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The p-adic Local Monodromy Theorem for Fake Annuli.

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ABSTRACT - We establish a generalization of the p-adic local monodromy theorem (of André, Mebkhout, and the author) in which differential equations on rigid analytic annuli are replaced by differential equations on so-called "fake annuli". The latter correspond loosely to completions of a Laurent polynomial ring with respect to a monomial valuation. The result represents a step towards a higher-dimensional version of the p-adic local monodromy theorem (the "problem of semistable reduction"); it can also be viewed as a novel presentation of the original p-adic local monodromy theorem.

1. Introduction.

This paper proves a generalization of the p-adic local monodromy theorem of André [1], Mebkhout [21], and the present author [11]. That theorem, originally conjectured by Crew [4] as an analogue in rigid (p-adic) cohomology of Grothendieck's local monodromy theorem in étale (ℓ-adic) cohomology, asserts the quasi-unipotence of differential modules with Frobenius structure on certain one-dimensional rigid analytic annuli.

The p-adic local monodromy theorem has far-reaching consequences in the theory of rigid cohomology, particularly for curves [4]. However, although one can extend it to a relative form [15, Theorem 5.1.3] to obtain some higher-dimensional results, for some applications one needs a version of the monodromy theorem which is truly higher-dimensional.

This theorem takes an initial step towards producing such a higher-dimensional monodromy theorem, by proving a generalization in which the role of the annulus is replaced by a somewhat mysterious space, called a *fake annulus*, described by certain rings of multivariate power series. When there is only one variable, the space is a true annulus, so the result

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truly generalizes the original monodromy theorem. Indeed, this paper has the side effect of giving an exposition of the original theorem, albeit one somewhat encumbered by extra notation needed for the fake case.

In the remainder of this introduction, we explain further the context in which the p-adic local monodromy theorem arises, introduce and justify the fake analogue, and outline the structure of the paper.

1.1 – *Monodromy of p-adic differential equations.*

Let K be a field of characteristic zero complete with respect to a non-archimedean absolute value, whose residue field k has characteristic $p > 0$. Suppose we are given a rigid analytic annulus over K and a differential equation on the annulus, i.e., a module equipped with an integrable connection. We now wish to define the "monodromy around the puncture" of this connection, despite not having recourse to the analytic continuation we would use in the analogous classical setting. In particular, we would like to construct a representation of an appropriate étale fundamental group, whose triviality or unipotence amounts to the existence of a full set of horizontal sections or log-sections; the latter relates closely to the existence of an extension or logarithmic extension of the connection across the puncture (e.g., [16, Theorem 6.4.5]).

We can define a monodromy representation associated to a connection if we can find enough horizontal sections on some suitable covering space. In particular, we are mainly interested in connections which become unipotent, i.e., can be filtered by submodules whose successive quotients are trivial for the connection, on some cover of the annulus which is "finite étale near the boundary" (in a sense that can be made precise). One can then construct a monodromy representation, using the Galois action on horizontal sections, giving an equivalence of categories between such *quasi-unipotent* modules with connection and a certain representation category. (Beware that if the field k is not algebraically closed, these representations are only semilinear over the relevant field, namely the maximal unramified extension of K. See [13, Theorem 4.45] for a precise statement.)

In order for such an equivalence to be useful, we need to be able to establish conditions under which a module with connection is forced to be quasi-unipotent. As suggested in the introduction to [16], a natural, geometrically meaningful candidate restriction (analogous to the existence of a variation of Hodge structure for a complex analytic connection) is the existence of a *Frobenius structure* on the connection. For K discretely valued, the fact that connections with a Frobenius structure are quasi-unipotent is the content of the p-adic local monodromy theorem (pLMT) of André [1], Mebkhout [21], and this author [11].

Note that in this paper, we will not go all the way to the construction of monodromy representations. These appear directly in André's proof of the pLMT (at least for k algebraically closed); for a direct construction (of Fontaine type) assuming the pLMT, see [13]. See Remark 6.2.7 for more details.

1.2 – *Fake annuli.*

The *semistable reduction problem* (or *global quasi-unipotence problem*) for overconvergent F-isocrystals, as formulated by Shiho [23, Conjecture 3.1.8] and reformulated in [16, Conjecture 7.1.2], is essentially to give a higher dimensional version of the pLMT. From the point of view of [16], this can be interpreted as proving a uniform version of the pLMT across all divisorial valuations on the function field of the original variety. This interpretation immediately suggests that one needs to exploit the quasi-compactness of the Riemann-Zariski space associated to the function field of an irreducible variety; this observation is developed in more detail in [17].

The upshot is that one must prove the pLMT uniformly for the divisorial valuations in a neighborhood (in Riemann-Zariski space) of an arbitrary valuation, not just a divisorial one. Ideally, one could proceed by first verifying whether the pLMT itself makes sense and continues to hold true when one passes from a divisorial valuation to a more general one. This entails replacing the annuli in the pLMT with some sort of "fake annuli" which cannot be described as rigid analytic spaces in the usual sense. Nonetheless, one can still sensibly define rings of analytic functions in a neighborhood of an irrational point, and thus set up a ring-theoretic framework in which an analogue of the p-adic local monodromy theorem can be formulated. (This allows us to get away with the linguistic swindle of speaking meaningfully about "p-adic differential equations on fake annuli" without giving the noun phrase "fake annulus" an independent meaning!)

Indeed, this framework fits naturally into the context of the slope filtration theorem of [11]. That theorem, which gives a structural decomposition of a semilinear endomorphism on a finite free module over the Robba ring (of germs of rigid analytic functions on an open annulus with outer radius 1), does not make any essential use of the fact that the

Robba ring is described in terms of power series. Indeed, as presented in [14], the theorem applies directly to our fake annuli; thus to prove the analogue of the p-adic monodromy theorem, one need only analogize Tsuzuki's unit-root monodromy theorem from [24]. With a bit of effort, this can indeed be done, thus illustrating some of the power of the Frobenius-based approach to the monodromy theorem. Note that our definition of fake annuli will actually include true rigid analytic annuli, so the monodromy theorem given here will strictly generalize the p-adic local monodromy theorem.

Unfortunately, it is not so clear how to prove a form of the pLMT for arbitrary valuations; in this paper, we restrict to a somewhat simpler class. These are the monomial valuations, which correspond to monomial orderings in a polynomial ring. For instance, these include valuations on $k(x, y)$ in which the valuations of x and y are linearly independent over the rational numbers. There are many valuations that do not take this form, namely the *infinitely singular* valuations; the semistable reduction problem for these must be treated in a more roundabout fashion, which we will not discuss further here.

1.3 – *Structure of the paper.*

We conclude this introduction with a summary of the various sections of the paper.

In Section 2, we define the rings corresponding to fake annuli, and verify that they fit into the formalism within which slope filtrations are constructed in [14].

In Section 3, we define F-modules, ∇-modules, and (F, ∇)-modules on fake annuli, and verify that the category of (F, ∇)-modules is invariant of the choice of a Frobenius lift (Proposition 3.4.7).

In Section 4, we give a fake annulus generalization of Tsuzuki's theorem on unit-root (F, ∇)-modules (Theorem 4.5.2).

In Section 5, we invoke the technology of slope filtrations from [11] (via [14]), and apply it to deduce from Theorem 4.5.2 a form of the p-adic local monodromy theorem for (F, ∇)-modules on fake annuli (Theorem 5.2.4).

In Section 6, we deduce some consequences of the p-adic local monodromy theorem. Namely, we calculate some extension groups in the category of (F, ∇)-modules, establish a local duality theorem, and generalize some results from [5] and [12] on the full faithfulness of overconvergent-to-convergent restriction.

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2. Fake annuli.

In this section, we describe ring-theoretically the fake annuli to which we will be generalizing the p-adic local monodromy theorem, deferring to [14] for most of the heavy lifting.

First, we put in some notational conventions that will hold in force throughout the paper; for the most part, these hew to the notational régime of [14] (which in turn mostly follows [11]), with a few modifications made for greater consistency with [16].

CONVENTION 2.0.1. Throughout this paper, let K be a complete *discretely valued* field of characteristic 0, whose residue field k has characteristic $p > 0$. Let $\mathfrak{o} = \mathfrak{o}_K$ be the ring of integers of K, and let π denote a uniformizer of K. Let w be the valuation on $\mathfrak{o}$ normalized so that $w(\pi) = 1$. Let q be a power of p, and assume extant and fixed a ring endomorphism $\sigma_K : K \to K$, continuous with respect to the π-adic valuation, and lifting the q-power endomorphism on k. Let K_q be the fixed field of K under σ_K, and let $\mathfrak{o}_q$ be the fixed ring of $\mathfrak{o}$ under σ_K. Finally, let I_n denote the $n \times n$ identity matrix over any ring.

