_{k=1}^{n}a_k,$$

for every positive integer n. We know that $\mathcal{S}(n,\theta,b)/n$ tends to $(b-1)/2$ with n for almost all real numbers θ. It is believed that the same holds when θ is an arbitrary irrational algebraic number. In that case, we are only able to prove a much weaker result, namely that $n\mapsto\mathcal{S}(n,\xi,b)$ cannot increase too slowly, nor too rapidly.

COROLLARY 2. *Let $b\geq 2$ be an integer. Let ξ be an irrational algebraic number. Then, for n large enough, we have*

$$\mathcal{S}(n,\xi,b)\geq(\log n)^{1+1/(\omega(b)+4)}\cdot(\log\log n)^{-1/4} \tag{2.2}$$

and

$$\mathcal{S}(n,\xi,b)\leq n(b-1)-(\log n)^{1+1/(\omega(b)+4)}\cdot(\log\log n)^{-1/4}.$$

Inequality (2.2) should be compared with Theorem 4.1 from [3], which gives an upper bound for the product of the first n partial quotients of an irrational, algebraic number. Actually, the proofs of both results are very similar.

3. An auxiliary result.

For a prime number ℓ and a non-zero rational number x, we set $|x|_\ell:=\ell^{-u}$, where u is the exponent of ℓ in the prime decomposition of x. Furthermore, we set $|0|_\ell=0$. With this notation, the main result of [16] reads as follows.

THEOREM (Ridout, 1957). *Let S_1 and S_2 be disjoint finite sets of prime numbers. Let θ be a real algebraic number. Let ε be a positive real number. Then there are only finitely many rational numbers p/q with $q\geq 1$ such that*

$$\left|\theta-\frac{p}{q}\right|\cdot\prod_{\ell\in S_1}|p|_\ell\cdot\prod_{\ell\in S_2}|q|_\ell<\frac{1}{q^{2+\varepsilon}}\cdot \tag{3.1}$$

For the proof of Theorem 1, we need an explicit upper bound for the number of solutions to (3.1). The best one for our purpose has been established by Locher [13].

We normalize absolute values and heights as follows. Let $\boldsymbol{K}$ be an algebraic number field of degree k. Let $\mathcal{M}(\boldsymbol{K})$ denote the set of places on $\boldsymbol{K}$. For x in $\boldsymbol{K}$ and a place v in $\mathcal{M}(\boldsymbol{K})$, define the absolute value $|x|_v$ by

(i) $|x|_v = |\sigma(x)|^{1/k}$ if v corresponds to the embedding $\sigma : \boldsymbol{K} \hookrightarrow \boldsymbol{R}$;

(ii) $|x|_v = |\sigma(x)|^{2/k} = |\overline{\sigma}(x)|^{2/k}$ if v corresponds to the pair of conjugate complex embeddings $\sigma, \overline{\sigma} : \boldsymbol{K} \hookrightarrow \boldsymbol{C}$;

(iii) $|x|_v = (N\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(x)/k}$ if v corresponds to the prime ideal $\mathfrak{p}$ of $O_{\boldsymbol{K}}$.

Here, $N\mathfrak{p}$ is the norm of $\mathfrak{p}$ and $\mathrm{ord}_{\mathfrak{p}}(x)$ is the exponent of $\mathfrak{p}$ in the decomposition of the ideal (x) into prime ideals. These absolute values satisfy the product formula

$$\prod_{v \in \mathcal{M}(\boldsymbol{K})} |x|_v = 1 \qquad \text{for } x \text{ in } \boldsymbol{K}^*.$$

The *height* of x is then defined by

$$H(x) = \prod_{v \in \mathcal{M}(\boldsymbol{K})} \max\{1, |x|_v\}.$$

It does not depend on the choice of the number field $\boldsymbol{K}$ containing x. The Mahler measure of x is given by

$$M(x) = H(x)^{k'},$$

where $k' \leq k$ is the degree of x.

We denote by $|\cdot|_v$ an extension of $|\cdot|_v$ to the algebraic closure of $\boldsymbol{K}$.

