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Some New Formulas Involving Γ_q Functions.

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ABSTRACT - In a recent paper we found some new results for q-functions of many variables with the aid of the Γ_q function. The Heine notation reminding of the hypergeometric case was used throughout, and some relations between Γ_q functions were presented. This paper aims at giving the promised longer exposition of Γ_q - revealing also the connection between this and the Jacobi-theta functions which appear in context. We will give a slightly generalized definition of the Heine series with more general tilde operators. 4 q-summation formulas of Andrews will be given in the new notation. The close affinity to q-binomial coefficient formulas will be stressed by expressing the finite q-hypergeometric formulas, the canonical form, in two ways. Two further q-analogues of Kummer's ${}_2F_1(-1)$ formula will be given. An ancient q-analogue of the Euler reflection formula will be used for the proof of a special case of the Bailey-Daum summation formula, conjectured in the previous paper.

Multiple extensions of Gauss' formula will be given by a similar technique. All this will explain the utility of the Heine notation.

1. Introduction.

In this section we present the necessary definitions together with useful related formulas. The notation from [13] will be used whenever possible. The umbral calculus from [15] will play a significant role in the definition of the q-hypergeometric series. The Jackson Γ_q function fits nicely together with the q-shifted facorial. In Section two, 2 q-summation formulas of Andrews are given in terms of Γ_q functions, and 2 q-summation formulas of Andrews will be proved by Watson's [46] transformation formula for a terminating very-well-poised ${}_8\phi_7$ series. In Section three two further q-analogues of Kummer's ${}_2F_1(-1)$ formula will be given. We will prove a special case of the Bailey-Daum summation formula, conjectured in

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the previous paper [13], by using Jacobi-theta functions. In Section four multiple extensions of Gauss' formula will be given by a similar technique.

1.1 – *Tilde operators.*

For preliminary definitions the reader is referred to the paper [13]. In that paper we defined a q–shifted factorial and a tilde operator.

DEFINITION 1. The operator

$$\widetilde{}: \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{\pi i}{\log q}. \tag{1}$$

By (1) it follows that

$$\langle \widetilde{a;q} \rangle_n = \prod_{m=0}^{n-1} (1 + q^{a+m}), \tag{2}$$

It turns out that in certain formulas it is not easy to cope with the case $a = 0$. That's why we define

$$\langle \widetilde{0}; q \rangle_n \equiv \langle \widetilde{1}; q \rangle_{n-1}. \tag{3}$$

This means that if $a = 0$ we skip the first factor $1 + 1$, which is not really a q-shifted facorial, and just compute the last $n - 1$ factors. The reason is the affinity with the factor $1 - 1$, which causes trouble in denominators.

Because

$$\langle 1 - \widetilde{n;q} \rangle_n = \langle \widetilde{0}; q \rangle_n q^{-\binom{n}{2}}, \tag{4}$$

we also define

$$\langle 1 - \widetilde{n;q} \rangle_n \equiv \langle 1 - \widetilde{n;q} \rangle_{n-1}. \tag{5}$$

This formula was already used by Watson [47, p. 64].

To be able to treat a general root of unity, it is desirable to generalize this operator in the following way. In equations (6) to (16) we assume that $(m, l) = 1$.

DEFINITION 2. The operator

$$\widetilde{\tfrac{m}{l}} : \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{2\pi i m}{l \log q}. \tag{6}$$

This means

$$\langle \widetilde{\tfrac{m}{l}a}; q\rangle_n = \prod_{m=0}^{n-1}(1 - e^{2\pi i \frac{m}{l}} q^{a+m}), \tag{7}$$

Furthermore we define

$$\tfrac{m}{l}\widetilde{\langle a; q\rangle}_n \equiv \langle \widetilde{\tfrac{m}{l}a}; q\rangle_n. \tag{8}$$

We will also need another generalization of the tilde operator.

$$\langle \widetilde{_k a}; q\rangle_n \equiv \prod_{m=0}^{n-1}\left(\sum_{i=0}^{k-1} q^{i(a+m)}\right). \tag{9}$$

$$\langle \widetilde{_2 a}; q\rangle_n \equiv \langle \widetilde{a}; q\rangle_n. \tag{10}$$

$$\langle \widetilde{_1 a}; q\rangle_n \equiv 1. \tag{11}$$

$$_k\widetilde{\langle a; q\rangle}_n \equiv \langle \widetilde{_k a}; q\rangle_n. \tag{12}$$

The following simple rules follow from (6). Some of them were previously known in other notation from [17].

THEOREM 1.1.

$$\widetilde{\tfrac{m}{l}a} \pm b \equiv \tfrac{m}{l}\widetilde{(a \pm b)} \ mod \ \frac{2\pi i}{\log q}, \tag{13}$$

$$\sum_{k=1}^{l} \tfrac{1}{l}\widetilde{\pm a_k} \equiv \sum_{k=1}^{l} \pm a_k \ mod \ \frac{2\pi i}{\log q}, \tag{14}$$

$$\frac{m}{l} \times \widetilde{a} \equiv \tfrac{m}{2l}\widetilde{\frac{am}{l}} \ mod \ \frac{2\pi i}{\log q}, \tag{15}$$

$$\mathrm{QE}(\widetilde{\tfrac{m}{l}a}) = \mathrm{QE}(a) e^{\frac{2\pi i m}{l}}, \tag{16}$$

where the second equation is a consequence of the fact that we work mod $\frac{2\pi i}{\log q}$.

Furthermore,

THEOREM 1.2.

$$\langle \widetilde{a}; q^2 \rangle_n = \langle \widetilde{\tfrac{1}{4}a}, \widetilde{\tfrac{3}{4}a}; q \rangle_n. \tag{17}$$

$$\langle a; q^p \rangle_n = \prod_{k=0}^{p-1} \langle \widetilde{\tfrac{k}{p}a}; q \rangle_n, \text{ where } p \text{ is an odd prime.} \tag{18}$$

$$\langle a; q^k \rangle_n = \langle a; q \rangle_n \times {}_k \langle \widetilde{a; q} \rangle_n. \tag{19}$$

This leads to the following q-analogue of [36, p. 22, (2)].

THEOREM 1.3.

$$\langle a; q \rangle_{kn} = \prod_{m=0}^{k-1} \Big\langle \frac{a+m}{k}; q \Big\rangle_n \times_k \Big\langle \widetilde{\frac{a+m}{k}}; q \Big\rangle_n. \tag{20}$$

1.2 – *Heine's series*

The q-hypergeometric series was developed by Heine 1846 [24] as a generalization of the hypergeometric series.

DEFINITION 3. Generalizing Heine's series, we shall define a q-hypergeometric series by (compare [17, p. 4], [22, p. 345]):

$$\begin{aligned} &{}_{p+p'}\phi_{r+r'}(\hat{a}_1, \ldots, \hat{a}_p; \hat{b}_1, \ldots, \hat{b}_r | q, z \| s_1, \ldots, s_{p'}; t_1, \ldots, t_{r'}) \equiv \\ &{}_{p+p'}\phi_{r+r'}\left[\begin{matrix} \hat{a}_1, \ldots, \hat{a}_p \\ \hat{b}_1, \ldots, \hat{b}_r \end{matrix} | q, z \| \begin{matrix} s_1, \ldots, s_{p'} \\ t_1, \ldots, t_{r'} \end{matrix} \right] = \\ &= \sum_{n=0}^{\infty} \frac{\langle \hat{a}_1; q \rangle_n \ldots \langle \hat{a}_p; q \rangle_n}{\langle 1; q \rangle_n \langle \hat{b}_1; q \rangle_n \ldots \langle \hat{b}_r; q \rangle_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+r+r'-p-p'} \times \\ &z^n \prod_{k=1}^{p'} (s_k; q)_n \prod_{k=1}^{r'} (t_k; q)_n^{-1}, \end{aligned} \tag{21}$$

where $q \neq 0$ when $p + p' > r + r' + 1$, and

$$\widehat{a} = \begin{cases} a, \\ \widetilde{a}, \\ \widetilde{\tfrac{m}{l}a}, \\ {}_k\widetilde{a} \end{cases} \tag{22}$$

In a few cases the parameter $\hat{a}$ in (21) will be the real plus infinity ($0 < |q| < 1$). They correspond to multiplication by 1. If we want to be formal, we could introduce a symbol ∞_H, with property

$$\langle \infty_H; q \rangle_n = \langle \infty_H + \alpha; q \rangle_n = 1, \ \alpha \in \mathbb{C}, \ 0 < |q| < 1. \tag{23}$$

The symbol ∞_H corresponds to parameter 0 in [17, p. 4]. We will denote ∞_H by ∞ in the rest of the paper.

The terms to the left of | in (21) are thought to be exponents, they are periodic mod $\dfrac{2\pi i}{\log q}$.

The first term to the right of | in (21) is the base, and the second is a letter in the spirit of the umbral calculus in [15]. The terms to the right of || in (21) are Watson q-shifted factorials [17]. There will be a certain redundance in notation here, but this is no problem.