REMARK 2.0.2. As noted in the introduction, the restriction to K discretely valued is endemic to the methods of this paper; see Remark 2.4.4 for further discussion.

2.1 – *Monomial fields*

DEFINITION 2.1.1. Let k be a field. A *nearly monomial field (of height* 1*) over* k is a field E equipped with a valuation $v : E^* \to \mathbb{R}$ (also written $v : E \to \mathbb{R} \cup \{+\infty\}$) satisfying the following restrictions.

(a) The field E is a separable extension of k, i.e., $k \subseteq E$ and $k \cap E^p = k^p$.

(b) The image $v(E^*)$ of v is a finitely generated $\mathbb{Z}$-submodule of $\mathbb{R}$, and $v(k^*) = \{0\}$.

(c) The field E is complete with respect to v.

(d) With the notations

$$\mathfrak{o}_E = \{x \in E : v(x) \geq 0\}$$
$$\mathfrak{m}_E = \{x \in E : v(x) > 0\}$$
$$\kappa_E = \mathfrak{o}_E/\mathfrak{m}_E,$$

the natural map $k \to \kappa_E$ is finite.

If $\kappa_E = k$, we say E is a *monomial field* (or *fake power series field*) over k; in that case, k is integrally closed in E. We define the *rational rank* of E to be the rank of $v(E^*)$ as a $\mathbb{Z}$-module.

REMARK 2.1.2. One can also speak of monomial fields of height greater than 1, by allowing the valuation v to take values in a more general totally ordered abelian group. The techniques used in this paper do not apply to that case, so we will ignore it; in the semistable reduction context, one can eliminate the case of height greater than 1 by an inductive argument [17, Proposition 4.2.4].

EXAMPLE 2.1.3. A monomial field of rational rank 1 is just a power series field, by the Cohen structure theorem. This characterization generalizes to arbitrary monomial fields; see Definition 2.1.9. Note also that nearly monomial fields are examples of *Abhyankar valuations*, i.e., valuations in which equality holds in Abhyankar's inequality [26, Théorème 9.2].

REMARK 2.1.4. If E is a nearly monomial field and E'/E is a finite separable extension, then E' is also nearly monomial: E' is separable over k, the valuation v extends uniquely to a valuation v' on E', E' is complete with respect to v', and the index $[v'((E')^*) : v(E^*)]$ and degree $[\kappa_{E'} : \kappa_E]$ are both finite since their product is at most $\deg(E'/E)$ [26, Proposition 5.1]. Conversely, every nearly monomial field can be written as a finite separable extension of a monomial field; see Definition 2.1.9.

REMARK 2.1.5. If E is a nearly monomial field over k and κ_E/k is separable, then by Hensel's lemma, the integral closure k' of k in E is isomorphic to κ_E; in other words, E is a monomial field over k'. In particular, if k is perfect, then any finite extension of a monomial field over k is a monomial field over some finite separable extension of k. This fails if k is not perfect, even for finite separable extensions of the monomial field: the field k is integrally closed in

$$k((t))[z]/(z^p - z - ct^{-p}) \qquad (c \in k \setminus k^p),$$

but the latter has residue field $k(c^{1/p}) \neq k$.

It will frequently be convenient to work with monomial fields in terms of coordinate systems.

DEFINITION 2.1.6. Let m be a nonnegative integer. Let L be a lattice in $\mathbb{R}^m$, i.e., a $\mathbb{Z}$-submodule of $\mathbb{R}^m$ which is free of rank m, and which spans $\mathbb{R}^m$ over $\mathbb{R}$. Let $L^\vee \subseteq (\mathbb{R}^m)^\vee$ denote the lattice dual to L:

$$L^\vee = \{\mu \in (\mathbb{R}^m)^\vee : \mu(z) \in \mathbb{Z} \quad \forall z \in L\}.$$

Given a formal sum $\sum\limits_{z \in L} c_z\{z\}$, with the c_z in some ring, define the *support* of the sum to be the set of $z \in L$ such that $c_z \neq 0$; define the support of a matrix of formal sums to be the union of the supports of the entries. If $S \subseteq L$ and a formal sum or matrix has support contained in S, we also say that the element or matrix is "supported on S". For R a ring, let $R[L]$ denote the group algebra of L over R, i.e., the set of formal sums $\sum\limits_{z \in L} c_z\{z\}$ with coefficients in R and finite support.

REMARK 2.1.7. It is more typical to denote the class in $R[L]$ of a lattice element $z \in L$ by $[z]$, rather than $\{z\}$. However, we need to use brackets to denote Teichmüller lifts, so we will stick to braces for internal consistency.

DEFINITION 2.1.8. For any ring R and any $\lambda \in (\mathbb{R}^n)^\vee$, let v_λ denote the valuation on $R[L]$ given by

$$v_\lambda\left(\sum_{z \in L} c_z\{z\}\right) = \min\{\lambda(z) : z \in L,\ c_z \neq 0\}.$$

Let $P_\lambda \subseteq L$ denote the submonoid of $z \in L$ for which $\lambda(z) \geq 0$, and let $R[L]_\lambda$ denote the monoid algebra $R[P_\lambda]$. Let $R[\![L]\!]_\lambda$ and $R((L))_\lambda$ denote the v_λ-adic completions of $R[L]_\lambda$ and $R[L]$, respectively.

DEFINITION 2.1.9. Given a lattice L and some $\lambda \in (\mathbb{R}^m)^\vee$, we say λ is *irrational* if $L \cap \ker(\lambda) = \{0\}$. In this case, $k((L))_\lambda$ is a monomial field over

k. Conversely, given a nearly monomial field E over a field k, with valuation v, a *coordinate system* for E is a sequence $x_1, \dots, x_m$ of elements of E such that $v(x_1), \dots, v(x_m)$ freely generate $v(E^*)$ as a $\mathbb{Z}$-module. Given a coordinate system, put $L = \mathbb{Z}^m$ with generators $z_1, \dots, z_m$, and define $\lambda \in (\mathbb{R}^m)^\vee$ by $\lambda(z_i) = v(x_i)$; then λ is irrational, and the continuous map $k((L))_\lambda \to E$ given by $\{z_i\} \mapsto x_i$ is injective. If we identify $k((L))_\lambda$ with its image in E, then E is finite separable over $k((L))_\lambda$; if E is monomial over k, then in fact $E = k((L))_\lambda$ by Proposition 2.1.10 below. This fact may be viewed as a monomial version of the Cohen structure theorem in equal characteristics.

Proposition 2.1.10. *Let L be a lattice in $\mathbb{R}^m$, choose $\lambda \in (\mathbb{R}^m)^\vee$ irrational, and let E be a finite separable extension of $k((L))_\lambda$ with value group $\lambda(L)$ and residue field k. Then $E = k((L))_\lambda$.*

Proof. The claim is equivalent to showing that $k((L))_\lambda$ is separably defectless in the sense of Ostrowski's lemma [22, Théorème 2, p. 236]. In particular, since a tamely ramified extension of E is necessarily without defect, it suffices to check that there is no Artin-Schreier defect extension. If $E = k((L))_\lambda[z]/(z^p - z - x)$ were such an extension, we could rewrite it as $k((L))_\lambda[z]/(z^p - z - y)$ with the leading term of y being either an element of pL times a non-p-th power c in k, or an element of $L \setminus pL$ times a nonzero element of k. But in the first case the residue field of E would be $k(c^{1/p})$, and in the second case we would have $[v(E) : \lambda(L)] = p$; in either case, we would contradict the assumption that E is a defect extension. This contradiction yields the claim. $\square$

Remark 2.1.11. The term "monomial field" is modeled on the use of the term "monomial valuation", e.g., in [6], to refer to a valuation v of the sort considered in Definition 2.1.1. (Such valuations, each of which endows the lattice L with a total ordering, are more common in mathematics than one might initially realize: for example, they are used to define highest weights in the theory of Lie algebras, and they are sometimes used to construct term orders in the theory of Gröbner bases.) In a previous version of this paper, the term "fake power series field" was used instead; we have decided that it would be better to save this term for describing the completion of a finitely generated field extension of k with respect to *any* valuation of height 1. (See Remark 2.3.7 for some reasons why we are not considering such valuations here.)

2.2 – *Witt rings and Cohen rings*

We now enter the formalism of [14, § 2].