THEOREM L. *Let $\boldsymbol{K}$ be an algebraic number field. Let ε be real with $0 < \varepsilon < 1$. Let $\boldsymbol{F}/\boldsymbol{K}$ be an extension of number fields of degree f. Let S be a finite set of places on $\boldsymbol{K}$ of cardinality s. Suppose that for each v in S we are given a fixed element θ_v in $\boldsymbol{F}$. Let H be a real number with $H \geq H(\theta_v)$ for all v in S. Consider the inequality*

$$\prod_{v \in S} \min\{1, |\theta_v - \gamma|_v\} < H(\gamma)^{-2-\varepsilon} \tag{3.2}$$

to be solved in elements γ in $\boldsymbol{K}$. Then there are at most

$$e^{7s+19} \varepsilon^{-s-4} \log(6f) \cdot \log\left(\varepsilon^{-1} \log(6f)\right)$$

solutions γ in $\boldsymbol{K}$ to (3.2) *with*

$$H(\gamma) \geq \max\{H, 4^{4/\varepsilon}\}.$$

Proof. This is Theorem 2 from [13] in the case $d = 1$, with different notation. □

4. Proof of Theorem 1.

• *Preliminaries.*

Without any loss of generality, let us assume that

$$(b-1)/b < \xi < 1. \tag{4.1}$$

Write

$$\xi = \sum_{k \geq 1} \frac{a_k}{b^k} = 0.a_1 a_2 \ldots,$$

and define the increasing sequence of positive integers $(n_j)_{j\geq 1}$ by $a_1 = \ldots = a_{n_1}$, $a_{n_1} \neq a_{n_1+1}$ and $a_{n_j+1} = \ldots = a_{n_{j+1}}$, $a_{n_{j+1}} \neq a_{n_{j+1}+1}$ for every $j \geq 1$. Observe that

$$\mathrm{nbdc}(n, \xi, b) = \max\{j : n_j \leq n\}$$

for $n \geq n_1$, and that $n_j \geq j$ for $j \geq 1$. To construct good rational approximations to ξ we simply truncate its b-ary expansion at rank a_{n_j+1} and then complete with repeating the digit a_{n_j+1}. Precisely, for $j \geq 1$, we define the rational number

$$\xi_j := \sum_{k=1}^{n_j} \frac{a_k}{b^k} + \sum_{k=n_j+1}^{+\infty} \frac{a_{n_j+1}}{b^k} = \sum_{k=1}^{n_j} \frac{a_k}{b^k} + \frac{a_{n_j+1}}{b^{n_j}(b-1)}.$$

Set

$$\begin{aligned} P_j(X) &= a_{n_j+1} - a_{n_j} + (a_{n_j} - a_{n_j-1})X + \ldots + (a_2 - a_1)X^{n_j-1} + a_1 X^{n_j} \\ &= a_{n_j+1} - a_{n_j} + (a_{n_{j-1}+1} - a_{n_{j-1}})X^{n_j - n_{j-1}} + \ldots + \\ &\qquad\qquad (a_{n_1+1} - a_{n_1})X^{n_j - n_1} + a_1 X^{n_j}, \end{aligned}$$

and observe that

$$\xi_j = \frac{P_j(b)}{b^{n_j}(b-1)}. \tag{4.2}$$

Let p_j and q_j be the coprime positive integers such that

$$\xi_j = \frac{p_j}{q_j}. \tag{4.3}$$

Since $1 \le |a_{n_j+1} - a_{n_j}| \le b-1$, there exists a prime divisor ℓ of b such that $\mathrm{ord}_\ell(a_{n_j+1} - a_{n_j}) < \mathrm{ord}_\ell(b)$, thus

$$\mathrm{ord}_\ell\left(\frac{a_{n_j+1} - a_{n_j}}{b^{n_j}(b-1)}\right) \le -(n_j - 1)\mathrm{ord}_\ell(b) - 1. \tag{4.4}$$