1.3 – *The Γ_q function*

In the same way as the Γ function plays a basic role in complex analysis, the Γ_q function is fundamental for q-calculus. During the last years there has been an increasing interest in the Γ_q function. However there are certain restrictions in our knowledge of this function as the following exposition shows. The Γ_q function is defined in the unit disk $0 < |q| < 1$ by

DEFINITION 4.

$$\Gamma_q(x) = \frac{\langle 1; q \rangle_\infty}{\langle x; q \rangle_\infty} (1 - q)^{1-x}. \tag{24}$$

Here we deviate from the usual convention $q < 1$, because we want to work with meromorphic functions of several variables. The simple poles are located at $x = -n \pm \dfrac{2k\pi i}{\log q}$, $n, k \in \mathbb{N}$.

Heine, Ashton and Daum [11] used another function without the factor $(1-q)^{1-x}$, which they called the Heine Ω-function. The main difference between the two functions is that Ω has zeros, in contrast to the Γ_q function which has no zeros, and therefore $\frac{1}{\Gamma_q}$ is entire. Ashton [5] in his thesis supervised by Lindemann, showed its connection to elliptic functions. Daum [11] tried to find all the basic analogues of Thomae's ${}_3F_2$ transformation formula [42, eq. 11], using a notation analogous to that used by Whipple [48], and by essentially replacing the Γ_q function by the Heine Ω-function.

Daum's work was continued when in 1987 Beyer - Louck - Stein [9] and in 1992 Srinivasa Rao - Van der Jeugt - Raynal - Jagannathan - Rajeswari [38] showed that certain two-term transformation formulas between hypergeometric series easily can be described by means of invariance groups. In other words, they explained Whipple's [48] discovery in group language. In 1999 Van der Jeugt - Srinivasa Rao [43] found q-analogues of these results.

Daum [11] concludes his thesis by saying *It is hoped, however, that the use of the modified Heine Ω-function, will serve to emphasize the analogy between hypergeometric series and q-hypergeometric series and simplify the notation generally.*

Let's now return to the Γ_q function: To save space, the following notation for quotients of Γ_q functions will often be used.

DEFINITION 5. The generalized Γ_q function is given by

$$\Gamma_q\begin{bmatrix} a_1,\ldots,a_p \\ b_1,\ldots,b_r \end{bmatrix} \equiv \frac{\Gamma_q(a_1)\ldots\Gamma_q(a_p)}{\Gamma_q(b_1)\ldots\Gamma_q(b_r)}. \tag{25}$$

DEFINITION 6. This generalized Γ_q function is called balanced if

$$\sum_{k=1}^{p} a_k = \sum_{k=1}^{r} b_k, \quad p=r. \tag{26}$$

DEFINITION 7. A quotient of infinite q-shifted factorials

$$\frac{\prod_{k=1}^{p}\langle a_k;q\rangle_\infty}{\prod_{k=1}^{r}\langle b_k;q\rangle_\infty} \tag{27}$$

is called balanced if

$$\sum_{k=1}^{p} a_k = \sum_{k=1}^{r} b_k, \quad p=r. \tag{28}$$

These definitions are easily seen to be equivalent.

Balanced quotients of Γ_q's occur in many equations of q-calculus. To see why this is the case, we need to go back 100 years to the work of Mellin [31]. Mellin discovered that every hypergeometric function can be written as a function of generalized Γ functions. In the same way, every q-hypergeometric function can be written as a function of generalized Γ_q functions [23, p. 258]. In the presence of a balanced quotient of Γ_q's, limits when the parameters tend to $\pm\infty$ usually pose no problem. It would therefore be agreeable if we could find bounds for quotients of balanced Γ_q's. The first step in this direction was made by Ismail and Muldoon [25, p. 320]. This was generalized by Alzer [1].

The following lemmas are just a few examples of Γ_q formulas.

LEMMA 1.4. *A relation between Γ_q functions with different bases.*

$$\frac{\Gamma_{q^2}(x)}{\Gamma_q(x)} = \frac{\langle\widetilde{1};q\rangle_\infty}{\langle\widetilde{x};q\rangle_\infty}(1+q)^{1-x} \equiv (1+q)^{1-x}\langle\widetilde{1};q\rangle_{x-1}. \tag{29}$$

LEMMA 1.5.

$$\frac{\left\langle \frac{1+a}{2}, 1-b+\frac{a}{2}; q^2\right\rangle_\infty}{\langle 1+a-b;q^2\rangle_\infty} = \Gamma_q\begin{bmatrix} 1+a-b, 1+\frac{a}{2} \\ 1+a, 1+\frac{a}{2}-b\end{bmatrix} \frac{\left\langle 1+\widetilde{\frac{a}{2}}-b;q\right\rangle_\infty}{\left\langle 1+\widetilde{\frac{a}{2}}, 1+\widetilde{a}-b;q\right\rangle_\infty}. \tag{30}$$

PROOF. Use the Bailey-Daum theorem. $\square$

We are going to find q-analogues of hypergeometric summation formulas in this paper. One example is the Whipple formula [48] for a terminating series, which was first given by Watson.

$${}_3F_2\left(a,-n,\frac{c}{2};\frac{1+a-n}{2},c;1\right) = \Gamma\begin{bmatrix} \frac{1+a-n}{2}, \frac{1+c}{2}, \frac{1+c-a+n}{2}, \frac{1}{2} \\ \frac{1+a}{2}, \frac{1-n}{2}, \frac{1-a+c}{2}, \frac{1+c+n}{2}\end{bmatrix} \tag{31}$$

2. Four q-summation formulas of Andrews.

In recent years, combinatorics has developed quickly. Combinatorial identities can be expressed either as hypergeometric formulas - the canonical form-, or as binomial coefficient identities. Combinatorial results are often expressed as q-formulas. Here we have a similar duality: q-hypergeometric formulas - the canonical form-, or q-binomial coefficient identities. Now q-binomial coefficient computations are easier to handle by hand, or by computer. Some mathematicians, like for example Catalan, Bateman and Gould, have tried (in vain) to make systematic treatments of binomial coefficient identities. Nonetheless, q-hypergeometric formulas and q-binomial coefficient identities belong together. Therefore we represent many theorems of this chapter both ways.

As a basis for the following calculations, we are going to use 4 q-formulas by Andrews. The first is

THEOREM 2.1. *Andrews's q-analogue of Kummer's formula* [28, (2), p. 134] *from* [2, 1.8 p. 526].

$$
{}_2\phi_2\left[\begin{matrix} a,b \\ \frac{1+a+b}{2}, \frac{\widetilde{1+a+b}}{2} \end{matrix}\,|q,-q\right] = \Gamma_q\left[\begin{matrix}\frac{1+a+b}{2},\frac{1}{2}\\ \frac{1+b}{2},\frac{1+a}{2}\end{matrix}\right] \frac{\left\langle \widetilde{\frac{1+b}{2}};q\right\rangle_{\frac{a}{2}}}{\left\langle \widetilde{\frac{1}{2}};q\right\rangle_{\frac{a}{2}}} \equiv \tag{32}
$$
$$
\Gamma_{q^2}\left[\begin{matrix}\frac{1+a+b}{2},\frac{1}{2}\\ \frac{1+b}{2},\frac{1+a}{2}\end{matrix}\right].
$$

PROOF. By [2, 1.8 p. 526]

$$
LHS \overset{\text{by (30)}}{=} \Gamma_q\left[\begin{matrix}\frac{1+a+b}{2},1+\frac{a}{2}\\ 1+a,\frac{1+b}{2}\end{matrix}\right] \frac{\left\langle \widetilde{\frac{1+b}{2}},\widetilde{1};q\right\rangle_{\infty}}{\left\langle \frac{\widetilde{1+a+b}}{2},1+\widetilde{\frac{a}{2}};q\right\rangle_{\infty}} \overset{\text{by (29)}}{=}
$$
$$
\Gamma_q\left[\begin{matrix}\frac{1+a+b}{2}\\ \frac{1+b}{2},1+a\end{matrix}\right]\Gamma_{q^2}\left[\begin{matrix}1+\frac{a}{2}\\ -\end{matrix}\right](1+q)^{\frac{a}{2}}\frac{\left\langle \widetilde{\frac{1+b}{2}};q\right\rangle_{\infty}}{\left\langle \frac{\widetilde{1+a+b}}{2};q\right\rangle_{\infty}} = \tag{33}
$$
$$
\Gamma_q\left[\begin{matrix}\frac{1+a+b}{2}\\ \frac{1+b}{2}\end{matrix}\right]\Gamma_{q^2}\left[\begin{matrix}\frac{1}{2}\\ \frac{1+a}{2}\end{matrix}\right](1+q)^{-\frac{a}{2}}\frac{\left\langle \widetilde{\frac{1+b}{2}};q\right\rangle_{\infty}}{\left\langle \frac{\widetilde{1+a+b}}{2};q\right\rangle_{\infty}} \overset{\text{by (29)}}{=} RHS,
$$

where we have used the q-analogue of the Legendre duplication formula in the penultimate step. □

REMARK 1. The name Gauss second summation theorem for Kummer's formula [28, (2), p. 134] was coined by Slater. This formula has recently received a lot of attention. We will generalize the q-analogue later.