DEFINITION 2.2.1. Let K^{perf} be the completion of the direct limit $K \xrightarrow{\sigma_K} K \xrightarrow{\sigma_K} \cdots$ for the π-adic topology; this is a complete discretely valued field of characteristic 0 with residue field k^{perf}, so it contains $W(k^{\mathrm{perf}})$ by Witt vector functoriality. For E a perfect field of characteristic p containing k, put $\Gamma^E = W(E) \otimes_{W(k^{\mathrm{perf}})} \mathcal{O}_{K^{\mathrm{perf}}}$; note that the valuation w extends naturally to Γ^E.

DEFINITION 2.2.2. For E a perfect field of characteristic p containing k, complete for a valuation v trivial on k, define the *partial valuations* v_n on $\Gamma^E[\pi^{-1}]$ as follows. Given $x \in \Gamma^E[\pi^{-1}]$, write $x = \sum_i [\overline{x_i}]\pi^i$, where each $\overline{x_i} \in E$ and the brackets denote Teichmüller lifts. Set

$$v_n(x) = \min_{i \leq n}\{v(\overline{x_i})\}.$$

As in [14, Definition 2.1.5], the partial valuations satisfy some useful identities (here using σ to denote the q-power Frobenius):

$$\begin{aligned}
v_n(x+y) &\geq \min\{v_n(x), v_n(y)\} && (x, y \in \Gamma^E[\pi^{-1}],\ n \in \mathbb{Z}) \\
v_n(xy) &\geq \min_{m \in \mathbb{Z}}\{v_m(x) + v_{n-m}(y)\} && (x, y \in \Gamma^E[\pi^{-1}],\ n \in \mathbb{Z}) \\
v_n(x^\sigma) &= q v_n(x) && (x \in \Gamma^E[\pi^{-1}],\ n \in \mathbb{Z}) \\
v_n([\overline{x}]) &= v(\overline{x}) && (\overline{x} \in E,\ n \geq 0).
\end{aligned}$$

In the first two cases, equality holds whenever the minimum is achieved exactly once. Define the *levelwise topology* (or *weak topology*) on Γ^E by declaring that a sequence $\{x_i\}$ converges to zero if and only if for each n, $v_n(x_i) \to \infty$ as $i \to \infty$.

DEFINITION 2.2.3. For $r > 0$, write $v_{n,r}(x) = r v_n(x) + n$; for $r = 0$, write conventionally

$$v_{n,0}(x) = \begin{cases} n & v_n(x) < \infty \\ \infty & v_n(x) = \infty. \end{cases}$$

Let Γ^E_r be the subring of Γ^E for which $v_{n,r}(x) \to \infty$ as $n \to \infty$; then σ sends Γ^E_r to $\Gamma^E_{r/q}$. Define the map w_r on Γ^E_r by

$$w_r(x) = \min_n\{v_{n,r}(x)\};$$

then w_r is a valuation on Γ_r by [14, Lemma 2.1.7], and $w_r(x) = w_{r/q}(x^\sigma)$. Put

$$\Gamma^E_{\mathrm{con}} = \cup_{r>0}\Gamma^E_r;$$

this is a henselian discrete valuation ring with maximal ideal $\pi\Gamma^E_{\mathrm{con}}$ and residue field E (see discussion in [14, Definition 2.2.13]).

Convention 2.2.4. For E a not necessarily perfect field complete for a valuation v trivial on k, we write E^{perf} and E^{alg} for the *completed* (with respect to v) perfect and algebraic closures of E. When E is to be understood, we abbreviate $\Gamma^{E^{\mathrm{perf}}}$ and $\Gamma^{E^{\mathrm{alg}}}$ to Γ^{perf} and Γ^{alg}, respectively. Note that this is consistent with the conventions of [14] but *not* with those of [11], where the use of these superscripts is taken not to imply completion.

Since we are interested in constructing Γ^E for E a monomial field, which is not perfect, we must do a bit more work, as in [14, § 2.3].

Definition 2.2.5. Let E be a nearly monomial field over k with valuation v. Let Γ^E be a complete discrete valuation ring of characteristic 0 containing $\mathfrak{o}$ and having residue field E, such that π generates the maximal ideal of Γ^E. Suppose that Γ^E is equipped with a *Frobenius lift*, i.e., a ring endomorphism σ extending σ_K on $\mathfrak{o}_K$ and lifting the q-power Frobenius map on E. We may then embed Γ^E into Γ^{perf} by mapping Γ^E into the first term of the direct system $\Gamma^E \stackrel{\sigma}{\to} \Gamma^E \stackrel{\sigma}{\to} \cdots$, completing the direct system, and mapping the result into Γ^{perf} via Witt vector functoriality. In particular, we may use this embedding to induce partial valuations and a levelwise topology on Γ^E, taking care to remember that these depend on the choice of σ. If E' is a finite separable extension of E, and we start with a suitable Γ^E equipped with a Frobenius lift σ, we may form the unramified extension of Γ^E with residue field E'; this will be a suitable $\Gamma^{E'}$, and carries a unique Frobenius lift extending σ.

Definition 2.2.6. Let E be a nearly monomial field over k, and fix a pair (Γ^E, σ) as in Definition 2.2.5. Write

$$\Gamma^E_{\mathrm{con}} = \Gamma^E \cap \Gamma^{\mathrm{perf}}_{\mathrm{con}},$$

with the intersection taking place within Γ^{perf}. For $r > 0$, we say that Γ^E *has enough r-units* if $\Gamma^E \cap \Gamma^{\mathrm{perf}}_r$ contains units lifting all nonzero elements of E. We say that Γ^E *has enough units* (or more properly, the pair (Γ^E, σ) has enough units) if Γ^E has enough r-units for some $r > 0$; this implies that Γ^E_{con} is a henselian discrete valuation ring with maximal ideal $\pi\Gamma^E_{\mathrm{con}}$ and residue field E. If Γ^E has enough units, then so does $\Gamma^{E'}$ for any finite separable extension E' of E [14, Lemma 2.2.12].

2.3 – *Toroidal interpretation.*

The condition of having enough units is useful in the theory of slope filtrations, but is not convenient to check in practice. Fortunately, it has a more explicit interpretation in terms of certain "naïve" analogues of the functions v_n and w_r, as in [11, § 2] or [14, § 2.3].

DEFINITION 2.3.1. Let L be a lattice in $\mathbb{R}^m$ and let $\lambda \in (\mathbb{R}^m)^\vee$ be an irrational linear functional. Let Γ^λ denote the π-adic completion of $\mathfrak{o}((L))_\lambda$; its elements may be viewed as formal sums $\sum_{z \in L} c_z\{z\}$ with $w(c_z) \to \infty$ as $\lambda(z) \to -\infty$. Define the *naïve partial valuations* on $\Gamma^\lambda[\pi^{-1}]$ by the formula

$$v_n^{\mathrm{naive}}\left(\sum c_z\{z\}\right) = \min\{\lambda(z) : z \in L,\, w(c_z) \leq n\},$$

where the minimum is infinite if the set of candidate z's is empty. These functions satisfy the identities

$$v_n^{\mathrm{naive}}(x+y) \geq \min\{v_n^{\mathrm{naive}}(x), v_n^{\mathrm{naive}}(y)\} \qquad (x, y \in \Gamma^\lambda[\pi^{-1}])$$

$$v_n^{\mathrm{naive}}(xy) \geq \min_{m \in \mathbb{Z}}\{v_m^{\mathrm{naive}}(x) + v_{m-n}^{\mathrm{naive}}(y)\} \qquad (x, y \in \Gamma^\lambda[\pi^{-1}]),$$

with equality in each case if the minimum is achieved only once. Define the *naïve levelwise topology* (or *naïve weak topology*) on Γ^λ by declaring that a sequence $\{x_i\}$ converges to zero if and only if for each n, $v_n^{\mathrm{naive}}(x_i) \to \infty$ as $i \to \infty$.

DEFINITION 2.3.2. For $r > 0$ and $n \in \mathbb{Z}$, write

$$v_{n,r}^{\mathrm{naive}}(x) = r v_n^{\mathrm{naive}}(x) + n;$$

extend the definition to $r = 0$ by setting

$$v_{n,0}^{\mathrm{naive}}(x) = \begin{cases} n & v_n^{\mathrm{naive}}(x) < \infty \\ \infty & v_n^{\mathrm{naive}}(x) = \infty. \end{cases}$$

Let $\Gamma_r^{\mathrm{naive}}$ be the set of $x \in \Gamma^\lambda$ such that $v_{n,r}^{\mathrm{naive}}(x) \to \infty$ as $n \to \infty$. Define the map w_r^{naive} on $\Gamma_r^{\mathrm{naive}}$ by

$$w_r^{\mathrm{naive}}(x) = \min_{n \in \mathbb{Z}}\{v_{n,r}^{\mathrm{naive}}(x)\};$$

then w_r^{naive} is a valuation on $\Gamma_r^{\mathrm{naive}}[\pi^{-1}]$, as in [14, Lemma 2.1.7]. Put

$$\Gamma_{\mathrm{con}}^{\mathrm{naive}} = \cup_{r>0}\Gamma_r^{\mathrm{naive}}.$$

REMARK 2.3.3. The ring $\Gamma_r^{\mathrm{naive}}$ is a principal ideal domain; this will follow from [14, Proposition 2.6.5] in conjunction with Definition 2.3.6 below.