We infer from (4.4) and

$$\mathrm{ord}_\ell\left(\frac{(a_{n_{j-1}+1} - a_{n_{j-1}})b^{n_j - n_{j-1} - 1} + \ldots + (a_2 - a_1)b^{n_j - 2} + a_1 b^{n_j - 1}}{b^{n_j - 1}(b-1)}\right) \ge -(n_j - 1)\,\mathrm{ord}_\ell(b)$$

that $q_j \ge 2^{n_j - 1}$. Consequently, we get

$$2^{n_j - 1} \le H(\xi_j) = q_j \le b^{n_j}(b-1). \tag{4.5}$$

Observe that

$$|\xi - \xi_j| < \frac{1}{b^{n_{j+1}}}. \tag{4.6}$$

- *Proof of inequality* (1.1).

We first establish the result claimed in the introduction, that is, that Ridout's Theorem implies (1.1). Let ε be a real number with $0 < \varepsilon < 1$. Let j be a positive integer. It follows from Ridout's Theorem that there exists a positive constant $C(\varepsilon)$ such that

$$\left|\xi - \frac{P_j(b)}{b^{n_j}(b-1)}\right| \ge C(\varepsilon)\, b^{-n_j(1+\varepsilon)}.$$

We then deduce from (4.6) that there is a positive constant $C'(\varepsilon)$ such that $n_{j+1} \le (1+\varepsilon)n_j + C'(\varepsilon)$ for $j \ge 1$. This implies that $n_j \le C''(\varepsilon) \cdot (1+\varepsilon)^j$ holds for $j \ge 1$ and some positive constant $C''(\varepsilon)$. Consequently, we get

$$\mathrm{nbdc}(n, \xi, b) \ge \frac{\log n}{\varepsilon} - 2\,\frac{\log C''(\varepsilon)}{\log(1+\varepsilon)} \ge \frac{\log n}{2\varepsilon},$$

when n is sufficiently large. This proves (1.1).

The improvement upon (1.1) rests on the fact that, by Theorem L, we have an explicit estimate for the number of solutions to (3.1).

- *Preparation for the proof of Theorem* 1.

Let denote by d the degree of ξ.

The Liouville inequality as stated by Waldschmidt [20], p. 84, asserts that

$$\left|\xi - \frac{p}{q}\right| \geq \frac{1}{M(\xi) \cdot (2q)^d}, \quad \text{for any positive integers } p, q.$$

Consequently, we obtain that

$$b^{n_{j+1}} \leq (2b)^d \cdot M(\xi) \cdot b^{dn_j},$$

thus we have

$$n_{j+1} \leq 2dn_j, \tag{4.7}$$

for every integer j with $j \geq 2 \log(3M(\xi))$.

Let ε be a positive real number with $\varepsilon < 1/(\log 6d)$. Let j_1 be the smallest integer with

$$j_1 > \max\{2d\log\left(3M(\xi)\right), 10(2d+1)\varepsilon^{-1} \log b\}, \tag{4.8}$$

and let $j \geq j_1$ be an integer. Let $\ell_1, \ldots, \ell_{\omega(b)}$ denote the (distinct) prime divisors of b. We infer from (4.6), (4.7), (4.2), (4.3) and the assumption (4.1) that

$$\begin{aligned} \min\{1, |1/\xi - q_j/p_j|\} \cdot \prod_{i=1}^{\omega(b)} |q_j|_{\ell_i} &< 4 \min\{1, |\xi - p_j/q_j|\} \cdot \prod_{i=1}^{\omega(b)} |q_j|_{\ell_i} \\ &< 4b^{-n_{j+1}} q_j^{-1} (b-1) \\ &\leq 4(b-1)^{1+2d} \left((b-1)b^{n_j}\right)^{-n_{j+1}/n_j} q_j^{-1} \\ &\leq 4(b-1)^{1+2d} H(\xi_j)^{-1-n_{j+1}/n_j}. \end{aligned} \tag{4.9}$$