If b is a negative integer, (32) may be reformulated in the form

THEOREM 2.2.

$$
\begin{aligned}
&{}_2\phi_2\left[\begin{matrix} a, -2N \\ \frac{1+a-2N}{2}, \frac{1+\widetilde{a}-2N}{2} \end{matrix} | q, -q\right] \equiv \\
&\sum_{k=0}^{2N} \binom{2N}{k}_q q^{2\binom{k}{2}+k(1-2N)} \frac{(-1)^k \langle a;q\rangle_k}{\left\langle \frac{1+a-2N}{2}; q^2\right\rangle_k} = \\
&\frac{\left\langle \frac{1}{2}; q^2\right\rangle_N q^{-Na}}{\left\langle \frac{1-a}{2}; q^2\right\rangle_N} \equiv \frac{\left\langle \frac{1-2N}{2}; q^2\right\rangle_N}{\left\langle \frac{1+a-2N}{2}; q^2\right\rangle_N}.
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
&{}_2\phi_2\left[\begin{matrix} a, -N \\ \frac{1+a-N}{2}, \frac{1+\widetilde{a}-N}{2} \end{matrix} | q, -q\right] \equiv \\
&\sum_{k=0}^{N} \binom{N}{k}_q q^{2\binom{k}{2}+k(1-N)} \frac{(-1)^k \langle a;q\rangle_k}{\left\langle \frac{1+a-N}{2}; q^2\right\rangle_k} = 0, N \text{ odd}.
\end{aligned}
\tag{35}
$$

The following corollary was influenced by [27]. The proof is the same. This idea to use the contiguity relations goes back to Kummer [28, p. 134-36].

COROLLARY 2.3.

$$
\begin{aligned}
&{}_2\phi_2\left[\begin{matrix} a,b \\ \frac{2+a+b}{2},\frac{2+\widetilde{a}+b}{2}\end{matrix}|q,-q\right]=\frac{q^a(1-q^b)}{q^a-q^b}\Gamma_q\begin{bmatrix}\frac{2+a+b}{2},\frac{1}{2}\\ \frac{2+b}{2},\frac{1+a}{2}\end{bmatrix}\frac{\left\langle\frac{\widetilde{2+b}}{2};q\right\rangle_{\frac{a}{2}}}{\left\langle\widetilde{\frac{1}{2}};q\right\rangle_{\frac{a}{2}}}\\
&-\frac{q^b(1-q^a)}{q^a-q^b}\Gamma_q\begin{bmatrix}\frac{2+a+b}{2},\frac{1}{2}\\ \frac{1+b}{2},\frac{2+a}{2}\end{bmatrix}\frac{\left\langle\frac{\widetilde{1+b}}{2};q\right\rangle_{\frac{a+1}{2}}}{\left\langle\widetilde{\frac{1}{2}};q\right\rangle_{\frac{a+1}{2}}}.
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
&{}_2\phi_2\left[\begin{matrix} a,b \\ \frac{3+a+b}{2},\frac{3+\widetilde{a}+b}{2}\end{matrix}|q,-q\right]=\\
&\frac{q^a(1-q^b)}{q^a-q^b}\left[\frac{q^a(1-q^{b+1})}{q^a-q^{b+1}}\Gamma_q\begin{bmatrix}\frac{3+a+b}{2},\frac{1}{2}\\ \frac{3+b}{2},\frac{1+a}{2}\end{bmatrix}\frac{\left\langle\frac{\widetilde{3+b}}{2};q\right\rangle_{\frac{a}{2}}}{\left\langle\widetilde{\frac{1}{2}};q\right\rangle_{\frac{a}{2}}}\right.\\
&\left.-\frac{q^{b+1}(1-q^a)}{q^a-q^{b+1}}\Gamma_q\begin{bmatrix}\frac{3+a+b}{2},\frac{1}{2}\\ \frac{2+b}{2},\frac{2+a}{2}\end{bmatrix}\frac{\left\langle\frac{\widetilde{2+b}}{2};q\right\rangle_{\frac{a+1}{2}}}{\left\langle\widetilde{\frac{1}{2}};q\right\rangle_{\frac{a+1}{2}}}\right]-\\
&\frac{q^b(1-q^a)}{q^a-q^b}\left[\frac{q^{a+1}(1-q^b)}{q^{a+1}-q^b}\Gamma_q\begin{bmatrix}\frac{3+a+b}{2},\frac{1}{2}\\ \frac{2+b}{2},\frac{2+a}{2}\end{bmatrix}\frac{\left\langle\frac{\widetilde{2+b}}{2};q\right\rangle_{\frac{a+1}{2}}}{\left\langle\widetilde{\frac{1}{2}};q\right\rangle_{\frac{a+1}{2}}}\right.\\
&\left.-\frac{q^b(1-q^{a+1})}{q^{a+1}-q^b}\Gamma_q\begin{bmatrix}\frac{3+a+b}{2},\frac{1}{2}\\ \frac{1+b}{2},\frac{3+a}{2}\end{bmatrix}\frac{\left\langle\frac{\widetilde{1+b}}{2};q\right\rangle_{\frac{a+2}{2}}}{\left\langle\widetilde{\frac{1}{2}};q\right\rangle_{\frac{a+2}{2}}}\right].
\end{aligned}
\tag{37}
$$

THEOREM 2.4. *Andrews's q-analogue of Kummer's formula* [28, p. 134] *from* [2, 1.8 p. 526] *and* [39, (15), p. 9].

$$
(38)\quad {}_2\phi_2\left[\begin{matrix} a, 1-a \\ c, \widetilde{1} \end{matrix} |q, -q^c\right] = \Gamma_q\left[\begin{matrix} \frac{c}{2}, \frac{1+c}{2} \\ \frac{1+c-a}{2}, \frac{a+c}{2} \end{matrix}\right] \frac{\left\langle \widetilde{\frac{1+c-a}{2}}; q\right\rangle_{\frac{a}{2}}}{\left\langle \frac{\widetilde{c}}{2}; q\right\rangle_{\frac{a}{2}}} \equiv \Gamma_{q^2}\left[\begin{matrix} \frac{c}{2}, \frac{1+c}{2} \\ \frac{1+c-a}{2}, \frac{a+c}{2} \end{matrix}\right].
$$

Proof. By [2, 1.9 p. 526]

$$
LHS = \frac{\left\langle \frac{1+c-a}{2}, \frac{a+c}{2}; q^2\right\rangle_\infty}{\langle c; q\rangle_\infty} =
$$

$$
(39)\quad \frac{\left\langle \frac{1+c-a}{2}, \widetilde{\frac{1+c-a}{2}}, \frac{a+c}{2}, \widetilde{\frac{a+c}{2}}; q\right\rangle_\infty}{\left\langle \frac{1+c}{2}, \widetilde{\frac{1+c}{2}}, \frac{c}{2}, \frac{\widetilde{c}}{2}; q\right\rangle_\infty} = RHS.
$$

□

If a is a negative integer, (38) may be reformulated in the form

Theorem 2.5.

$$
(40)\quad {}_2\phi_2\left[\begin{matrix} -2N, 1+2N \\ c, \widetilde{1} \end{matrix} |q, -q^c\right] \equiv \sum_{k=0}^{2N} \binom{2N}{k}_q (-1)^k q^{2\binom{k}{2}+k(c-2N)} \frac{\langle 1+2N; q\rangle_k}{\langle c, \widetilde{1}; q\rangle_k} = \frac{\left\langle \frac{c-2N}{2}; q^2\right\rangle_N}{\left\langle \frac{1+c}{2}; q^2\right\rangle_N}.
$$

$$
(41)\quad {}_2\phi_2\left[\begin{matrix} -N, 1+N \\ c, \widetilde{1} \end{matrix} |q, -q^c\right] = \frac{\left\langle \frac{c-N}{2}; q^2\right\rangle_{\frac{1+N}{2}}}{\left\langle \frac{c}{2}; q^2\right\rangle_{\frac{1+N}{2}}}, N \text{ odd}.
$$

We have expressed two of Andrew's formulas by the Γ_q function, and also expressed the corresponding finite sums by quotients of q-shifted factorials. We will continue with generalizations of these two formulas.

We need a lemma for the following proof.