We may view Γ^λ as an instance of the definition of Γ^E in the case $E = k((L))_\lambda$; this gives sense to the following result.

PROPOSITION 2.3.4. *Let σ be a Frobenius lift on $\Gamma^E = \Gamma^\lambda$ for $E = k((L))_\lambda$. Then for $r > 0$, the following are equivalent.*

(a) *σ is continuous for the naïve levelwise topology (i.e., that topology coincides with the levelwise topology induced by σ), and for each $z \in L$ nonzero, $\{z\}^\sigma/\{z\}^q$ is a unit in $\Gamma_r^{\mathrm{naive}}$.*

(b) *For $s \in (0, qr]$, $n \geq 0$, and $x \in \Gamma^E$,*

$$\min_{j\leq n}\{v_{j,s}(x)\} = \min_{j\leq n}\{v_{j,s}^{\mathrm{naive}}(x)\}. \tag{2.3.4.1}$$

(c) *Γ^E has enough qr-units, and for each $z \in L$ nonzero, $\{z\}$ is a unit in Γ_{qr}^E.*

In particular, in each of these cases, for $s \in (0, qr]$, $\Gamma_s^{\mathrm{naive}} = \Gamma_s^E$ and $w_s(x) = w_s^{\mathrm{naive}}(x)$ for all $x \in \Gamma_s$.

PROOF. Given (a), for $s \in (0, qr]$, we have

$$\min_{j\leq n}\{v_{j,s}^{\mathrm{naive}}(x)\} = \min_{j\leq n}\{v_{j,s/q}^{\mathrm{naive}}(x^\sigma)\} \tag{2.3.4.2}$$

for each $n \geq 0$ and each $x \in \Gamma$, as in the proof of [14, Lemma 2.3.3]. We then obtain (b) as in the proof of [14, Lemma 2.3.5], from which (c) follows immediately.

Given (c), the equation (2.3.4.1) holds for $x = \{z\}$ for any $z \in L$, since the minima both occur for $j = 0$. For $x = \sum c_z\{z\}$ a finite sum, we have

$$\min_{j\leq n}\{v_{j,s}^{\mathrm{naive}}(x)\} = \min_{j\leq n}\{\min_{z\in L}\{v_{j,s}^{\mathrm{naive}}(c_z\{z\})\}\} \tag{2.3.4.3}$$

and so the left side of (2.3.4.1) is greater than or equal to the right side. On the other hand, if j is taken to be the smallest value for which the outer minimum is achieved on the right side of (2.3.4.3), then the inner minimum is achieved by a unique value of z. Thus we actually may deduce equality in

(2.3.4.1), again for $x = \sum c_z\{z\}$ a finite sum. For general x, we may obtain the desired equality by replacing x by a finite sum x' such that $x - x' = y + z$ for some $y \in \Gamma^E$ with $w(y)$ greater than n, and some $z \in \Gamma^E_{qr}$ with $w_{qr}(z)$ greater than either side of (2.3.4.1). Hence (c) implies (b).

Finally, note that (b) implies (a) straightforwardly. $\square$

REMARK 2.3.5. Note that in Proposition 2.3.4, conditions (a) and (c) may be checked for z running over a basis of L. Note also that Proposition 2.3.4 implies that for $E = k((L))_\lambda$, if Γ^E has enough units, then Γ^E is isomorphic to Γ^λ.

DEFINITION 2.3.6. By the *standard extension* of σ_K to Γ^λ, we will mean the Frobenius lift σ defined by

$$\sum_{z \in L} c_z\{z\} \mapsto \sum_{z \in L} c_z^{\sigma_K}\{z\}^q.$$

(We will also refer to such a σ as a *standard Frobenius lift.*) When equipped with a standard Frobenius lift, Γ^λ has enough r-units for every $r > 0$; by Proposition 2.3.4, it follows that $v_n(x) = v_n^{\mathrm{naive}}(x)$ for all $n \in \mathbb{Z}$ and all $x \in \Gamma^\lambda[\pi^{-1}]$. Thus many of the results of [14, § 2], proved in terms of the Frobenius-based valuations, also apply verbatim to the naïve valuations.

REMARK 2.3.7. For applications to semistable reduction, one would also like to consider a similar situation in which the residue field $k((L))_\lambda$ is replaced by the completion of a finitely generated field extension of k with respect to an arbitrary valuation of height (real rank) 1, at least in the case where the transcendence degree over k is equal to 2. This would require a slightly more flexible set of foundations: one must work only with finitely generated k-subalgebras of the complete field, so that one has hope of having enough units. A more serious problem is how to perform Tsuzuki's method (a/k/a Theorem 4.5.2) in this context.

2.4 – *Analytic rings.*

We now introduce "analytic rings", citing into [14] for their structural properties.

DEFINITION 2.4.1. Let E be a nearly monomial field over k, or the completed perfect or algebraic closure thereof. In the first case, suppose that Γ^E has enough r_0-units for some $r_0 > 0$ (otherwise take $r_0 = \infty$). Let I be a subinterval of $[0, r_0)$ bounded away from r_0 (i.e., I is a subinterval of $[0, r]$ for some $r \in (0, r_0)$). Let Γ_I^E denote the Fréchet completion of $\Gamma_{r_0}^E[\pi^{-1}]$ under the valuations w_s for $s \in I$; this ring is an integral domain [14, Lemma 2.4.6]. If I is closed, then Γ_I^E is a principal ideal domain [14, Proposition 2.6.9]. Put

$$\Gamma_{\mathrm{an},r}^E = \Gamma_{(0,r]}^E;$$

this ring is a Bézout ring, i.e., a ring in which every finitely generated ideal is principal [14, Theorem 2.9.6]. Put $\Gamma_{\mathrm{an,con}}^E = \cup_{r>0} \Gamma_{\mathrm{an},r}^E$; then $\Gamma_{\mathrm{an,con}}^E$ is also a Bézout ring. The group of units in $\Gamma_{\mathrm{an,con}}^E$ consists of the nonzero elements of $\Gamma_{\mathrm{con}}[\pi^{-1}]$ [14, Corollary 2.5.12]. For E' finite separable over E, $E' = E^{\mathrm{perf}}$, or $E' = E^{\mathrm{alg}}$, by [14, Proposition 2.4.10] one has

$$\Gamma_{\mathrm{an,con}}^{E'} = \Gamma_{\mathrm{an,con}}^E \otimes_{\Gamma_{\mathrm{con}}^E} \Gamma_{\mathrm{con}}^{E'}.$$

Remark 2.4.2. It is likely that Γ_I^E is a Bézout ring for any I as above. However, this statement is not verified in [14], and we will not need it anyway, so we withhold further comment on it.

Remark 2.4.3. If $E = k((t))$ is a power series field, then the ring Γ_I^E is the ring of rigid analytic functions on the annulus $w(t) \in I$ in the t-plane. Thus our construction of fake annuli includes "true" one-dimensional rigid analytic annuli over K, and most of our results on fake annuli (like the p-adic local monodromy theorem) generalize extant theorems on true annuli. On the other hand, if $E = k((L))_\lambda$ and $\mathrm{rank}(L) > 1$, then the ring Γ_I^E is trying to be the ring of rigid analytic functions on a subspace of the rigid affine plane in the variables $\{z_1\}, \dots, \{z_m\}$ for some basis $z_1, \dots, z_m$ of L, consisting of points for which there exists $r \in I$ with $w(\{z_i\}) = r\lambda(z_i)$ for $i = 1, \dots, m$. If $I = [r, r]$, then this space is an affinoid space in the sense of Berkovich, but otherwise it is not (because one can only cut out an analytic subspace of the form $w(x) = \alpha w(y)$ for α rational). Indeed, as far as we can tell, this space is not a p-adic analytic space in either of the Tate or Berkovich senses, despite the fact that it has a sensible ring of analytic functions; hence the use of the adjective "fake" in the phrase "fake annulus", and the absence of an honest definition of that phrase.

