REND. SEM. MAT. UNIV. PADOVA, Vol. 118 (2007)

Centralizers of Involutions in Locally Finite-Simple Groups.

A. BERKMAN (*) - M. KUZUCUOĞLU (*) - E. ÖZYURT (**)

Dedicated to Prof. Dr. Mehpare Bilhan on her 63rd birthday.

ABSTRACT - We consider infinite locally finite-simple groups (that is, infinite groups in which every finite subset lies in a finite simple subgroup). We first prove that in such groups, centralizers of involutions either are soluble or involve an infinite simple group, and we conclude that in either case centralizers of involutions are not inert subgroups. We also show that in such groups, the centralizer of an involution is linear if and only if the ambient group is linear.

1. Introduction.

Centralizers of involutions played an important role in the classification of finite simple groups. In infinite periodic simple groups, centralizers of involutions are a natural source for finding infinite proper subgroups. The question of whether the centralizer of every involution in an infinite locally finite simple group involves an infinite simple group is answered negatively by Meierfrankenfeld in [13, Chapter 16]. To be more precise, he constructed infinite non-linear locally finite simple groups in which the centralizer of every involution is almost locally soluble.

If $\mathcal{P}$ is a property of groups, then a group G is called a *locally $\mathcal{P}$ group*, if every finite subset of G lies in a subgroup of G with the property $\mathcal{P}$. In this context, a *locally finite-simple group* is a group in which every finite subset lies in a finite simple subgroup. Clearly such groups are simple, locally finite and have a local system consisting of finite simple subgroups. However, not all infinite simple locally finite groups are locally finite-

(*) Indirizzo degli A.: Mathematics Department, Middle East Technical University, Ankara, 06531, Turkey.

(**) Indirizzo dell'A.: Mathematics Department, Adnan Menderes University, Aydın, 09010, Turkey.

simple (see [20] and [8, Corollary 3.8]).

The main aim of this work is to show that in an infinite locally finite-simple group, centralizers of involutions either are soluble or involve an infinite non-abelian simple group. A more technical and detailed result is stated as Theorem 4 below. This theorem can be considered as a generalization and unification of Theorems B and D in [6] in the context of involutions.

One should note that the condition on the local system cannot be removed from the statement of Theorem 4, since there are infinite locally finite non-linear simple groups in which the centralizer of at least one involution is almost locally soluble, but not soluble [13].

A corollary of Theorem 4 is: In an infinite locally finite-simple group, if the centralizer of an involution is linear, then the ambient group is also linear.

By [4, Theorem B] and [14], Theorem 7 below covers all simple periodic groups of finitary linear transformations acting on a space over a field of odd characteristic. In fact, by a remark of J. Hall in [8, page 165], the restriction on characteristic may be removed.

As an application of Theorem 4, we also prove that the centralizers of involutions are not inert in infinite locally finite-simple groups. (Recall that a subgroup H in a group G is *inert* in G, if $|H : H \cap H^g| < \infty$ for any $g \in G$.)

2. Some remarks on groups of Lie type.

In infinite simple locally finite groups of Lie type, centralizers of involutions either involve infinite simple groups or are soluble. The former case occurs in groups of higher rank and the latter in small rank. Interestingly, there is only one type of infinite simple locally finite group that contains both kinds of centralizers, namely $PSp(4, \mathbb{F})$ where $\mathbb{F}$ is an infinite locally finite field of characteristic 2.

LEMMA 1. *Let $\mathbb{F}$ and $\mathbb{K}$ be infinite locally finite fields. Assume $\mathbb{F}$ is of characteristic 2 and $\mathbb{K}$ is of arbitrary characteristic. In the simple groups $PSL(2, \mathbb{K})$, $PSL(3, \mathbb{F})$, $PSU(3, \mathbb{F})$ and $Sz(\mathbb{F})$, all centralizers of involutions are soluble. The simple group $PSp(4, \mathbb{F}) \cong P\Omega(5, \mathbb{F})$ contains three conjugacy classes of involutions. Centralizers of two classes involve infinite simple groups, and centralizers of the remaining class are soluble.*

PROOF. The proof of the lemma can be easily extracted from [1, 15, 16] and elementary observations. $\square$