It follows from (4.5) and (4.8) that $H(\xi_j)^\varepsilon \geq 2^{\varepsilon j - 1} > 4(b-1)^{1+2d}$. Consequently, we infer from (4.9) that, if $n_{j+1} \geq (1+2\varepsilon)n_j$ holds, then

$$\begin{aligned} \min\{1, |1/\xi - q_j/p_j|\} \cdot \prod_{i=1}^{\omega(b)} |q_j|_{\ell_i} &\leq 4(b-1)^{1+2d} H(\xi_j)^{-\varepsilon} H(\xi_j)^{-2-\varepsilon} \\ &< H(\xi_j)^{-2-\varepsilon}. \end{aligned} \tag{4.10}$$

Observe that all the rational numbers ξ_j are distinct. Furthermore, we have $H(\xi_j) \geq \max\{H(\xi), 4^{4/\varepsilon}\}$ as soon as $j \geq j_1$.

We now apply Theorem L with $\boldsymbol{K} = \boldsymbol{Q}$, $S = \{\infty, \ell_1, \ldots, \ell_{\omega(b)}\}$, $\theta_\infty = 1/\xi$, $\theta_{\ell_i} = 0$ for $i = 1, \ldots, \omega(b)$. Let $s = 1 + \omega(b)$ be the cardinality of S and set

$$T(\varepsilon) := 2\, e^{7s+19} \varepsilon^{-s-4} (\log \varepsilon^{-1}) \log(6d). \tag{4.11}$$

It then follows from (4.10), our choice of ε, and Theorem L that for every integer $j \ge j_1$ outside a set of cardinality at most $T(\varepsilon)$ we have

$$n_{j+1} \le (1+2\varepsilon)n_j.$$

- *Completion of the proof of Theorem* 1.

Let J be an integer with

$$J > \max\{\big(2d \log \big(3M(\xi)\big)\big)^2, (4db)^6\}.$$

Let j_2 be the smallest integer with

$$j_2 > \max\{2d \log \big(3M(\xi)\big), 4db\sqrt{J}\},$$

and check that $J \ge j_2^{3/2}$.

Let $k \ge 3$ be an integer. It is convenient to set $\sigma = s+4 = \omega(b)+5$. For $h = 1, \ldots, k$, define

$$\varepsilon_h = J^{-(\sigma^k - \sigma^{h-1})/(\sigma^{k+1}-1)} (\log J)^{1/\sigma} (\log 4d)^{(\sigma^{k-h}+1)/\sigma^{k+1-h}}, \tag{4.12}$$

and check that

$$J^{-1/4} < \varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_k < 1/(\log 6d).$$

Put $\mathcal{S}_0 = \{j_2, j_2+1, \ldots, J\}$. Observe that our choices of j_2 and $\varepsilon_1, \ldots, \varepsilon_k$ imply the analogue of (4.8), namely that

$$j_2 > \max\{2d \log \big(3M(\xi)\big), 10(2d+1)\varepsilon_h^{-1} \log b\}, \quad (1 \le h \le k).$$

For $h = 1, \ldots, k$, let $\mathcal{S}_h$ denote the set of positive integers j such that $j_2 \le j \le J$ and $n_{j+1} \ge (1+2\varepsilon_h)n_j$. Observe that $\mathcal{S}_0 \supset \mathcal{S}_1 \supset \ldots \supset \mathcal{S}_k$. It follows from (4.10) that, for any j in $\mathcal{S}_h$, the rational number p_j/q_j satisfies

$$\min\{1, |1/\xi - q_j/p_j|\} \cdot \prod_{i=1}^{\omega(b)} |q_j|_{\ell_i} < H(\xi_j)^{-2-\varepsilon_h}.$$

Consequently, the cardinality of $\mathcal{S}_h$ is at most $T(\varepsilon_h)$.