Remark 2.4.4. Since one can sensibly define rigid analytic annuli over arbitrary complete nonarchimedean fields, Remark 2.4.3 suggests the possibility of working with fake annuli over more general complete K. However, the algebraic issues here get more complicated, and we have not straightened them out to our satisfaction. For example, the analogue of the ring $\Gamma_r^{\mathrm{naive}}$ fails to be a principal ideal ring if the valuation on K is not discrete; it probably still has the Bézout property (that finitely generated ideals are principal), but we have not checked this. In any case, the formalism of [14] completely breaks down when K is not discretely valued, so an attempt here to avoid a discreteness hypothesis now would fail to improve upon our ultimate results; we have thus refrained from making such an attempt.

3. Frobenius and connection structures.

We now introduce a notion which should be thought of as a p-adic differential equation with Frobenius structure on a fake annulus. We start with some notational conventions.

CONVENTION 3.0.1. Throughout this section, assume that E is a monomial field and that Γ^E is equipped with a Frobenius lift such that Γ^E has enough r_0-units for some $r_0 > 0$; we view Γ^E as being equipped with a levelwise topology via the choice of a coordinate system. (This choice does not matter, as the topology can be characterized as the coarsest one under which the $v_{n,r}$ are continuous for all $n \in \mathbb{Z}$ and all $r \in (0, r_0)$.) We suppress E from the notation, writing Γ for Γ^E, Γ_{con} for Γ^E_{con}, and so on.

CONVENTION 3.0.2. When a valuation is applied to a matrix, it is defined to be the minimum value over entries of the matrix.

We also make a definition of convenience.

DEFINITION 3.0.3. Under Convention 3.0.1, we will mean by an *admissible ring* any one of the following topological rings.

- The ring Γ or $\Gamma[\pi^{-1}]$ with its levelwise topology.
- The ring Γ_r or $\Gamma_r[\pi^{-1}]$ with the Fréchet topology induced by w_s for all $s \in (0, r]$, for $r \in (0, r_0)$. Note that for Γ_r, this coincides with the topology induced by w_r alone.
- The ring Γ_{con} or $\Gamma_{\mathrm{con}}[\pi^{-1}]$ topologized as the direct limit of the Γ_r or $\Gamma_r[\pi^{-1}]$.

- The ring Γ_I with the Fréchet topology induced by the w_s for $s \in I$, for some $I \subseteq [0, r_0)$ bounded away from r_0.
- The ring $\Gamma_{\mathrm{an,con}}$ topologized as the direct limit of the $\Gamma_{\mathrm{an},r}$.

By a *nearly admissible ring*, we mean one of the above rings with E replaced by a finite separable extension.

3.1 – *Differentials.*

DEFINITION 3.1.1. Let S/R be an extension of topological rings. A *module of continuous differentials* is a topological S-module $\Omega^1_{S/R}$ equipped with a continuous R-linear derivation $d : S \to \Omega^1_{S/R}$, having the following universal property: for any topological S-module M equipped with a continuous R-linear derivation $D : S \to M$, there exists a unique morphism $\phi : \Omega^1_{S/R} \to M$ of topological S-modules such that $D = \phi \circ d$. Since the definition is via a universal property, the module of continuous differentials is unique up to unique isomorphism if it exists at all.

Constructing modules of continuous differentials is tricky in general (imitating the usual construction of the module of Kähler differentials requires a topological tensor product, which is a rather delicate matter); however, for fake annuli, the construction is straightforward.

DEFINITION 3.1.2. By a *coordinate system* for Γ, we will mean a lattice L in some $\mathbb{R}^m$, an irrational linear functional $\lambda \in (\mathbb{R}^m)^\vee$, an isomorphism $\Gamma^\lambda \cong \Gamma$ carrying $z \in L$ to a unit in Γ_{r_0} for each nonzero $z \in L$, and a basis $z_1, \dots, z_m$ of L. Such data always exist thanks to Proposition 2.3.4.

DEFINITION 3.1.3. For the remainder of this subsection, choose a coordinate system for Γ, and let $\mu_1, \dots, \mu_m \in L^\vee$ denote the basis dual to $z_1, \dots, z_m$. For $\mu \in L^\vee$ and S an admissible ring, let ∂_μ be the continuous derivation on S defined by the formula

$$\partial_\mu \left(\sum_z c_z \{z\} \right) = \sum_z \mu(z) c_z \{z\};$$

note that $\mu(z) \in \mathbb{Z}$, so it may sensibly be viewed as an element of $\mathfrak{o}$. (The continuity of ∂_μ is clear in terms of naïve partial valuations, so Proposition 2.3.4 implies continuity in terms of the Frobenius-based valuations.) For $\mu = \mu_i$, write ∂_i for ∂_{μ_i}. Define $\Omega^1_{S/\mathfrak{o}}$ to be the free S-module $S\,d\{z_1\} \oplus \cdots \oplus S\,d\{z_m\}$, equipped with the natural induced topology and with the continuous $\mathfrak{o}$-linear derivation $d : S \to \Omega^1_{S/\mathfrak{o}}$ given by

$$dx = \sum_{i=1}^{m} \partial_i(x)\, d\log\{z_i\}$$

(where $d\log(f) = df/f$).

PROPOSITION 3.1.4. *The module $\Omega^1_{S/\mathfrak{o}}$ is a module of continuous derivations for S over $\mathfrak{o}$. In particular, the construction does not depend on the choice of the coordinate system.*

PROOF. This is a straightforward consequence of the fact that one of $\mathfrak{o}[\{z_i\}^{\pm 1}]$ or $\mathfrak{o}[\pi^{-1}, \{z_i\}^{\pm 1}]$ is dense in S. $\square$

REMARK 3.1.5. Note that Proposition 3.1.4 also allows us to construct the module of continuous differentials $\Omega^1_{S/\mathfrak{o}}$ when S is only nearly admissible.

REMARK 3.1.6. For $\mathrm{rank}(L) = 1$ and $\mu \in L$ nonzero, the image of ∂_μ is closed; however, this fails for $\mathrm{rank}(L) > 1$, because bounding $\lambda(z)$ does not in any way limit the p-adic divisibility of z within L. This creates a striking difference between the milieux of true and fake annuli, from the point of view of the study of differential equations. On true annuli, one has the rich theory of p-adic differential equations due to Dwork-Robba, Christol-Mebkhout, et al. On fake annuli, much of that theory falls apart; the parts that survive are those that rest upon Frobenius structures, whose behavior differs little in the two settings.

3.2 – ∇-*modules.*

DEFINITION 3.2.1. Let S be a nearly admissible ring. Define a ∇-*module* over S to be a finite free S-module M equipped with an integrable $\mathfrak{o}$-linear connection $\nabla : M \to M \otimes \Omega^1_{S/\mathfrak{o}}$; here integrability means that, letting ∇_1 denote the induced map

$$M \otimes_S \Omega^1_{S/\mathfrak{o}} \stackrel{\nabla\otimes 1}{\to} M \otimes_S \Omega^1_{S/\mathfrak{o}} \otimes_S \Omega^1_{S/\mathfrak{o}} \stackrel{1\otimes\wedge}{\to} M \otimes_S \wedge^2_S \Omega^1_{S/\mathfrak{o}},$$

the composite map $\nabla_1 \circ \nabla$ is zero. We say $\boldsymbol{v} \in M$ is *horizontal* if $\nabla(\boldsymbol{v}) = 0$.

DEFINITION 3.2.2. Suppose S is admissible, and fix a coordinate system for Γ. Given a ∇-module M over S, for $\mu \in L^\vee$, define the map $\Delta_\mu : M \to M$

by writing $\nabla(\boldsymbol{v}) = \sum_{i=1}^{m} \boldsymbol{w}_i \otimes d\log\{z_i\}$ with $\boldsymbol{w}_i \in M$, and setting

$$\Delta_\mu(\boldsymbol{v}) = \sum_{i=1}^{m} \mu(z_i)\boldsymbol{w}_i.$$

Also, write Δ_i for Δ_{μ_i}.

REMARK 3.2.3. The maps Δ_μ satisfy the following properties.

- The map $L^\vee \times M \to M$ given by $(\mu, \boldsymbol{v}) \mapsto \Delta_\mu(\boldsymbol{v})$ is additive in each factor.
- For all $\mu \in L^\vee$, $s \in S$, and $\boldsymbol{v} \in M$, we have the Leibniz rule

$$\Delta_\mu(s\boldsymbol{v}) = s\Delta_\mu(\boldsymbol{v}) + \partial_\mu(s)\boldsymbol{v}.$$

- For $\mu_1, \mu_2 \in L^\vee$, the maps $\Delta_{\mu_1}, \Delta_{\mu_2}$ commute.

Conversely, given a finite free S-module M equipped with maps $\Delta_\mu : M \to M$ for each $\mu \in L^\vee$ satisfying these conditions, one can uniquely reconstruct a ∇-module structure on M that gives rise to the Δ_μ.