LEMMA 2. *Let q be a prime power.*

(1) $O^\varepsilon(2,q)$ *is dihedral group of order* $2(q-\varepsilon)$, *where* $\varepsilon = \pm 1$.
(2) $P\Omega(3,q) \cong PSL(2,q) \cong PSp(2,q)$
(3) $P\Omega^+(4,q) \cong PSL(2,q) \times PSL(2,q)$, $P\Omega^-(4,q) \cong PSL(2,q^2)$
(4) $P\Omega(5,q) \cong PSp(4,q)$
(5) $P\Omega^-(6,q) \cong PSU(4,q)$, $P\Omega^+(6,q) \cong PSL(4,q)$
(6) $O(2m+1,2^k) \cong Sp(2m,2^k)$
(7) $SO^\varepsilon(n,2^k) \cong \Omega^\varepsilon(n,2^k)$

PROOF. These are well-known results. One can see [17, Sections 11, 12] for example. □

3. Proof of the Theorem.

The class of all locally finite groups having a series of finite length in which there are at most n non-abelian simple factors and the rest are locally soluble is denoted by $\mathfrak{F}_n$. The following will be helpful in proving the main theorem below.

ASSERTION 3 [6, Lemma 2.3]. *If all finitely generated subgroups of a locally finite group lie in $\mathfrak{F}_n$, then the group also lies in $\mathfrak{F}_n$.*

The next theorem is that in 'most' infinite locally finite-simple groups, centralizers of involutions involve infinite simple groups. We use the expression *to involve a simple group* in what follows, but we have more than the existence of a simple section: Our simple section appears as a factor in the composition series described below.

THEOREM 4. *Let G be an infinite locally finite-simple group. Then the centralizer of every involution $i \in G$ belongs to $\mathfrak{F}_4$; that is, it has a series of finite length in which each factor is either locally soluble or non-abelian simple, and the number $n(i)$ of non-abelian simple factors is at most four. Moreover we have the following.*

1) *For every involution $i \in G$, $n(i) = 0$ if and only if G is one of the following: $PSL(2,\mathbb{K})$ where $\mathbb{K}$ is an infinite locally finite field of arbitrary characteristic, $PSL(3,\mathbb{F})$, $PSU(3,\mathbb{F})$, $Sz(\mathbb{F})$ where $\mathbb{F}$ is an infinite locally finite field of characteristic 2. In this case, centralizers of involutions are soluble.*

2) *There exist involutions* $i, j \in G$ *such that* $n(i) = 0$ *and* $n(j) \neq 0$ *if and only if* $G \cong PSp(4, \mathbb{F})$ *and the characteristic of* $\mathbb{F}$ *is* 2.

PROOF. Since G is locally finite-simple, G has a local system consisting of finite simple subgroups. By the classification of the finite simple groups, there are only finitely many sporadic simple groups, so we may discard them from the local system. Moreover, by [10, 1.A.10], we may assume that G has a local system consisting entirely of finite alternating groups or entirely of finite simple groups of Lie type.

Case 1. Assume that G has a local system consisting of finite alternating groups. In this case, centralizers of involutions are in $\mathfrak{F}_2$, and $n(i) \neq 0$ for every involution $i \in G$ by [6, Proposition 2.5].

Case 2. Assume that G has a local system consisting of finite simple groups of Lie type. By [10, 1.A.10] again, G has a local system consisting of finite simple groups of a fixed Lie type. Now, these groups may be of bounded or unbounded (Lie) rank.

Case 2.1. If there is a bound on the rank of finite simple groups of fixed Lie type, then by a theorem (proved independently by Belyaev [2], Hartley–Shute [7], Borovik [5], Thomas [18]), we obtain that G is a simple linear group of Lie type over an infinite locally finite field $\mathbb{F}$.

Case 2.1.1. Assume that $\mathbb{F}$ has odd characteristic. If the rank of G is greater than or equal to 2, then by [6, Theorem D], the centralizer of every involution is in $\mathfrak{F}_4$ and involves an infinite simple group. Note that $P\Omega(8, \mathbb{F})$ has a centralizer of an involution which involves four simple groups.