LEMMA 2.6. *A q-analogue of the Legendre duplication formula for the Γ-function written in q-shifted factorial form is*

$$\frac{\langle 1-n;q^2\rangle_{\frac{n}{2}}}{\langle 1-\widetilde{n};q\rangle_{n-1}\left\langle\frac{1}{2};q^2\right\rangle_{\frac{n}{2}}}=(-1)^{\frac{n}{2}}\,\mathrm{QE}\left(-\frac{n^2}{4}\right), n \text{ even.} \tag{42}$$

THEOREM 2.7. *Compare* [18, (II 17), p. 355]. *A q-analogue of* [45], [8, (1.2), p. 237].

$$ {}_4\phi_3\left[\begin{matrix}\frac{c}{2},\frac{\widetilde{c}}{2},a,-2N\\ \frac{-2N+1+a}{2},\frac{-2N\widetilde{+1}+a}{2},c\end{matrix}|q,q\right]=\frac{\left\langle\frac{1}{2},\frac{1+c-a}{2};q^2\right\rangle_N}{\left\langle\frac{1-a}{2},\frac{1+c}{2};q^2\right\rangle_N}. \tag{43}$$

$$ {}_4\phi_3\left[\begin{matrix}\frac{c}{2},\frac{\widetilde{c}}{2},a,-N\\ \frac{-N+1+a}{2},\frac{-N\widetilde{+1}+a}{2},c\end{matrix}|q,q\right]=0,\ N \text{ odd.} \tag{44}$$

PROOF. The proof of (44) is relegated to the next proof, i.e. (54). We will use the method in Andrews [3, p. 334]. For convenience, we will denote the

LHS of (43) $ {}_4\phi_3\left[\begin{matrix}\frac{c}{2},\frac{\widetilde{c}}{2},a,-n\\ \frac{-n+1+a}{2},\frac{-n\widetilde{+1}+a}{2},c\end{matrix}|q,q\right]$.

By Watson's [46] transformation formula for a terminating very-well-poised ${}_8\phi_7$ series, denoting

$$(\alpha)\equiv\left(a,b,c,d,1+\frac{1}{2}a,1\widetilde{+\frac{1}{2}}a,e,-n\right), \tag{45}$$

$$(\beta)\equiv\left(1+a-b,1+a-c,1+a-d,1+a-e,\frac{1}{2}a,\widetilde{\frac{1}{2}}a,1+a+n\right), \tag{46}$$

this formula takes the following form

$$\begin{aligned}&{}_8\phi_7\left[\begin{matrix}(\alpha)\\(\beta)\end{matrix}|q,q^{2a+2+n-b-c-d-e}\right]=\frac{\langle 1+a,1+a-d-e;q\rangle_n}{\langle 1+a-d,1+a-e;q\rangle_n}\times\\ &{}_4\phi_3\left[\begin{matrix}d,e,1+a-b-c,-n\\1+a-b,1+a-c,d+e-n-a\end{matrix}|q,q\right].\end{aligned} \tag{47}$$

Now make the substitution

$$a \to \widetilde{-n}, b \to \frac{-n+1-a}{2}, c \to \frac{\widetilde{-n+1}-a}{2}, d \to \frac{c}{2}, e \to \frac{\widetilde{c}}{2}, \tag{48}$$

to obtain

$$\begin{aligned} &{}_8\phi_7\left[\begin{matrix}(\alpha')\\(\beta')\end{matrix}|q, q^{1+a-c}\right]\frac{\left\langle \widetilde{-n+1}-\frac{c}{2}, -n+1-\frac{c}{2}; q\right\rangle_n}{\langle \widetilde{-n+1}, -n+1-c; q\rangle_n} = \\ &{}_4\phi_3\left[\begin{matrix}\frac{c}{2}, \frac{\widetilde{c}}{2}, a, -n\\ \frac{-n+1+a}{2}, \frac{\widetilde{-n+1}+a}{2}, c\end{matrix}|q, q\right]. \end{aligned} \tag{49}$$

where

$$(\alpha') \equiv \left(\widetilde{-n}, \frac{-n+1-a}{2}, \frac{\widetilde{-n+1}-a}{2}, \frac{c}{2}, \frac{\widetilde{c}}{2}, {}^{\frac{1}{4}}\widetilde{1-\frac{1}{2}}n, {}^{\frac{3}{4}}\widetilde{1-\frac{1}{2}}n, -n\right), \tag{50}$$

$$(\beta') \equiv (\frac{-n+1+a}{2}, \frac{\widetilde{-n+1}+a}{2}, 1-n-\frac{c}{2}, \widetilde{1-n}-\frac{c}{2}, {}^{\frac{1}{4}}\frac{\widetilde{-n}}{2}, {}^{\frac{3}{4}}\frac{\widetilde{-n}}{2}, \widetilde{1}), \tag{51}$$

Now

$$\langle\widetilde{a}; q^2\rangle_n = \langle {}^{\frac{1}{4}}\widetilde{a}, {}^{\frac{3}{4}}\widetilde{a}; q\rangle_n. \tag{52}$$

By the q-Dixon theorem we have, assuming n even

$$\begin{aligned} &{}_4\phi_3\left[\begin{matrix}\frac{c}{2}, \frac{\widetilde{c}}{2}, a, -n\\ \frac{-n+1+a}{2}, \frac{\widetilde{-n+1}+a}{2}, c\end{matrix}|q, q\right] = \\ &{}_4\phi_3\left[\begin{matrix}\frac{-n+1-a}{2}, 1-\frac{\widetilde{n}}{2}, \frac{c}{2}, -n\\ \frac{1-n+a}{2}, 1-n-\frac{c}{2}, -\frac{\widetilde{n}}{2}\end{matrix}|q^2, q^{1+a-c}\right]\frac{\left\langle \widetilde{-n+1}-\frac{c}{2}, -n+1-\frac{c}{2}; q\right\rangle_n}{\langle \widetilde{-n+1}, -n+1-c; q\rangle_n} = \\ &\frac{\left\langle \widetilde{-n+1}-\frac{c}{2}, -n+1-\frac{c}{2}; q\right\rangle_n}{\langle \widetilde{-n+1}, -n+1-c; q\rangle_n}\frac{\left\langle 1-n, \frac{1-n+a-c}{2}; q^2\right\rangle_{\frac{n}{2}}}{\left\langle -n+1-\frac{c}{2}, \frac{1-n+a}{2}; q^2\right\rangle_{\frac{n}{2}}} \overset{\text{by (42)}}{=} \frac{\left\langle \frac{1}{2}, \frac{1+c-a}{2}; q^2\right\rangle_{\frac{n}{2}}}{\left\langle \frac{1-a}{2}, \frac{1+c}{2}; q^2\right\rangle_{\frac{n}{2}}}. \end{aligned} \tag{53}$$

□

Equation (43) may be reformulated in the form

THEOREM 2.8. *Compare* [3, p. 334].

$$
{}_4\phi_3\left[\begin{matrix} \frac{c}{2}, \widetilde{\frac{c}{2}}, a, -n \\ \frac{-n+1+a}{2}, \frac{-n+\widetilde{1}+a}{2}, c \end{matrix} |q, q\right]
$$

$$
= \frac{\langle \widetilde{1}, 1-c; q\rangle_\infty}{\langle 1\widetilde{-}n, -n+1-c; q\rangle_\infty} \times \tag{54}
$$

$$
\frac{\left\langle \frac{-2n+2}{2}, \frac{1+a}{2}, \frac{-n+2-c}{2}, \frac{a-n+1-c}{2}; q^2\right\rangle_\infty}{\left\langle \frac{2-c}{2}, \frac{1+a-n}{2}, \frac{-n+2}{2}, \frac{a+1-c}{2}; q^2\right\rangle_\infty}.
$$

PROOF.

$$
LHS = {}_4\phi_3\left[\begin{matrix} \frac{-n+1-a}{2}, 1-\widetilde{\frac{n}{2}}, \frac{c}{2}, -n \\ \frac{1-n+a}{2}, 1-n-\frac{c}{2}, -\widetilde{\frac{n}{2}} \end{matrix} |q^2, q^{1+a-c}\right] \frac{\left\langle -n+\widetilde{1}-\frac{c}{2}, -n+1-\frac{c}{2}; q\right\rangle_n}{\langle -\widetilde{n+}1, -n+1-c; q\rangle_n} =
$$

$$
\frac{\langle \widetilde{1}, 1-c; q\rangle_\infty \left\langle \frac{a+1}{2}, -n+1, \frac{-n+2-c}{2}, \frac{-n+a+1-c}{2}; q^2\right\rangle_\infty}{\left\langle -\widetilde{n+}1, -n+1-c; q\right\rangle_\infty \left\langle \frac{a-n+1}{2}, \frac{-n+2}{2}, \frac{2-c}{2}, \frac{a+1-c}{2}; q^2\right\rangle_\infty} =
$$

$$
\frac{\langle \widetilde{1}; q\rangle_\infty \left\langle \frac{a+1}{2}, \frac{-n+1}{2}, \frac{1-c}{2}, \frac{-n+a+1-c}{2}; q^2\right\rangle_\infty}{\left\langle \frac{a-n+1}{2}, \frac{-n+1-c}{2}, \frac{a+1-c}{2}; q^2\right\rangle_\infty} = \frac{\langle \widetilde{1}, 1-c; q\rangle_\infty}{\langle 1\widetilde{-}n, -n+1-c; q\rangle_\infty} \times \tag{55}
$$

$$
\frac{\left\langle \frac{-2n+2}{2}, \frac{1+a}{2}, \frac{-n+2-c}{2}, \frac{a-n+1-c}{2}; q^2\right\rangle_\infty}{\left\langle \frac{2-c}{2}, \frac{1+a-n}{2}, \frac{-n+2}{2}, \frac{a+1-c}{2}; q^2\right\rangle_\infty}.
$$

□

REMARK 2. The equivalent expression

$$
\frac{\left\langle \frac{a+1}{2}, \frac{-n+1}{2}, \frac{1-c}{2}, \frac{-n+a+1-c}{2}; q^2\right\rangle_\infty}{\left\langle \frac{1}{2}, \frac{a-n+1}{2}, \frac{-n+1-c}{2}, \frac{a+1-c}{2}; q^2\right\rangle_\infty} \tag{56}
$$

for the RHS of (54) was given in [21, 1.4.]. The simple proof uses the Euler formula [16, p. 271]

$$\langle \widetilde{1}; q\rangle_\infty \left\langle \frac{1}{2}; q^2 \right\rangle_\infty = 1. \tag{57}$$

This was the beginning of the investigation that led to the current paper.