REMARK 3.2.4. Note that for true annuli (i.e., $\operatorname{rank}(L) = 1$), the integrability restriction is empty because $\Omega^1_{S/\mathfrak{o}}$ has rank 1 over S. However, for fake annuli, integrability is a real restriction: even though the ring theory looks one-dimensional, the underlying "fake space" is really m-dimensional, inasmuch as $\Omega^1_{S/\mathfrak{o}}$ has rank m over S.

DEFINITION 3.2.5. Let M be a ∇-module over $\Gamma^{E'}_{\mathrm{an,con}}$, for E' a finite separable extension of E. We say M is:

- *constant* if M admits a horizontal basis (a basis of elements of the kernel of ∇);
- *quasi-constant* if there exists a finite separable extension E'' of E' such that $M \otimes \Gamma^{E''}_{\mathrm{an,con}}$ is constant;
- *unipotent* if M admits an exhaustive filtration by saturated ∇-submodules, whose successive quotients are constant;
- *quasi-unipotent* if M admits an exhaustive filtration by saturated ∇-submodules, whose successive quotients are quasi-constant.

We extend these definitions to (F, ∇)-modules by applying them to the underlying ∇-module.

REMARK 3.2.6. If M is quasi-unipotent, then there exists a finite separable extension E'' of E' such that $M \otimes \Gamma^{E''}_{\mathrm{an,con}}$ is unipotent. The converse is also true: if $M \otimes \Gamma^{E''}_{\mathrm{an,con}}$ is unipotent, then the shortest unipotent filtration of $M \otimes \Gamma^{E''}_{\mathrm{an,con}}$ is unique, so descends to $\Gamma^{E'}_{\mathrm{an,con}}$.

3.3 – *Frobenius structures*

DEFINITION 3.3.1. Let S be a nearly admissible ring stable under σ; for instance, $\Gamma, \Gamma_{\mathrm{con}}, \Gamma_{\mathrm{an,con}}$ are permitted, but Γ_r is not. Define an *F-module* over S (with respect to σ) to be a finite free S-module M equipped with a S-module homomorphism $F : \sigma^* M \to M$ which is an isogeny, i.e., which becomes an isomorphism upon tensoring with $S[\pi^{-1}]$. We typically view F as a σ-linear map from M to itself; we occasionally view M as a left module for the twisted polynomial ring $S\{\sigma\}$. Given an F-module M over S and an integer c, which must be nonnegative if $\pi^{-1} \notin S$, define the *twist* $M(c)$ of M to be a copy of M with the action of F multiplied by π^c.

DEFINITION 3.3.2. Let S be a nearly admissible ring stable under σ. Define an *(F, ∇)-module* over S to be a finite free S-module M equipped with the structures of both an F-module and a ∇-module, which are compatible in the sense of making the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{\nabla} & M \otimes \Omega^1_{S/\mathfrak{o}} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F \otimes d\sigma} \\ M & \xrightarrow{\nabla} & M \otimes \Omega^1_{S/\mathfrak{o}} \end{array}$$

REMARK 3.3.3. We may regard $\Omega^1_{S/\mathfrak{o}}$ itself as an F-module via $d\sigma$, in which case the compatibility condition asserts that $\nabla : M \to M \otimes \Omega^1_{S/\mathfrak{o}}$ is an F-equivariant map. The fact that $\partial_\mu(f^\sigma) \equiv 0 \bmod \pi$ for any $f \in \Gamma_{\mathrm{con}}$ means that $\Omega^1_{\Gamma_{\mathrm{con}}/\mathfrak{o}}$ is isomorphic as an F-module to $N(1)$, for some F-module N over Γ_{con}. In the language of [14], this means that the generic HN slopes of $\Omega^1_{\Gamma_{\mathrm{con}}/\mathfrak{o}}$ are positive [14, Proposition 5.1.3].

DEFINITION 3.3.4. For a a positive integer, define an *F^a-module* or *(F^a, ∇)-module* as an F-module or (F, ∇)-module relative to σ^a. Given an F-module M, viewed as a left $S\{\sigma\}$-module, define the F^a-module $[a]_* M$ to be the left $S\{\sigma^a\}$-module given by restriction along the inclusion $S\{\sigma^a\} \hookrightarrow S\{\sigma\}$; in other words, replace the Frobenius action by its a-th power. Given an F^a-module N, viewed as a left $S\{\sigma^a\}$-module, define the F-module $[a]^*M$ to be the left $S\{\sigma\}$-module

$$[a]^*M = S\{\sigma\} \otimes_{S\{\sigma^a\}} M;$$

then the functors $[a]^*$ and $[a]_*$ are left and right adjoints of each other. See [14, § 3.2] for more on these operations.

3.4 – *Change of Frobenius.*

The category of (F, ∇)-modules over $\Gamma_{\mathrm{an,con}}$ relative to σ turns out to be canonically independent of the choice of σ, by a Taylor series argument (as in [25, § 3.4]).

Convention 3.4.1. Throughout this subsection, fix a coordinate system on Γ. Given an m-tuple $J = (j_1, \dots, j_m)$ of nonnegative integers, write $J! = j_1! \cdots j_m!$; if $U = (u_1, \dots, u_m)$, write $U^J = u_1^{j_1} \cdots u_m^{j_m}$. Also, define the "falling factorials"

$$\partial^{\underline{J}} = \prod_{i=1}^{m} \prod_{l=0}^{j_i - 1} (\partial_i - l)$$

$$\Delta^{\underline{J}} = \prod_{i=1}^{m} \prod_{l=0}^{j_i - 1} (\Delta_i - l),$$

with the convention that $\partial^{\underline{0}}$ and $\Delta^{\underline{0}}$ are the respective identity maps. (The use of falling factorial notation is modeled on [8].)

Lemma 3.4.2. *Let M be a ∇-module over $\Gamma_{\mathrm{an,con}}$. Then for any $r \in \Gamma_{\mathrm{an,con}}$, any $\boldsymbol{v} \in M$, and any m-tuple J of nonnegative integers,*

$$\frac{1}{J!} \Delta^{\underline{J}}(r\boldsymbol{v}) = \sum_{J_1 + J_2 = J} \left(\frac{1}{J_1!} \partial^{\underline{J_1}}(r) \right) \left(\frac{1}{J_2!} \Delta^{\underline{J_2}}(\boldsymbol{v}) \right).$$

Proof. Since

$$\partial_i(\partial_i - 1) \cdots (\partial_i - j + 1) = \{z_i\}^j (\{z_i\}^{-1} \partial_i)^j$$

and similarly for Δ_i, this amounts to a straightforward application of the Leibniz rule. □

LEMMA 3.4.3. *For any $u_1, \dots, u_m \in \Gamma_{\mathrm{con}}$ with $w(u_i) > 0$ for $i = 1, \dots, m$, and any $x \in \Gamma_{\mathrm{an,con}}$, the series*

$$f(x) = \sum_{j_1, \dots, j_m = 0}^{\infty} \frac{1}{J!} U^J \partial^{\underline{J}}(x)$$

converges in $\Gamma_{\mathrm{an,con}}$, and the map $x \mapsto f(x)$ is a continuous ring homomorphism sending $\{z_i\}$ to u_i.

PROOF. Pick $r > 0$ such that $u_1, \dots, u_m, x \in \Gamma_{\mathrm{an},r}$ and $w_r(u_i) > 0$ for $i = 1, \dots, m$. Write $x = \sum_{z \in L} c_z\{z\}$; note that for each J,

$$\frac{1}{J!} \partial^{\underline{J}}(x) = \sum_{z \in L} \left(\prod_{i=1}^{m} \binom{\mu_i(z)}{j_i} \right) c_z \{z\},$$

so that $w_s(\partial^{\underline{J}}(x)/J!) \geq w_s(x)$ for $s \in (0, r]$. This yields the desired convergence, as well as continuity of the map $x \mapsto f(x)$. Moreover, f is a ring homomorphism on $\mathfrak{o}[\{z_1\}, \dots, \{z_m\}]$ by Lemma 3.4.2, so must be a ring homomorphism on $\Gamma_{\mathrm{an,con}}$ by continuity; the fact that it sends $\{z_i\}$ to u_i is apparent from the formula. $\square$

LEMMA 3.4.4. *Let M be a ∇-module over $\Gamma_{\mathrm{an},r}$ for some $r > 0$. Suppose that for some positive integer h, M admits a basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ such that the $n \times n$ matrices $N_1, \dots, N_m$ defined by $\Delta_i(\boldsymbol{e}_l) = \sum_j (N_i)_{jl} \boldsymbol{e}_j$ satisfy $w_r(N_i) > w((p^h)!)$ for $i = 1, \dots, m$. For J an m-tuple of nonnegative integers, define the $n \times n$ matrix N_J by*

$$\Delta^{\underline{J}}(\boldsymbol{e}_l) = \sum_j (N_J)_{jl} \boldsymbol{e}_j.$$