Hence it suffices to consider only the cases where the rank of G is 1. If G is isomorphic to $PSL(2, \mathbb{F})$, then the centralizers of involutions are soluble by Lemma 1. If G is isomorphic to $PSU(3, \mathbb{F})$, then the centralizers of involutions involve $PSU(2, \mathbb{F})$ by [16, Chapter 6 (5.15)]. Finally, let $G \cong {}^2G_2(\mathbb{K})$, where $\mathbb{K}$ is an infinite locally finite field of characteristic 3. By [19, p. 63] and [9, Lemma 2.1], all involutions in ${}^2G_2(3^{2n+1})$ are conjugate, hence all involutions in ${}^2G_2(\mathbb{K})$ are conjugate. Moreover, the centralizer of any involution in ${}^2G_2(3^{2n+1})$ is isomorphic to $\mathbb{Z}_2 \times PSL(2, 3^{2n+1})$ by [12, pp. 16–19], and hence the centralizer of any involution in ${}^2G_2(\mathbb{K})$ is isomorphic to $\mathbb{Z}_2 \times PSL(2, \mathbb{K})$.

Case 2.1.2. Assume that $\mathbb{F}$ has characteristic 2. Hence G is an infinite simple group of Lie type over a locally finite field of characteristic 2. Note that owing to Assertion 3, it is enough to prove the results for the corresponding finite simple groups of Lie type. We invoke [16] for the centralizers of involutions in finite classical groups and [1] for finite exceptional groups. Let $k > 1$.

In $PSL(n, 2^k)$, centralizers of involutions are in $\mathfrak{F}_2$ for every $n \geq 2$ and involve $PSL(r, 2^k)$ for some $1 < r < n$ if $n \geq 4$ by [16, Chapter 6 (5.3)]. Centralizers of involutions are soluble in $PSL(2, \mathbb{F})$ and $PSL(3, \mathbb{F})$ by Lemma 1.

In $PSp(2m, 2^k)$, centralizers of involutions are in $\mathfrak{F}_2$ for every $m \geq 2$ and involve $PSp(2r, 2^k)$ for some $1 < r < m$ if $m \geq 3$ by [16, Chapter 6 (5.14)]. Centralizers of involutions in $PSp(4, \mathbb{F})$ were already discussed in Lemma 1.

In $PSU(n, 2^k)$, centralizers of involutions are in $\mathfrak{F}_2$ for every $n \geq 3$ and involve $PSU(r, 2^k)$ for some $1 < r < n$ if $n \geq 4$ by [16, Chapter 6 (5.16)]. Centralizers of involutions are soluble in $PSU(3, \mathbb{F})$ by Lemma 1.

Owing to Lemma 2, it is enough to consider the orthogonal groups $P\Omega^{\varepsilon}(n, 2^k)$ for the cases where $n \geq 8$ and $n = 2m$ is even. In such groups, centralizers of involutions all lie in $\mathfrak{F}_2$ and involve either $P\Omega^{\varepsilon}(r, 2^k)$ or $PSp(2s, 2^k)$ for some $1 < r < n$ or $1 < s < m$ by [16, Chapter 6 (5.18)].

In $E_6(2^k)$, there are three conjugacy classes of involutions with representatives x, y, z [1, 12.8], and their centralizers involve $PSL(6, 2^k)$, $PSp(6, 2^k)$, $PSL(2, 2^k) \times PSL(3, 2^k)$ respectively [1, 15.5, corrections].

In $F_4(2^k)$, there are four conjugacy classes of involutions with representatives t, u, tu and v [1, 12.6], and their centralizers involve $PSp(6, 2^k)$, $PSp(6, 2^k)$, $PSp(4, 2^k)$, and $PSL(2, 2^k)$ respectively [1, p. 45 and 13.3].

In $E_7(2^k)$, there are five conjugacy classes of involutions with representatives x, y, z, u and v [1, 12.9], and their centralizers involve $P\Omega^{+}(12, 2^k)$, $PSp(8, 2^k) \times PSL(2, 2^k)$, $PSL(2, 2^k) \times PSp(6, 2^k)$ $F_4(2^k)$ and $PSp(6, 2^k)$ respectively [1, 16.20].