Write

$$\mathcal{S}_0 = (\mathcal{S}_0 \setminus \mathcal{S}_1) \cup (\mathcal{S}_1 \setminus \mathcal{S}_2) \cup \ldots \cup (\mathcal{S}_{k-1} \setminus \mathcal{S}_k) \cup \mathcal{S}_k.$$

Combining (4.7) with the above estimates, we obtain that

$$\begin{aligned}\frac{n_J}{n_{j_2}} &= \frac{n_J}{n_{J-1}} \times \frac{n_{J-1}}{n_{J-2}} \times \ldots \times \frac{n_{j_2+1}}{n_{j_2}} \\ &\le (1+2\varepsilon_1)^J \prod_{h=2}^{k} (1+2\varepsilon_h)^{T(\varepsilon_{h-1})} (2d)^{T(\varepsilon_k)}.\end{aligned}$$

Taking the logarithm and using the fact that $\log(1+u) < u$ for any positive real number u, we get

$$\log n_J - \log n_{j_2} \le 2J\,\varepsilon_1 + 2\sum_{h=2}^{k} \varepsilon_h\, T(\varepsilon_{h-1}) + T(\varepsilon_k)\cdot(\log 2d). \tag{4.13}$$

We easily infer from (4.11), (4.12), and (4.13) that

$$\begin{aligned}\log n_J - \log n_{j_2} \le 3ke^{7\omega(b)+26}\, J^{(\sigma-1)/\sigma}\, J^{(\sigma-1)/\sigma^k}\,(\log J)^{1/\sigma}\,(\log 4d)^{1/\sigma} \\ + J^{(\sigma-1)/\sigma}\, J^{(\sigma-1)/\sigma^k}\,(\log J)^{1/\sigma}\,(\log 4d)^{(\sigma^{k-1}+1)/\sigma^k}.\end{aligned} \tag{4.14}$$

Let $\log^+$ be the function defined on the positive real numbers by $\log^+ x = \max\{\log x, 1\}$. Choosing for k the smallest integer greater than $(\log\log J)\cdot(\log^+\log\log 4d)$, we get from (4.14) that

$$\log n_J - \log n_{j_2} \ll J^{(\sigma-1)/\sigma}\,(\log J)^{1/\sigma}\,(\log\log J)\,(\log 4d)^{1/\sigma}\,(\log^+\log\log 4d),$$

where, as in the sequel of the proof, the constant implied by $\ll$ depend only on $\omega(b)$. Since $n_{j_2} \le j_2 \le J^{2/3} \le J^{(\sigma-1)/\sigma}$, we get that

$$\log n_J \ll J^{(\sigma-1)/\sigma}\,(\log J)^{1/\sigma}\,(\log\log J)\,(\log 4d)^{1/\sigma}\,(\log^+\log\log 4d),$$

thus

$$\begin{aligned}J \gg (\log n_J)^{1+1/(\sigma-1)}\,(\log\log n_J)^{-1/(\sigma-1)}\,(\log\log\log n_J)^{-1-1/(\sigma-1)} \\ \times (\log 4d)^{-1/(\sigma-1)}\,(\log^+\log\log 4d)^{-1-1/(\sigma-1)}.\end{aligned}$$

Since $\sigma - 1 = \omega(b) + 4 > 4$, this completes the proof of Theorem 1. □

5. On the Hensel expansion of irrational algebraic *p*-adic numbers.

Let p be a prime number. As usual, we denote by $\boldsymbol{Q}_p$ the field of p-adic numbers, and we call algebraic (resp. transcendental) any element of $\boldsymbol{Q}_p$ which is algebraic (resp. transcendental) over $\boldsymbol{Q}$. If ξ_p is a p-adic number, we denote by

$$\xi_p = \sum_{k=-k_0}^{+\infty} a_k p^k, \quad (a_k \in \{0,1,\dots,p-1\}, k_0 \ge 0, a_{-k_0} \ne 0 \text{ if k0} > 0),$$

its Hensel expansion. It is proved in [2], Section 6, that the sequence $(a_k)_{k\ge -k_0}$ cannot be 'too simple' when ξ_p is algebraic irrational. Using the methods developed in the present paper, we get the p-adic analogues of