Then

$$w_r(N_J) \geq w(J!) - w(p)(j_1 + \dots + j_m)/(p^h(p-1)).$$

PROOF. The condition that $w_r(N_i) > w((p^h)!)$ means that for any $a \in \mathbb{Z}$ and any $b \in \{0, \dots, p^h - 1\}$, if we write

$$\boldsymbol{v} = \sum_{j=1}^{m} x_j \boldsymbol{e}_j \qquad (x_j \in \Gamma_{\mathrm{an},r})$$

$$(\Delta_i - ap^h)(\Delta_i - ap^h - 1) \cdots (\Delta_i - ap^h - b)\boldsymbol{v} = \sum_{j=1}^{m} y_{ijab} \boldsymbol{e}_j \qquad (y_{ijab} \in \Gamma_{\mathrm{an},r}),$$

then $\min_j\{w_r(y_{ijab})\} \geq \min_j\{w_r(x_j)\} + w(b!)$ (i.e., the same bound as for the trivial connection with $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ horizontal). This gives the bound

$$\begin{aligned} w_r(N_J) &\geq w(J!) + \sum_{i=1}^m \bigl(-w(j_i!) + \lfloor j_i/p^h \rfloor w((p^h)!) + w((j_i - p^h \lfloor j_i/p^h \rfloor)!)\bigr) \\ &\geq w(J!) - \sum_{i=1}^m w(p) j_i/(p^h(p-1)) \end{aligned}$$

using the fact that $w(j_i!) = \sum_{g=1}^{\infty} w(p)\lfloor j_i/p^g \rfloor$. This yields the claim. $\square$

LEMMA 3.4.5. *Let M be an (F,∇)-module over $\Gamma_{\mathrm{an,con}}$ or over $\Gamma_{\mathrm{con}}[\pi^{-1}]$, and let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ be a basis of M. For each nonnegative integer g, define the $n \times n$ matrices $N_{g,1}, \dots, N_{g,m}$ by $\Delta_i(F^g \boldsymbol{e}_l) = \sum_j (N_{g,i})_{jl}(F^g \boldsymbol{e}_j)$. Then there exist $r_1 \in (0, r_0)$ and $c > 0$ such that for each nonnegative integer g and for each of $i = 1, \dots, m$, $N_{g,i}$ has entries in $\Gamma_{\mathrm{an},r_1 q^{-g}}$ and*

$$w_{rq^{-g}}(N_{g,i}) \geq g - c \qquad (r \in [r_1/q, r_1]).$$

Moreover, if M is defined over $\Gamma_{\mathrm{con}}[\pi^{-1}]$, we can also ensure that $w(N_{g,i}) \geq g - c$.

PROOF. Define $a_{hi} \in \Gamma_{\mathrm{con}}$ by the formula

$$\partial_i(x^\sigma) = \sum_{h=1}^m a_{hi}(\partial_h x)^\sigma \qquad (x \in \Gamma_{\mathrm{con}});$$

then $w(a_{hi}) \geq 1$ as in Remark 3.3.3. In particular, we can choose $r_1 \in (0, r_0)$ as in Proposition 2.3.4 such that for $i = 1, \dots, m$, $a_i \in \Gamma_{r_1}$, $w_{r_1}(a_{hi}) \geq 1$, and $N_{0,i}$ has entries in Γ_{an,r_1}. Then the formula

$$N_{g+1,i} = \sum_{h=1}^m a_{hi} N_{g,h}^\sigma$$

yields the claim for any c with $\min_i\{\min_{r \in [r_1/q, r_1]}\{w_r(N_{0,i})\}\} \geq -c$ and (in case M is defined over $\Gamma_{\mathrm{con}}[\pi^{-1}]$) $\min_i\{w(N_{0,i})\} \geq -c$.

LEMMA 3.4.6. *Let M be an (F,∇)-module over $\Gamma_{\mathrm{an,con}}$ (resp. over $\Gamma_{\mathrm{con}}[\pi^{-1}]$). Then for any $u_1, \dots, u_m \in \Gamma_{\mathrm{con}}$ with $w(u_i) > 0$ for $i = 1, \dots, m$, and any $\boldsymbol{v} \in M$, the series*

$$f(\boldsymbol{v}) = \sum_{j_1, \dots, j_m = 0}^{\infty} \frac{1}{J!} U^J \Delta^{\underline{J}}(\boldsymbol{v})$$

converges for the natural topology of M, and the map $\boldsymbol{v} \mapsto f(\boldsymbol{v})$ is semilinear for the map defined by Lemma 3.4.3.

PROOF. Pick a basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ of M; for each nonnegative integer g, define the $n \times n$ matrices $N_{g,1}, \dots, N_{g,m}$ by $\Delta_i(F^g \boldsymbol{e}_l) = \sum_j (N_{g,i})_{jl}(F^g \boldsymbol{e}_j)$. By Lemma 3.4.5, we can choose $r_1 \in (0, r_0)$ such that for some $c > 0$, $w_{rq^{-g}}(N_{g,i}) \geq g - c$ for all nonnegative integers g and all $r \in [r_1/q, r_1]$; moreover, if we are working over $\Gamma_{\text{con}}[\pi^{-1}]$, we can ensure that $w(N_{g,i}) \geq$ $\geq g - c$.

Now choose a positive integer h with $w(p)/(p^h(p-1)) < 1/2$. Then by the previous paragraph, for each sufficiently small $r > 0$, there exists a basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ of M (depending on r) on which each Δ_i acts via a matrix N_i with $w_r(N_i) > w((p^h)!)$. By Lemma 3.4.4, the matrix N_J defined by

$$\Delta^{\underline{J}}(\boldsymbol{v}_l) = \sum_j (N_J)_{jl} \boldsymbol{v}_j$$

satisfies $w_r(N_J) \geq w(J!) - (j_1 + \dots + j_m)/2$.

On the other hand, since $w(u_i) \geq 1$ for $i = 1, \dots, m$, we have that $w_r(u_i) > 1/2$ for r sufficiently small. We conclude that for each sufficiently small $r > 0$, there exists a basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ such that the series defining each of $f(\boldsymbol{v}_1), \dots, f(\boldsymbol{v}_n)$ converges under w_r. By Lemma 3.4.2 and Lemma 3.4.3, for each $\Gamma_{\text{an},r}$-linear combination $\boldsymbol{v}$ of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$, the series defining $f(\boldsymbol{v})$ converges under w_r. By the same token, in case M is defined over $\Gamma_{\text{con}}[\pi^{-1}]$, for each $\Gamma_r[\pi^{-1}]$-linear combination $\boldsymbol{v}$ of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$, the series defining $f(\boldsymbol{v})$ converges under w. This yields the desired convergence of f; again, the semilinearity follows from Lemma 3.4.2 and Lemma 3.4.3. $\square$

PROPOSITION 3.4.7. *Let σ_1 and σ_2 be Frobenius lifts on Γ such that Γ has enough units with respect to each of σ_1 and σ_2, and for each $z \in L$ nonzero, $\{z\}$ is a unit in Γ_{con} under both definitions. (By Proposition* 2.3.4, *it is equivalent to require that the definitions of Γ_{con} with respect to σ_1 and to σ_2 coincide.) Then there is a canonical equivalence of categories between (F, ∇)-modules over $\Gamma_{\text{an,con}}$ (resp. over $\Gamma_{\text{con}}[\pi^{-1}]$) relative to σ_1 and relative to σ_2, acting as the identity on the underlying ∇-modules.*

PROOF. Put $u_i = \{z_i\}^{\sigma_2}/\{z_i\}^{\sigma_1} - 1$. Let M be a ∇-module admitting a compatible Frobenius structure F_1 relative to σ_1. For $\boldsymbol{v} \in M$, define

$$F_2(\boldsymbol{v}) = \sum_{j_1, \dots, j_m = 0}^{\infty} \frac{1}{J!} U^J F_1(\Delta^{\underline{J}}(\boldsymbol{v}));$$

this series converges thanks to Lemma 3.4.6. Moreover, the result is σ_2-linear thanks to Lemma 3.4.2. $\square$

REMARK 3.4.8. By tweaking the proof of Proposition 3.4.7, one can also obtain the analogous independence from the choice of σ for the category of (F, ∇)-modules over Γ_{con}. We will not use this result explicitly, though a related construction will occur in Subsection 4.2.