Now let $c \to -\infty$ in (54) and (43) to arrive at (32) and (34).

COROLLARY 2.9.

$$ {}_3\phi_2 \left[\begin{matrix} a, -2N, \infty \\ \dfrac{1+a-2N}{2}, \widetilde{\dfrac{1+a-2N}{2}} \end{matrix} |q, q \right] = \frac{\left\langle \frac{1}{2}; q^2 \right\rangle_N}{\left\langle \frac{1-a}{2}; q^2 \right\rangle_N}. \tag{58}$$

PROOF. Let $c \to \infty$ in (43). □

COROLLARY 2.10.

$$\begin{aligned} &{}_3\phi_2 \left[\begin{matrix} \frac{c}{2}, \frac{\widetilde{c}}{2}, -2N \\ c, \infty \end{matrix} |q, q \right] \\ &\equiv \sum_{k=0}^{2N} \binom{2N}{k}_q q^{\binom{k}{2}+k(1-2N)} \frac{(-1)^k \left\langle \frac{c}{2}; q^2 \right\rangle_k}{\langle c; q\rangle_k} \\ &= \frac{\left\langle \frac{1}{2}; q^2 \right\rangle_N q^{Nc}}{\left\langle \frac{1+c}{2}; q^2 \right\rangle_N}. \end{aligned} \tag{59}$$

$$\begin{aligned} &{}_3\phi_2 \left[\begin{matrix} \frac{c}{2}, \frac{\widetilde{c}}{2}, -N \\ c, \infty \end{matrix} |q, q \right] \\ &\equiv \sum_{k=0}^{N} \binom{N}{k}_q q^{\binom{k}{2}+k(1-N)} \frac{(-1)^k \left\langle \frac{c}{2}; q^2 \right\rangle_k}{\langle c; q\rangle_k} = 0, N \text{ odd}. \end{aligned} \tag{60}$$

PROOF. Let $a \to \infty$ in (43) and (44). □

COROLLARY 2.11. *Compare* [18, ex. 3.4, p. 101].

(61)
$$
{}_3\phi_1\left[\begin{matrix}\frac{c}{2},\frac{\widetilde{c}}{2},-2N \\ c\end{matrix}|q,-q^{2N}\right] \equiv \sum_{k=0}^{2N}\binom{2N}{k}_q \frac{(-1)^k\left\langle\frac{c}{2};q^2\right\rangle_k}{\langle c;q\rangle_k} = \frac{\left\langle\frac{1}{2};q^2\right\rangle_N}{\left\langle\frac{1+c}{2};q^2\right\rangle_N}.
$$

(62)
$$
{}_3\phi_1\left[\begin{matrix}\frac{c}{2},\frac{\widetilde{c}}{2},-N \\ c\end{matrix}|q,-q^{N}\right] \equiv \sum_{k=0}^{N}\binom{N}{k}_q \frac{(-1)^k\left\langle\frac{c}{2};q^2\right\rangle_k}{\langle c;q\rangle_k} = 0, N \text{ odd}.
$$

Proof. Let $a \to -\infty$ in (43) and (44). □

Theorem 2.12. [3, p. 333]

(63)
$$
{}_4\phi_3\left[\begin{matrix}\frac{c}{2},\frac{\widetilde{c}}{2},1+n,-n \\ c+1-e,\widetilde{1},e\end{matrix}|q,q\right] = \frac{\langle \widetilde{e}, e-c;q\rangle_\infty}{\langle \widetilde{e-n}, -n+e-c;q\rangle_\infty} \times \frac{\left\langle\frac{e+1+n}{2},\frac{-n+e+1-c}{2},-n+e;q^2\right\rangle_\infty}{\left\langle\frac{-n+e+1}{2},\frac{n+1+e-c}{2},e;q^2\right\rangle_\infty}.
$$

Proof. We will again use Watson's 1929 [46] transformation formula for a terminating very-well-poised ${}_8\phi_7$ series. Now make the substitution

(64)
$$
a \to -n\widetilde{+e}-1, b \to -n+e-1, c \to \widetilde{-n}, d \to \frac{c}{2}, e \to \frac{\widetilde{c}}{2}.
$$

to obtain

(65)
$$
{}_8\phi_7\left[\begin{matrix}(\alpha') \\ (\beta')\end{matrix}|q,q^{e+1-a-c}\right] \frac{\left\langle -n\widetilde{+e}-\frac{c}{2}, -n+e-\frac{c}{2};q\right\rangle_n}{\langle -\widetilde{n+}e, -n+e-c;q\rangle_n} = {}_4\phi_3\left[\begin{matrix}\frac{c}{2},\frac{\widetilde{c}}{2},1+n,-n \\ e,\widetilde{1},c+1-e\end{matrix}|q,q\right].
$$

where

$$(\alpha') \equiv \left(\widetilde{-n}, -n+e-1, -n\widetilde{+e}-1, \frac{c}{2}, \frac{\widetilde{c}}{2}, {}_{\frac{1}{4}}\frac{1\widetilde{+e}-n}{2}, {}_{\frac{3}{4}}\frac{1\widetilde{+e}-n}{2}, -n\right), \tag{66}$$

$$(\beta') \equiv (e, \widetilde{e}, e-n-\frac{c}{2}, e-\widetilde{n}-\frac{c}{2}, {}_{\frac{1}{4}}\frac{-n\widetilde{+e}-1}{2}, {}_{\frac{3}{4}}\frac{-n\widetilde{+e}-1}{2}, \widetilde{1}). \tag{67}$$

Now

$$\langle \widetilde{a}; q^2 \rangle_n = \langle \widetilde{{}_{\frac{1}{4}}a}, \widetilde{{}_{\frac{3}{4}}a}; q \rangle_n. \tag{68}$$

And we have

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} \frac{c}{2}, \frac{\widetilde{c}}{2}, 1+n, -n \\ e, \widetilde{1}, c+1-e \end{matrix} |q, q\right] = \\
&{}_4\phi_3\left[\begin{matrix} -n-1+e, \frac{1-\widetilde{n}+e}{2}, \frac{c}{2}, -n \\ e, e-n-\frac{c}{2}, \frac{-n\widetilde{+e}-1}{2} \end{matrix} |q^2, q^{e+1+n-c}\right] \frac{\left\langle -n\widetilde{+e}-\frac{c}{2}, -n+e-\frac{c}{2}; q\right\rangle_n}{\left\langle -\widetilde{n+}e, -n+e-c; q\right\rangle_n} = \\
&\frac{\langle \widetilde{e}, e-c; q\rangle_\infty \left\langle \frac{n+e+1}{2}, -n+e, \frac{-n+e+1-c}{2}; q^2\right\rangle_\infty}{\langle -\widetilde{n+}e, -n+e-c; q\rangle_\infty \left\langle e, \frac{-n+e+1}{2}, \frac{n+e+1-c}{2}; q^2\right\rangle_\infty} = \frac{\langle \widetilde{e}, e-c; q\rangle_\infty}{\langle e\widetilde{-n}, -n+e-c; q\rangle_\infty} \times \\
&\frac{\left\langle \frac{-2n+2e-c}{2}, \frac{e+1+n}{2}, \frac{-n+e+1-c}{2}, \frac{2e-c}{2}, -n+e; q^2\right\rangle_\infty}{\left\langle \frac{2e-c}{2}, \frac{-2n+2e-c}{2}, \frac{-n+e+1}{2}, \frac{n+1+e-c}{2}, e; q^2\right\rangle_\infty} = RHS.
\end{aligned} \tag{69}$$

□

Equation (63) may be reformulated in the form

THEOREM 2.13. *A q-analogue of the Whipple formula* [48, p. 114], [8, (1.3), p. 237].