In $E_8(2^k)$, there are four conjugacy classes of involutions with representatives x, y, z and u [1, 12.11], and their centralizers involve $E_7(2^k)$, $PSp(12, 2^k)$, $F_4(2^k) \times PSL(2, 2^k)$ and $PSp(8, 2^k)$ respectively [1, 17.15].

In $G_2(2^k)$, there are two conjugacy classes of involutions [1, 18.2], and their centralizers involve $PSL(2, 2^k)$ in both cases [1, 18.4].

In $Sz(\mathbb{F})$, centralizers of involutions are soluble by [15, Proposition 1].

In ${}^3D_4(2^k)$, there are two conjugacy classes of involutions with representatives z and t [1, 18.2], and their centralizers involve $PSL(2, 2^{3k})$ and $PSL(2, 2^k)$ respectively [1, 18.5].

In ${}^2E_6(2^k)$, there are three conjugacy classes of involutions with representatives t, u and v [1, 12.7], and their centralizers involve $PSU(6, 2^k)$, $P\Omega(7, 2^k)$ and $PSL(2, 2^k)$ respectively [1, 14.3].

In ${}^2F_4(2^k)$, there are two conjugacy classes of involutions with representatives z and t [1, 18.2], and their centralizers involve $Sz(2^k)$ and $PSL(2,2^k)$ respectively [1, 18.6].

Case 2.2. By [10, 1.A.10] we may assume that G is a non-linear locally finite simple group with a local system consisting of a unique type of classical Lie group and the ranks of these groups are not bounded. In this case, the result follows from the analysis done in Case 2.1 and Assertion 3. (One can also use [6, Theorem D] for the odd characteristic case.) Note that $n(i) \neq 0$ for every involution $i \in G$ in this case. $\square$

The following corollary is an easy consequence of the proof of the previous theorem.

COROLLARY 5. *Let G be an infinite locally finite-simple group. Then the following statements hold.*

1) *Let $i \in G$ be an involution. Then $C_G(i)$ is linear if and only if G is linear.*

2) *If $C_G(i)$ involves a finite non-abelian simple group for some involution $i \in G$, then $C_G(i)$ involves an infinite simple group.*

COROLLARY 6. *In an infinite locally finite-simple group, no centralizer of an involution is an inert subgroup.*

PROOF. Let G be a group satisfying the hypothesis. Then for the groups in the statement of Theorem 4 part (1), centralizers of involutions involve infinite simple groups. Clearly, in a simple group, inert subgroups are residually finite, but by [11, Lemma 4.1], any residually finite, locally finite group in $\mathfrak{F}_n$ is locally solvable, a contradiction. Hence it is enough to consider the five types of groups excluded in part (1) of Theorem 4. But by [3, Corollary 5.2], there exists no infinite proper inert subgroup in locally finite simple groups of Lie type. $\square$

A *Kegel cover* $\mathcal{K}$ of a locally finite group G is a set of pairs (H, M) such that H is a finite subgroup of G, M is a maximal normal subgroup of H and for each finite subgroup K of G there exists $(H, M) \in \mathcal{K}$ with $K \leq H$ and $K \cap M = 1$. The simple groups H/M are called factors of $\mathcal{K}$.

Following the line of proof in [6, Theorem B′], we may restate Theorem 4, Corollary 5 and Corollary 6 for a more general class of locally finite groups. As mentioned in the Introduction, in this general form, the following theorem covers all simple locally finite finitary linear groups.

THEOREM 7. *Let G be an infinite locally finite simple group with a Kegel cover $\mathcal{K}$ such that for each $(H,M) \in \mathcal{K}$, $M/O_{2'}M$ is hypercentral in $H/O_{2'}M$. Then the following statements hold.*

1) *The centralizer of every involution $i \in G$ belongs to $\mathfrak{F}_4$, that is, it has a series of finite length in which each factor is either locally soluble or non-abelian simple, and the number $n(i)$ of non-abelian simple factors is at most four.*

2) (*a*) *For every involution $i \in G$, $n(i) = 0$ if and only if G is one of the following: $PSL(2,\mathbb{K})$ where $\mathbb{K}$ is an infinite locally finite field of arbitrary characteristic, $PSL(3,\mathbb{F})$, $PSU(3,\mathbb{F})$ and $Sz(\mathbb{F})$ where $\mathbb{F}$ is an infinite locally finite field of characteristic 2. In this case, centralizers of involutions are soluble.*