Theorem 1 and Corollaries 1 and 2. With the above notation, for a positive integer n, set

$$\mathrm{nbdc}(n, \xi_p, p) = \mathrm{Card}\{1 \le k \le n : a_k \ne a_{k+1}\},$$

and

$$\mathcal{S}(n, \xi_p, p) = \sum_{k=1}^{n} a_k.$$

THEOREM 2. *Let p be a prime number. Let ξ_p be an algebraic irrational number in $\boldsymbol{Q}_p$. For any positive real number δ with $\delta < 1/9$ and any sufficiently large integer n, we have*

$$\mathrm{nbdc}(n, \xi_p, p) \ge (\log n)^{1+\delta},$$

and there are at least $(\log n)^{1+\delta}$ non-zero digits among the first n digits of the Hensel expansion of ξ_p, and, moreover,

$$(\log n)^{1+\delta} \le \mathcal{S}(n, \xi_p, p) \le n(p-1) - (\log n)^{1+\delta}.$$

The proof of Theorem 2 uses the same idea as that of Theorem 1. The good approximations to ξ_p are obtained by truncating its Hensel expansion and repeating the last digit. Theorem L, however, cannot be applied in the present context: we need a statement involving rational approximation to a p-adic algebraic number that gives an explicit upper bound for the number of solutions to an inequality first studied by Ridout [17]. A suitable statement is Theorem 3.1 from Evertse and Schlickewei [12], that we apply with $n = s = 2$ to get Theorem 2. We leave the details of the proof to the reader.

6. On β-expansions.

Let $\beta > 1$ be a real number. The β-transformation T_β is defined on $[0,1]$ by $T_\beta : x \longmapsto \beta x \bmod 1$. Rényi [15] introduced the β-expansion of a real x in $[0,1]$, denoted by $d_\beta(x)$, and defined by

$$d_\beta(x) = 0.x_1 x_2 \ldots x_n \ldots,$$

where $x_i = [\beta T_\beta^{i-1}(x)]$. For $x < 1$, this expansion coincides with the representation of x computed by the 'greedy algorithm'. If β is an integer, the digits x_i of x lie in the set $\{0, 1, \ldots, \beta - 1\}$ and $d_\beta(x)$ corresponds to the β-ary expansion of x defined in Section 1. When β is not an integer, the digits

x_i lie in the set $\{0, 1, \ldots, [\beta]\}$. We direct the reader to [4] and to the references quoted therein for more on β-expansions. We stress that the β-expansion of 1 has been extensively studied.

It turns out that the method of proof of Theorem 1 applies to β-expansions, when β is a Pisot or a Salem number. Recall that a Pisot (resp. Salem) number is a real algebraic integer > 1, whose complex conjugates lie inside the open unit disc (resp. inside the closed unit disc, with at least one of them on the unit circle). In particular, every integer $b \geq 2$ is a Pisot number.

For a real algebraic number θ, let $r(\theta)$ denote the sum of the number of infinite places on the number field $\boldsymbol{Q}(\theta)$ plus the number of distinct prime ideals in $\boldsymbol{Q}(\theta)$ that divide the norm of θ. In particular, $r(b) = \omega(b) + 1$ for every integer $b \geq 2$.

THEOREM 3. *Let $\beta > 1$ be a Pisot or a Salem number. Let M be a positive integer and let $(a_k)_{k \geq 1}$ be a non-ultimately periodic sequence of integers from $\{0, 1, \ldots, M\}$, and set*

$$\xi = \sum_{k \geq 1} \frac{a_k}{\beta^k} \cdot$$

Let δ be a positive real number with $\delta < 1/(r(\beta) + 3)$. If there are arbitrarily large integers n for which

$$\mathrm{Card}\{1 \leq k \leq n : a_k \neq a_{k+1}\} < (\log n)^{1+\delta},$$

then ξ is transcendental.

We omit the proof of Theorem 3 since it is very similar to that of Theorem 1. Note that Ridout's Theorem has already been used in this context, see [1].

We display a consequence of Theorem 3 on the number on non-zero digits in the β-expansion of 1.