$$\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} \frac{c}{2}, \frac{\widetilde{c}}{2}, 1+2N, -2N \\ c+1-e, \widetilde{1}, e \end{matrix} |q, q\right] \\
&\equiv \sum_{k=0}^{2N} \binom{2N}{k}_q q^{\binom{k}{2}+k(1-2N)} \frac{(-1)^k \langle 1+2N; q\rangle_k \left\langle \frac{c}{2}; q^2\right\rangle_k}{\langle c+1-e, \widetilde{1}, e; q\rangle_k} \\
&= \frac{(-1)^N \ \mathrm{QE}(N^2+N+Nc-Ne) \left\langle \frac{e-2N}{2}, \frac{e+1-c}{2}; q^2\right\rangle_N}{\left\langle \frac{e+1}{2}, \frac{2+c-e}{2}; q^2\right\rangle_N}.
\end{aligned} \tag{70}$$

$$
{}_4\phi_3\left[\begin{matrix}\frac{c}{2},\frac{\widetilde{c}}{2},2+2N,-2N-1\\ c+1-e,\widetilde{1},e\end{matrix}|q,q\right]
$$

$$
\equiv\sum_{k=0}^{2N+1}\binom{2N+1}{k}_q q^{\binom{k}{2}-2kN}\frac{(-1)^k\langle 2+2N;q\rangle_k\left\langle\frac{c}{2};q^2\right\rangle_k}{\langle c+1-e,\widetilde{1},e;q\rangle_k} \tag{71}
$$

$$
=\frac{(-1)^{N+1}\ \mathrm{QE}((N+1)^2+(c-e)(N+1))\left\langle\frac{e-2N-1}{2},\frac{e-c}{2};q^2\right\rangle_{N+1}}{\left\langle\frac{e}{2},\frac{1+c-e}{2};q^2\right\rangle_{N+1}}.
$$

Now let $c\to-\infty$ in (70) and (71) to arrive at (40) and (41).

Corollary 2.14. *Another q-analogue of Kummers formula.*

$$
{}_3\phi_2\left[\begin{matrix}-2N,1+2N,\infty\\ c,\widetilde{1}\end{matrix}|q,q\right]\equiv
$$

$$
\sum_{k=0}^{2N}\binom{2N}{k}_q(-1)^k q^{\binom{k}{2}+k-2Nk}\frac{\langle 1+2N;q\rangle_k}{\langle c,\widetilde{1};q\rangle_k}=\frac{\left\langle\frac{c-2N}{2};q^2\right\rangle_N}{\left\langle\frac{1+c}{2};q^2\right\rangle_N}q^{2N^2+N}. \tag{72}
$$

$$
{}_3\phi_2\left[\begin{matrix}-N,1+N,\infty\\ c,\widetilde{1}\end{matrix}|q,q\right]\equiv\sum_{k=0}^{N}\binom{N}{k}_q(-1)^k q^{\binom{k}{2}+k-Nk}\frac{\langle 1+N;q\rangle_k}{\langle c,\widetilde{1};q\rangle_k}
$$

$$
=\frac{\left\langle\frac{c-N}{2};q^2\right\rangle_{\frac{1+N}{2}}}{\left\langle\frac{c}{2};q^2\right\rangle_{\frac{1+N}{2}}}q^{\binom{N+1}{2}},N\text{ odd}. \tag{73}
$$

Proof. Let $c\to\infty$ in (70) and (71). □

3. Kummer's ${}_2F_1(-1)$ formula and Jacobi's theta function.

In [13] we used a q–analogue of the Dixon-Schafheitlin theorem

$$
(74)\quad {}_4\phi_3\left[\begin{matrix} a,b,c,\widetilde{1+\frac{1}{2}a} \\ 1+a-b,1+a-c,\widetilde{\frac{1}{2}a} \end{matrix} |q,q^{1+\frac{a}{2}-b-c}\right] = \\
= \Gamma_q\left[\begin{matrix} 1+a-b,1+a-c,1+\frac{a}{2},1+\frac{a}{2}-b-c \\ 1+a,1+a-b-c,1+\frac{a}{2}-b,1+\frac{a}{2}-c \end{matrix}\right]
$$

to prove a q-analogue of Kummer's ${}_2F_1(-1)$ formula [28], the Bailey–Daum summation formula,

$$
(75)\quad {}_2\phi_1(a,b;1+a-b|q,-q^{1-b}) = \Gamma_q\left[\begin{matrix} 1+a-b,1+\frac{a}{2} \\ 1+a,1+\frac{a}{2}-b \end{matrix}\right] \frac{\left\langle \widetilde{1+\frac{a}{2}}-b,\widetilde{1};q\right\rangle_\infty}{\left\langle \widetilde{1+\frac{a}{2}},\widetilde{1-b};q\right\rangle_\infty},
$$

where $1+a-b \neq 0,-1,-2\ldots.$

We can easily find two related formulas.

THEOREM 3.1. *A second q-analogue of Kummer's* ${}_2F_1(-1)$ *formula.*

$$
(76)\quad {}_3\phi_3\left(a,b,\widetilde{1+\frac{a}{2}};1+a-b,\frac{\widetilde{a}}{2},\infty|q,q^{1+\frac{a}{2}-b}\right) = \Gamma_q\left[\begin{matrix} 1+a-b,1+\frac{a}{2} \\ 1+a,1+\frac{a}{2}-b \end{matrix}\right],
$$

where $1+a-b \neq 0,-1,-2\ldots.$

PROOF. Let $c \to -\infty$ in (74). This proof is a q-analogue of [7, p. 13]. □

THEOREM 3.2. *A third q-analogue of Kummer's* ${}_2F_1(-1)$ *formula.*

$$
(77)\quad {}_4\phi_2\left(a,b,\widetilde{1+\frac{a}{2}},\infty;1+a-b,\frac{\widetilde{a}}{2}|q,q^{-\frac{a}{2}-b}\right) = \Gamma_q\left[\begin{matrix} 1+a-b,1+\frac{a}{2} \\ 1+a,1+\frac{a}{2}-b \end{matrix}\right] q^{\frac{ab}{2}},
$$

where $1+a-b \neq 0,-1,-2\ldots.$

PROOF. Let $c \to +\infty$ in (74). This proof is a q-analogue of [4, p. 144]. □

In [13] we found the following special case of (75)

$$
\begin{aligned}
&{}_2\phi_1(-2N, b; 1-2N-b|q, -q^{1-b}) \equiv \\
&\sum_{k=0}^{2N} \binom{2N}{k}_q q^{\binom{k}{2}+k(1-2N-b)} \frac{\langle b;q\rangle_k}{\langle 1-2N-b;q\rangle_k} \equiv \\
&2\sum_{k=0}^{N-1} \binom{2N}{k}_q (-1)^k \frac{\langle b;q\rangle_k}{\langle 2N+b-k;q\rangle_k} + \binom{2N}{N}_q (-1)^N \frac{\langle b;q\rangle_N}{\langle N+b;q\rangle_N} = \\
&\frac{\langle \widetilde{b};q\rangle_N}{\langle N+b;q\rangle_N} \left\langle \frac{1}{2}; q^2 \right\rangle_N .
\end{aligned}
\tag{78}
$$

$$
{}_2\phi_1(-N, b; 1-N-b|q, -q^{1-b}) = 0, N \text{ odd}. \tag{79}
$$

The aim of the rest of this section is to give a proof of (78).

The proof for $q = 1$ [35, p. 43] uses the Euler reflection formula

$$
\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{80}
$$

The following crucial formula has an interesting history. It was known in the literature twice before it was given in English in 2001. It first appeared 1873 in a book by Thomae about among other things theta functions. Then it appeared in an Italian paper 1923 by Pia Nalli. The interesting thing is that both Thomae and Pia Nalli used notations for theta functions which are clearly different from the modern notation. The Thomae notation is reminiscent of Riemann theta functions and the Pia Nalli notation was probably influenced by nineteenth-century Italian books about elliptic functions. It is also rumoured that Andrews and Askey have known this formula for many years.

THEOREM 3.3. *The q-analogue of* (80) *is* [41, p. 183 (168a)], [6, p. 1326], [32, p. 338]

$$
\Gamma_q(x)\Gamma_q(1-x) = \frac{i\, q^{1/8}(1-q)(\langle 1;q\rangle_\infty)^3}{q^{x/2}\theta_1\left(\frac{-ix}{2}\log q, q^{1/2}\right)}, \tag{81}
$$

where the first Jacobi theta function is given by

$$
\theta_1(z,q) = 2\sum_{n=0}^{\infty} (-1)^n \mathrm{QE}\left(\left(n+\frac{1}{2}\right)^2\right) \sin(2n+1)z. \tag{82}
$$

This function has period 2π *and quasiperiod* $\dfrac{-i}{2}\log q$.

For the proof of (78) we need two further lemmata.