(*b*) *There exist involutions $i,j \in G$ such that $n(i) = 0$ and $n(j) \neq 0$ if and only if $G \cong PSp(4,\mathbb{F})$ and the characteristic of $\mathbb{F}$ is 2.*

3) *Let $i \in G$ be an involution. Then $C_G(i)$ is linear if and only if G is linear.*

4) *If $C_G(i)$ involves a finite non-abelian simple group for some involution $i \in G$, then $C_G(i)$ involves an infinite simple group.*

REFERENCES

[1] M. ASCHABACHER - G. M. SEITZ, *Involutions in Chevalley groups over fields of even order, Nagoya Math. J.*, Vol. **63** (1976), pp. 1–91.

[2] V.V. BELYAEV, *Locally finite Chevalley groups, Studies in group theory*, Academy of Sciences of the USSR, Urals Scientific Centre, 1984.

[3] V.V. BELYAEV, *Locally finite groups containing a finite inseperable subgroup*. Sibirskii Matematicheskii Z. (34) (1993), pp. 23–41. English Translation: Siberian Math. J., **34**, no. 2 (1993), pp. 218–232.

[4] V.V. BELYAEV, *Semisimple periodic groups of finitary transformations*, Algebra i Logika (32) (1993), pp. 17–33; English Translation: Algebra and Logic, (32) (1993), pp. 8–16.

[5] A.V. BOROVIK, *Embedding of finite Chevalley groups and periodic linear groups*, Sibirsk. Mat. Zh., **24** (1983) 26-35; English Translation: Siberian Math. J., **24** (1983), pp. 843–851.

[6] B. HARTLEY - M. KUZUCUOĞLU, *Centralizers of elements in locally finite simple groups*, Proc. London Math. Soc. (3) **62** (1991), pp. 301–324.

[7] B. HARTLEY - G. SHUTE, *Monomorphism and direct limits of finite groups of Lie type*, Quart. J. Math. Oxford (2) **33** (1982), pp. 309–323.

[8] B. HARTLEY et al (eds), *Finite and Locally finite groups*, NATO ASI series vol. **471**. Kluwer Academic, Dordrecht, 1995.

[9] Z. JANKO - J.C. THOMPSON, *On a class of simple groups of Ree*, J. Algebra, **4**, no. 2 (1966), pp. 274–292.

[10] Otto H. Kegel - B. A. F. Wehrfritz, *Locally Finite Groups*, North-Holland Publishing Company - Amsterdam, 1973.
[11] M. Kuzucuoğlu, *Barely Transitive permutation groups*, Arch. Math., **55** (1990), pp. 521–532.
[12] V. M. Levchuk - Ya. N. Nuzhin, *The structure of Ree groups*. (Russian) Algebra i Logika, **24**, no. 1 (1985), 26–41, 122. English translation: Algebra and Logic, **24**, no. 1 (1985), pp. 16–26.
[13] U. Meierfrankenfeld, *Locally Finite Simple Groups*, Lecture Notes on the Internet, http://www.mth.msu.edu/˜meier/Classnotes/LFG/LFG.dvi
[14] U. Meierfrankenfeld, *A simple subnormal subgroup for locally finite finitary linear groups*. Preprint, 1992.
[15] M. Suzuki, *On a class of doubly transitive groups*, Ann. of Math. (2) **75** (1962), pp. 105–145.
[16] M. Suzuki, *Group Theory Vol. I and Vol. II, 1986*, Springer-Verlag, New York inc.
[17] D. E. Taylor, *The Geometry of the Classical Groups*, Heldermann Verlag, 1992.
[18] S. Thomas, *The classification of the simple periodic linear groups*, Arch. Math., **41** (1983), pp. 103–116.
[19] H. N. Ward, *On Ree's series of simple groups*, Trans. AMS No: 121 (1966), pp. 62–89.
[20] A. E. Zaleskii - V. N. Serezhkin, *Finite linear groups generated by reflections*, Math. USSR-Izv., **17** (1981), pp. 477–503.

Manoscritto pervenuto in redazione il 23 agosto 2006.