COROLLARY 3. *Let $\beta > 1$ be a Pisot or a Salem number. Write*

$$d_\beta(1) = 0.a_1 a_2 \ldots$$

For any positive real number δ with $\delta < 1/(r(\beta) + 3)$ and any sufficiently large integer n, we have

$$a_1 + a_2 + \ldots + a_n > (\log n)^{1+\delta},$$

and there are at least $(\log n)^{1+\delta}$ indices j with $1 \leq j \leq n$ and $a_j \neq 0$.

7. On the series $\sum_{j\geq 1} b^{-n_j}$.

Let b be an integer with $b \geq 2$. Let $\boldsymbol{n} = (n_j)_{j\geq 1}$ be a non-decreasing sequence of positive integers. It is well-known that if $\boldsymbol{n}$ grows sufficiently rapidly, then the number

$$\xi_{\boldsymbol{n},b} = \sum_{j\geq 1} b^{-n_j}$$

is transcendental. For example, it easily follows from Ridout's Theorem that the assumption

$$\limsup_{j\to+\infty} \frac{n_{j+1}}{n_j} > 1$$

implies the transcendence of $\xi_{\boldsymbol{n},b}$, see e.g. Satz 7 from the monograph [19].

In particular, for any positive real number ε, the real number $\xi_{\boldsymbol{n},b}$ is transcendental when $n_j = 2^{[\varepsilon j]}$, where $[\ \cdot\ ]$ denotes the integer part. A much sharper statement follows from Theorem 1.

Corollary 4. *Let $b \geq 2$ be an integer. Let η be a real number with*

$$0 \leq \eta < \frac{1}{\omega(b)+5}.$$

Then, the sum of the series

$$\sum_{j\geq 1} b^{-n_j}, \quad \textit{where } n_j = 2^{[j^{1-\eta}]} \textit{ for } j \geq 1,$$

is transcendental.

Proof. Let υ be a real number with $0 \leq \upsilon < 1$. Let N be a large integer. We check that the number of positive integers j such that $2^{[j^{1-\upsilon}]} \leq N$ is less than some absolute constant times $(\log N)^{1/(1-\upsilon)}$, and we apply Theorem 1 to conclude. □

Actually, it is possible to derive from Theorem 1 the transcendence of the real number $\xi_{\boldsymbol{n},b}$ for other fast growing sequences $\boldsymbol{n}$. Like in Corollary 4, the precise statement would involve $\omega(b)$, the number of distinct prime factors of b. It turns out that it is possible to improve Corollary 4 and to get rid of the dependence on $\omega(b)$ by using the following version of the Cugiani–Mahler Theorem [10, 11, 14], that we extract from Bombieri and Gubler [7] (see also Bombieri and van der Poorten [8]).

Theorem BG. *Let S_1 and S_2 be disjoint finite sets of prime numbers. Let θ be a real algebraic number of degree d. For any positive real number t set*

$$f(t) = 7\,(\log 4d)^{1/2} \left(\frac{\log\log(t+\log 4)}{\log(t+\log 4)}\right)^{1/4}.$$

Let $(p_j/q_j)_{j\ge1}$ be the sequence of rational solutions of

$$\left|\theta - \frac{p}{q}\right| \cdot \prod_{\ell\in S_1} |p|_\ell \cdot \prod_{\ell\in S_2} |q|_\ell \le \frac{1}{q^{2+f(\log q)}},$$

ordered such that $1 \le q_1 < q_2 < \ldots$ Then either the sequence $(p_j/q_j)_{j\ge1}$ is finite or

$$\limsup_{j\to+\infty} \frac{\log q_{j+1}}{\log q_j} = +\infty.$$

Proof. This follows from Theorem 6.5.10 of [7]. □

We can then proceed exactly as Mahler did ([14], Theorem 3, page 178).

Theorem 4. *Let $b \ge 2$ be an integer. Let $(a_j)_{j\ge1}$ be a non-decreasing sequence of positive integers tending to infinity. Let $\boldsymbol{n} = (n_j)_{j\ge1}$ be an increasing sequence of positive integers satisfying $n_1 \ge 3$ and*

$$n_{j+1} \ge \left(1 + a_n \left(\frac{\log\log n_j}{\log n_j}\right)^{1/4}\right) n_j, \quad (j \ge 1).$$

Then the real number

$$\xi_{\boldsymbol{n},b} = \sum_{j\ge1} b^{-n_j}$$

is transcendental.