LEMMA 3.4. *Another q-analogue of the Legendre duplication formula written in q-shifted factorial form is*

$$\frac{\langle N;q\rangle_N}{\langle 1-\widetilde{N;q}\rangle_{N-1}\left\langle\frac{1}{2};q^2\right\rangle_N}=\mathrm{QE}\left(\binom{N}{2}\right). \tag{83}$$

LEMMA 3.5.

$$\frac{\theta_1(-iN\log q,q^{1/2})}{\theta_1\left(\frac{-iN}{2}\log q,q^{1/2}\right)}=(-1)^N q^{-(N/2)}. \tag{84}$$

PROOF. Use the quasiperiodicity of θ_1. □

PROOF. If we use the same method of proof as in [35, p. 43], but use (81) instead of (80), we arrive at

$$ {}_2\phi_1(-2N,b;1-2N-b|q,-q^{1-b})=\frac{\langle 1-\widetilde{N}-b,\widetilde{1};q\rangle_\infty}{\langle 1\widetilde{-N},1\widetilde{-b};q\rangle_\infty}\times$$

$$\frac{\langle 1-N-b,N;q\rangle_\infty}{\langle 1-2N-b,2N;q\rangle_\infty}\mathrm{QE}\left(\frac{N}{2}\right)\frac{\theta_1(-iN\log q,q^{1/2})}{\theta_1\left(\frac{-iN}{2}\log q,q^{1/2}\right)}=$$

$$\frac{\langle 1-\widetilde{N}-b,N;q\rangle_N}{\langle 1\widetilde{-N},1-2N-b;q\rangle_N}\mathrm{QE}\left(\frac{N}{2}\right)\frac{\theta_1(-iN\log q,q^{1/2})}{\theta_1\left(\frac{-iN}{2}\log q,q^{1/2}\right)}= \tag{85}$$

$$\frac{\langle\widetilde{b},N;q\rangle_N}{\langle 1\widetilde{-N},b+N;q\rangle_N}\mathrm{QE}\left(N^2+\frac{N}{2}\right)(-1)^N\frac{\theta_1(-iN\log q,q^{1/2})}{\theta_1\left(\frac{-iN}{2}\log q,q^{1/2}\right)}.$$

On the other hand, by (5), (83) and (84) we have

$$\frac{\langle N;q\rangle_N}{\langle \widetilde{1-N};q\rangle_N}\mathrm{QE}\left(N^2+\frac{N}{2}\right)(-1)^N\frac{\theta_1(-iN\log q,q^{1/2})}{\theta_1\left(\frac{-iN}{2}\log q,q^{1/2}\right)}=\left\langle\frac{1}{2};q^2\right\rangle_N. \tag{86}$$

The last two formulas together prove (78). □

4. Multiple extensions of Gauss' formula.

In this section we will look at several multiple q-formulas, so we start with some notation.

Definition 8. The notation $\sum_{\vec{m}}$ denotes a multiple summation with the indices $m_1,\dots,m_n$ running over all non-negative integer values. In this connection we put $|m|\equiv\sum_{j=1}^{n} m_j$.

If $\vec{m}$ and $\vec{k}$ are two arbitrary vectors with n elements, their q-binomial coefficient is defined as

$$\binom{\vec{m}}{\vec{k}}_q\equiv\prod_{j=1}^{n}\binom{m_j}{k_j}_q. \tag{87}$$

If $\{x_j\}_{j=1}^n$ and $\{y_j\}_{j=1}^n$ are two arbitrary sequences of complex numbers, then their scalar product is defined by

$$\vec{x}\vec{y}\equiv\sum_{j=1}^{n}x_jy_j. \tag{88}$$

In the same way we define the following vector versions of powers, etc.

$$\vec{x}^{\vec{\alpha}}\equiv\prod_{j=1}^{n}x_j^{\alpha_j}, \tag{89}$$

$$\delta_{\vec{m},\vec{k}}\equiv\begin{cases}1, & \text{if } \vec{m}=\vec{k};\\ 0, & \text{otherwise.}\end{cases} \tag{90}$$

(91) $$q^{\binom{\vec{k}}{2}} \equiv \prod_{j=1}^{n} q^{\binom{k_j}{2}},$$

(92) $$(\vec{u};q)_{\vec{s}} \equiv \prod_{j=1}^{n} (u_j;q)_{s_j}; \quad (-1)^{\vec{k}} \equiv (-1)^{|k|}.$$

The following theorem, for $n=1$ forms the basis of q-analysis.

According to Jensen [26, p. 30], Ward [44, p. 255] and Kupershmidt [29, p. 244], this theorem was obtained by Euler. It was also obtained by Gauss and published posthumously in 1876 [19]. It is proved by induction [12].

THEOREM 4.1.

(93) $$\sum_{\vec{m}=\vec{0}}^{\vec{k}} (-1)^{\vec{m}} \binom{\vec{k}}{\vec{m}}_q q^{\binom{\vec{m}}{2}} \vec{u}^{\vec{m}} = (\vec{u};q)_{\vec{k}}$$

COROLLARY 4.2.

(94) $$\sum_{\vec{m}=\vec{0}}^{\vec{k}} (-1)^{\vec{m}} \binom{\vec{k}}{\vec{m}}_q q^{\binom{\vec{m}}{2}} = \delta_{\vec{k},\vec{0}}$$

Our aim is now to find some further multidimensional variations of (94) and a couple of related formulas. For the proof we need some other formulas, starting with

THEOREM 4.3. *The Jackson q-derivative expressed as q-difference of a q-shifted factorial.*

(95) $$D^n_{q,q^\alpha} \langle \gamma+\alpha;q\rangle_k = (-1)^n \{k-n+1\}_{n,q} \langle \gamma+\alpha+n;q\rangle_{k-n} q^{\binom{n}{2}+n\gamma}.$$

We will only need one q-Lauricella function, which is defined by

DEFINITION 9.

(96) $$\Phi_D^{(n)}(a,b_1,\ldots,b_n;c|q;x_1,\ldots,x_n) = \\ = \sum_{\vec{m}} \frac{\langle a;q\rangle_{m_1+\ldots+m_n} \langle b_1;q\rangle_{m_1} \ldots \langle b_n;q\rangle_{m_n} \prod_{j=1}^{n} x_j^{m_j}}{\langle c;q\rangle_{m_1+\ldots+m_n} \prod_{j=1}^{n} \langle 1;q\rangle_{m_j}}.$$

The following lemma is a q-analogue of Chaundy [10, p. 164].

LEMMA 4.4.

$$\sum_{r=0}^{R}\sum_{s=0}^{S}\frac{(-1)^{r+s}\langle 1;q\rangle_R\langle 1;q\rangle_S}{\langle 1;q\rangle_r\langle 1;q\rangle_{R-r}\langle 1;q\rangle_s\langle 1;q\rangle_{S-s}}\times$$

$$\frac{\langle c-1;q\rangle_{r+s}\langle c;q\rangle_{2r+2s}}{\langle c-1;q\rangle_{2r+2s}\langle c;q\rangle_{R+S+r+s}}\mathrm{QE}\left(\binom{r}{2}+\binom{s}{2}+Sr\right)= \tag{97}$$

$$\begin{cases}1, & \textit{if } R=S=0;\\ 0, & \textit{otherwise.}\end{cases}$$

PROOF.

$$LHS=\sum_{r=0}^{R}\sum_{s=0}^{S}\frac{\langle -R;q\rangle_r\langle -S;q\rangle_s}{\langle 1;q\rangle_r\langle 1;q\rangle_s\left\langle\frac{c-1}{2},\frac{c}{2};q^2\right\rangle_{r+s}}\times$$

$$\frac{\langle c-1;q\rangle_{r+s}\left\langle\frac{c+1}{2},\frac{c}{2};q^2\right\rangle_{r+s}}{\langle c;q\rangle_{R+S}\langle c+R+S;q\rangle_{r+s}}\mathrm{QE}(Rr+S(r+s))$$

$$=\frac{1}{\langle c;q\rangle_{R+S}}\sum_{k=0}^{R+S}\frac{\langle c-1;q\rangle_k\left\langle\frac{c+1}{2},\frac{c}{2};q^2\right\rangle_k}{\langle c+R+S;q\rangle_k\left\langle\frac{c-1}{2},\frac{c}{2};q^2\right\rangle_k}\sum_{r+s=k}\times$$

$$\frac{\langle -R;q\rangle_r\langle -S;q\rangle_s}{\langle 1;q\rangle_r\langle 1;q\rangle_s}\mathrm{QE}(Rr+Sk)$$

(98)

$$=\frac{1}{\langle c;q\rangle_{R+S}}\sum_{k=0}^{R+S}\frac{\langle c-1;q\rangle_k\left\langle\frac{c+1}{2},\frac{c}{2};q^2\right\rangle_k}{\langle c+R+S;q\rangle_k\left\langle\frac{c-1}{2},\frac{c}{2};q^2\right\rangle_k}\sum_{r=0}^{k}\times$$

$$\frac{\langle -R,-k;q\rangle_r\langle -S;q\rangle_k}{\langle 1,1+S-k;q\rangle_r\langle 1;q\rangle_k}\mathrm{QE}(Rr+Sk+r(1+S))$$

$$=\frac{1}{\langle c;q\rangle_{R+S}}\sum_{k=0}^{R+S}\frac{\langle c-1;q\rangle_k\left\langle\frac{c+1}{2},\frac{c}{2};q^2\right\rangle_k}{\langle c+R+S;q\rangle_k\left\langle\frac{c-1}{2},\frac{c}{2};q^2\right\rangle_k}\times$$

$$\frac{\langle -S;q\rangle_k}{\langle 1;q\rangle_k}\,{}_2\phi_1(-k,-R;1+S-k|q,q^{1+S+R})=$$

$$=\frac{1}{\langle c;q\rangle_{R+S}}\sum_{k=0}^{R+S}\frac{\langle c-1;q\rangle_k\left\langle\frac{c+1}{2},\frac{c}{2};q^2\right\rangle_k}{\langle c+R+S;q\rangle_k\left\langle\frac{c-1}{2},\frac{c}{2};q^2\right\rangle_k}\times$$

$$\frac{\langle -S;q\rangle_k}{\langle 1;q\rangle_k}\Gamma_q\begin{bmatrix}1+S-k,1+S+R\\1+S,1+S-k+R\end{bmatrix}=$$

$$=\frac{1}{\langle c;q\rangle_{R+S}}\sum_{k=0}^{R+S}\frac{\langle c-1;q\rangle_k\left\langle\frac{c+1}{2},\frac{c}{2};q^2\right\rangle_k}{\langle c+R+S;q\rangle_k\left\langle\frac{c-1}{2},\frac{c}{2};q^2\right\rangle_k}\times$$

(98)

$$\frac{\langle -S;q\rangle_k}{\langle 1;q\rangle_k}\frac{\langle 1+S;q\rangle_{-k}}{\langle 1+R+S;q\rangle_{-k}}=$$

$$=\frac{1}{\langle c;q\rangle_{R+S}}\sum_{k=0}^{R+S}\frac{\left\langle c-1,-S-R,\frac{c+1}{2},\widetilde{\frac{c+1}{2}};q\right\rangle_k q^{k(S+R)}}{\left\langle c+R+S,1,\frac{c-1}{2},\widetilde{\frac{c-1}{2}};q\right\rangle_k}=$$

$$=\frac{1}{\langle c;q\rangle_{R+S}}\,{}_4\phi_3\left(c-1,-R-S,\frac{c+1}{2},\widetilde{\frac{c+1}{2}};c+R+S,\frac{c-1}{2},\widetilde{\frac{c-1}{2}}|q,q^{S+R}\right)=$$

$$=\frac{1}{\langle c;q\rangle_{R+S}}\Gamma_q\begin{bmatrix}R+S,c+R+S,\frac{c+1}{2},\frac{c-1}{2}\\\frac{c+1}{2}+R+S,\frac{c-1}{2}+R+S,0,c\end{bmatrix}.$$

Result 0, except for $R+S=0$. Then the ${}_3\phi_2$ equals 1 and we get 1. □

Corollary 4.5.

$$\text{(99)}\qquad\sum_{m=0}^{s}\sum_{n=0}^{t}(-1)^{m+n}\binom{s}{m}_q\binom{t}{n}_q q^{\binom{m}{2}+\binom{n}{2}+mt}=\begin{cases}1, & \textit{if } s=t=0;\\0, & \textit{otherwise.}\end{cases}$$

The following result from [13] leads to the same formula.

$$\begin{aligned}&\Phi_1(a;b,b';c|q;q^{c-a-b},q^{c-a-b-b'})=\\&\Phi_1(a;b,b';c|q;q^{c-a-b-b'},q^{c-a-b'})=\Gamma_q\begin{bmatrix}c,c-a-b-b'\\c-a,c-b-b'\end{bmatrix},\end{aligned}\tag{100}$$

where $|q^{c-a-b-b'}|<1$. Put $b=-s,b'=-t$ to obtain

$$\begin{aligned}&\sum_{m=0}^{s}\sum_{n=0}^{t}(-1)^{m+n}\binom{s}{m}_q\binom{t}{n}_q\langle a;q\rangle_{m+n}\langle c+m+n;q\rangle_{s+t-m-n}\\&\mathrm{QE}\left(\binom{m}{2}+\binom{n}{2}+m(c-a)+n(c-a+s)\right)=\langle c-a;q\rangle_{s+t}.\end{aligned}\tag{101}$$

Now multiply with $q^{a(s+t)}$ and apply D^{s+t}_{q,q^a} to both sides to obtain

$$\begin{aligned}&\sum_{m=0}^{s}\sum_{n=0}^{t}\binom{s}{m}_q\binom{t}{n}_q\langle c+m+n;q\rangle_{s+t-m-n}\\&\mathrm{QE}\left(\binom{m+n}{2}\binom{m}{2}+\binom{n}{2}+c(m+n)+ns\right)=1\end{aligned}\tag{102}$$

Finally, apply D^{s+t}_{q,q^c} to both sides to obtain

$$\begin{aligned}&\sum_{m=0}^{s}\sum_{n=0}^{t}(-1)^{m+n}\binom{s}{m}_q\binom{t}{n}_q\mathrm{QE}\left(\binom{m+n}{2}+\binom{m}{2}+\binom{n}{2}+ns\right)\\&\mathrm{QE}\left(\binom{s+t-m-n}{2}+(s+t-m-n)(m+n)\right)=0\end{aligned}\tag{103}$$

The following important formula is a q-analogue of [34] and [37].

THEOREM 4.6. [14]. *If* $\{C_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$ *are bounded sequences of complex numbers then*

$$\begin{aligned}&\sum_{\vec{m}}\frac{C_{m_1+\ldots+m_n}\prod_{j=1}^{n}x^{m_j}\langle\alpha_j;q\rangle_{m_j}}{\prod_{j=1}^{n}\langle 1;q\rangle_{m_j}}\mathrm{QE}\left(-\sum_{k=1}^{n}m_k\sum_{l=2}^{k}\alpha_l\right)=\\&\sum_{N=0}^{\infty}\frac{C_Nx^N\left\langle\sum_{k=1}^{n}\alpha_k;q\right\rangle_N}{\langle 1;q\rangle_N}\mathrm{QE}\left(-N\sum_{l=2}^{n}\alpha_l\right).\end{aligned}\tag{104}$$

COROLLARY 4.7.

$$\Phi_D^n(b,-k_1,\ldots,-k_n;c|q;q^{1-k_2-\ldots-k_n},q^{1-k_3-\ldots-k_n},\ldots,q)= \frac{\langle c-b;q\rangle_k}{\langle c;q\rangle_k}q^{bk},\quad \sum_{i=1}^n k_i=k. \tag{105}$$

PROOF. Put

$$x=\mathrm{QE}(1-k_2-\ldots-k_n),\ C_N=\frac{\langle b;q\rangle_N}{\langle c;q\rangle_N},\ \alpha_l=-k_l \tag{106}$$

in (104). □

COROLLARY 4.8.

$$\sum_{\vec{m}=\vec{0}}^{\vec{k}}\binom{\vec{k}}{\vec{m}}_q(-1)^{\vec{m}}q^{\binom{\vec{m}}{2}}\mathrm{QE}\left(-k\vec{m}+\sum_{j=1}^n m_j(1-\sum_{l=j+1}^n k_l)\right)=\delta_{\vec{k},\vec{0}} \tag{107}$$

PROOF. Put $c=b$ in (105). □

COROLLARY 4.9. *A q-analogue of* [30, p. 150].

$$\Phi_D^n(\beta,\alpha_1,\ldots,\alpha_n;\gamma|q;q^{\gamma-\alpha-\beta+\alpha_2+\ldots+\alpha_n},q^{\gamma-\alpha-\beta+\alpha_3+\ldots+\alpha_n},\ldots,q^{\gamma-\alpha-\beta})= \Gamma_q\begin{bmatrix}\gamma,\gamma-\alpha-\beta\\ \gamma-\alpha,\gamma-\beta\end{bmatrix},\ \mathrm{Re}\,(\gamma-\alpha-\beta)>0,\quad \sum_{i=1}^n\alpha_i=\alpha. \tag{108}$$

PROOF. Put

$$x=\mathrm{QE}(\gamma-\alpha-\beta+\alpha_2+\ldots+\alpha_n),\ C_N=\frac{\langle\beta;q\rangle_N}{\langle\gamma;q\rangle_N}. \tag{109}$$

in (104). □

COROLLARY 4.10. *A generalization of* (99) *and* (103).

$$\sum_{\vec{m}=\vec{0}}^{\vec{k}}\binom{\vec{k}}{\vec{m}}_q(-1)^{\vec{m}}q^{\binom{\vec{m}}{2}}\mathrm{QE}\left(-k\vec{m}+\sum_{j=1}^n m_j(\sum_{l=1}^j k_l)\right)=\delta_{\vec{k},\vec{0}} \tag{110}$$

PROOF. Put $\alpha_l=-k_l,\ \gamma=\beta$ in (108). □

The formulas (107) and (110) are the 2 multiple extensions of Gauss' formula.

5. Conclusion.

We have shown that hypergeometric formulas involving balanced generalized Γ functions can have q-analogues involving the corresponding generalized Γ_q function or the equivalent quotient of infinite q-shifted factorials. Limits when some parameters $\rightarrow \pm\infty$ have been established. These limits have the same form as in the case $q = 1$.

The importance of theta functions for Γ_q functions has been shown. In the last chapter we found again that Γ_q functions, q-difference operators and q-shifted factorials fit nicely in the context of multiple q-functions. Further results in the same spirit will follow.

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