We omit the proof since it is exactly the same as Mahler's one. Note that Theorem 1 yields a similar statement as Theorem 4, with however $1/4$ replaced by any real number smaller than $1/(\omega(b)+4)$.

It does not seem to us that Theorem 1 can be derived from Theorem BG.

An easy computation allows us to deduce from Theorem 4 the following improvement of Corollary 4.

COROLLARY 5. *Let $b \geq 2$ be an integer. Let η be a real number with*

$$0 \leq \eta < \frac{1}{5}.$$

Then, the sum of the series

$$\sum_{j \geq 1} b^{-n_j}, \quad \text{where } n_j = 2^{[j^{1-\eta}]} \text{ for } j \geq 1,$$

is transcendental.

More generally, one can consider the following question.

PROBLEM. *Let $\boldsymbol{n} = (n_j)_{j \geq 1}$ be a strictly increasing sequence of positive integers and set*

$$f_{\boldsymbol{n}}(z) = \sum_{j \geq 1} z^{n_j}. \tag{7.1}$$

If the sequence $\boldsymbol{n}$ increases sufficiently rapidly, then the function $f_{\boldsymbol{n}}$ takes transcendental values at every non-zero algebraic point in the open unit disc.

By a clever use of the Schmidt Subspace Theorem, Corvaja and Zannier [9] proved that the conclusion of the Problem holds for $f_{\boldsymbol{n}}$ given by (7.1) when the strictly increasing sequence $\boldsymbol{n}$ satisfies

$$\liminf_{j \to +\infty} \frac{n_{j+1}}{n_j} > 1. \tag{7.2}$$

It is perhaps possible to combine the method developed in [9] with the ideas of the present paper to relax the assumption (7.2).

8. Continued fraction expansions.

Despite some recent progress, the continued fraction expansion of a real algebraic number of degree greater than or equal to three remains very mysterious. In particular, we still do not know whether its sequence of partial quotients is bounded or not. Even an analogue of Theorem 1 for continued fraction expansions would be new.

In the sequel, we restrict our attention to expansions with partial quotients equal to 1 or 2. More precisely, for a strictly increasing sequence

$\boldsymbol{n} = (n_j)_{j\geq 1}$ of positive integers, we denote by $\xi_{\boldsymbol{n}} := [0; a_1, a_2, \dots]$ the real number whose continued fraction expansion is given, for $k \geq 1$, by $a_k = 1$ if $k \notin \boldsymbol{n}$ and $a_k = 2$ if $k \in \boldsymbol{n}$.

It is expected that $\xi_{\boldsymbol{n}}$ is transcendental unless $(a_k)_{k\geq 1}$ is ultimately periodic (in which case $\xi_{\boldsymbol{n}}$ is clearly a quadratic number). In particular, it is very likely that $\xi_{\boldsymbol{n}}$ is transcendental if $\boldsymbol{n}$ increases sufficiently rapidly. As a first result in this direction, we claim that, if $\xi_{\boldsymbol{n}}$ is algebraic, then

$$\log n_j = o(j). \tag{8.1}$$

Indeed, if (8.1) does not hold, then there are infinitely many positive integers j and a real number $c > 1$ such that $n_j > c^j$. Consequently, we have

$$\limsup_{j\to+\infty} \frac{n_{j+1}}{n_j} > 1,$$

and the transcendence of the associated real number $\xi_{\boldsymbol{n}}$ follows from Theorem 3.2 of [3].

With similar ideas as in the present paper, but with some additional technical difficulties due to the use of the Schmidt Subspace Theorem in place of the Ridout Theorem, we are able to improve the above result as follows:

There exists a positive real number δ such that, if $\xi_{\boldsymbol{n}}$ is algebraic, then

$$\log n_j \leq j^{1-\delta}$$

holds for every sufficiently large integer j.

This and related results will be the subject of a forthcoming paper.

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