4. Unit-root (F, ∇)-modules (after Tsuzuki).

In this section, we give the generalization to fake annuli of Tsuzuki's unit-root local monodromy theorem [24], variant proofs of which are given by Christol [2] and in the author's unpublished dissertation [10]. Our argument here draws on elements of all of these; its specialization to the case of true annuli constitutes a novel (if only slightly so) exposition of Tsuzuki's original result.

CONVENTION 4.0.1. Throughout this section, let E denote a nearly monomial field over k, viewed in a fixed fashion as a finite separable extension of a monomial field over k. We assume that any Frobenius lift σ considered on $\Gamma = \Gamma^E$ is chosen so that Γ has enough units. In particular, $\Gamma = \Gamma^E$ and $\Gamma_{\mathrm{con}} = \Gamma^E_{\mathrm{con}}$ are nearly admissible in the sense of Definition 3.0.3.

4.1 – *Unit-root F-modules*

DEFINITION 4.1.1. We say an F-module M over Γ^E or Γ^E_{con}, with respect to some Frobenius lift σ, is *unit-root* (or *étale*) if the map $F : \sigma^* M \to M$ is an isomorphism (not just an isogeny). We say an (F, ∇)-module over Γ^E or Γ_{con} is unit-root if its underlying F-module is unit-root.

We will frequently calculate on such modules in terms of bases, so it is worth making the relevant equations explicit.

REMARK 4.1.2. Assume that E is a monomial field, and fix a coordinate system for Γ. Let M be a ∇-module over Γ or Γ_{con} with basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$. Given $\mu \in L^\vee$, define the $n \times n$ matrix N_μ by $\Delta_\mu(\boldsymbol{e}_l) = \sum_j (N_\mu)_{jl} \boldsymbol{e}_j$; if we identify $\boldsymbol{v} = c_1 \boldsymbol{e}_1 + \dots + c_n \boldsymbol{e}_n \in M$ with the column vector with entries $c_1, \dots, c_n$, then we have

$$\Delta_\mu(\boldsymbol{v}) = N_\mu \boldsymbol{v} + \partial_\mu(\boldsymbol{v}).$$

Given an F-module with the same basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$, define the $n \times n$ matrix A by $F(\boldsymbol{e}_l) = \sum_j A_{jl}\boldsymbol{e}_j$; then with the same identification of $\boldsymbol{v}$ with a column vector, we have

$$F(\boldsymbol{v}) = A\boldsymbol{v}^\sigma.$$

In case the Frobenius lift σ is standard, the compatibility of Frobenius and connection structures is equivalent to the equations

$$N_\mu A + \partial_\mu(A) = qAN_\mu^\sigma \qquad (\mu \in L^\vee);$$

of course it is only necessary to check this on a basis of $L^\vee$.

REMARK 4.1.3. It is also worth writing out how the equations in Remark 4.1.2 transform under change of basis. First, if U is an invertible $n \times n$ matrix, then

$$N_\mu A + \partial_\mu(A) = 0 \iff (U^{-1}N_\mu U + U^{-1}\partial_\mu(U))(U^{-1}A) + \partial_\mu(U^{-1}A) = 0.$$

Second, in case σ is standard, the equations

$$N_\mu A + \partial_\mu(A) = qAN_\mu^\sigma \qquad \text{and} \qquad N'_\mu A' + \partial_\mu(A') = qA'(N'_\mu)^\sigma$$

are equivalent for

$$\begin{aligned} N'_\mu &= U^{-1}N_\mu U + U^{-1}\partial_\mu(U) \\ A' &= U^{-1}AU^\sigma. \end{aligned}$$

4.2 – *Unit-root F-modules and Galois representations*

We now consider unit-root F-modules over Γ, obtaining the usual Fontaine-style setup.

LEMMA 4.2.1. *Let ℓ be a separably closed field of characteristic $p > 0$, and let τ denote the q-power Frobenius on ℓ. Let A be an invertible $n \times n$ matrix over ℓ.*

(a) *There exists an invertible $n \times n$ matrix U over ℓ such that $U^{-1}AU^\tau$ is the identity matrix.*

(b) *For any $1 \times n$ column vector $\boldsymbol{v}$ over ℓ, there are exactly q^n distinct $1 \times n$ column vectors $\boldsymbol{w}$ over ℓ for which $A\boldsymbol{w}^\tau - \boldsymbol{w} = \boldsymbol{v}$.*

PROOF. Part (a) is [9, Proposition 1.1]; part (b) is an easy corollary of (a). □

We next introduce a "big ring" over which unit-root F-modules over Γ can be trivialized.

DEFINITION 4.2.2. Let $\tilde{\Gamma}$ be the π-adic completion of the maximal unramified extension of Γ; then any Frobenius lift σ on Γ extends uniquely to $\tilde{\Gamma}$, and the derivation d extends uniquely to a derivation $d : \tilde{\Gamma} \to (\Omega^1_{\Gamma/\mathfrak{o}} \otimes_\Gamma \tilde{\Gamma})$. Likewise, any ∇-module M over Γ induces a connection $\nabla : (M \otimes_\Gamma \tilde{\Gamma}) \to$ $\to (M \otimes_\Gamma \Omega^1_{\Gamma/\mathfrak{o}} \otimes_\Gamma \tilde{\Gamma})$. Let $\tilde{\mathfrak{o}}_q$ be the fixed subring of $\tilde{\Gamma}$ under σ; this is a complete discrete valuation ring with residue field $\mathbb{F}_q$ and maximal ideal generated by π.

PROPOSITION 4.2.3. *Let M be a unit-root F-module over $\tilde{\Gamma}$. Then M admits an F-invariant basis.*

PROOF. Applying Lemma 4.2.1(a) produces a basis which is fixed modulo π. Given a basis fixed modulo π^n, correcting it to a basis fixed modulo π^{n+1} amounts to solving a set of vector equations of the form of Lemma 4.2.1(b). The resulting sequence of bases converges to the desired F-invariant basis. □

DEFINITION 4.2.4. Assume that $\mathbb{F}_q \subseteq k$. Given a unit-root F-module M over Γ, let $D_\Gamma(M)$ denote the set of F-invariant elements of $M \otimes_\Gamma \tilde{\Gamma}$; then $D_\Gamma(M)$ is a finite free $\tilde{\mathfrak{o}}_q$-module equipped with a continuous action of $G = \mathrm{Gal}(E^{\mathrm{sep}}/E)$. By Proposition 4.2.3, the natural map $D_\Gamma(M) \otimes_{\tilde{\mathfrak{o}}_q} \tilde{\Gamma} \to M \otimes_\Gamma \tilde{\Gamma}$ is an isomorphism. Conversely, given a finite free $\tilde{\mathfrak{o}}_q$-module N equipped with a continuous action of G, let $V(N)$ denote the set of G-invariant elements of $N \otimes_{\tilde{\mathfrak{o}}_q} \tilde{\Gamma}$; by Galois descent, the natural map $V(N) \otimes_\Gamma \tilde{\Gamma} \to N \otimes_{\tilde{\mathfrak{o}}_q} \tilde{\Gamma}$ is an isomorphism. The functors D_Γ and V thus exhibit equivalences of categories between unit-root F-modules over Γ and finite free $\tilde{\mathfrak{o}}_q$-modules equipped with continuous G-action.

So far in this subsection, we have considered only unit-root F-modules over $\tilde{\Gamma}$, without connection structure. The reason is that the connection does not really add any extra structure in this case.

PROPOSITION 4.2.5. *Let M be a unit-root F-module over Γ (resp. over $\tilde{\Gamma}$). Then there is a unique integrable connection $\nabla : M \to M \otimes \Omega^1$ compatible with F.*

PROOF. We first check existence and uniqueness for the connection, without worrying about integrability. Let $\nabla_0 : M \to M \otimes \Omega^1$ be any connection (not necessarily integrable). Then a map $\nabla : M \to M \otimes \Omega^1$ is a connection if and only if $\nabla - \nabla_0$ is a Γ-linear map from M to $M \otimes \Omega^1$, i.e., if it corresponds to an element of $M^\vee \otimes M \otimes \Omega^1$. Moreover, ∇ is compatible with F if and only if

$$(\nabla - \nabla_0)F - (F \otimes d\sigma)(\nabla - \nabla_0) = (F \otimes d\sigma)\nabla_0 - \nabla_0 F;$$

in other words, we can write down a particular $\boldsymbol{w} \in M^\vee \otimes M \otimes \Omega^1$ such that ∇ is F-equivariant if and only if $\nabla - \nabla_0$ corresponds to an element $\boldsymbol{v} \in M^\vee \otimes M \otimes \Omega^1$ with $\boldsymbol{v} - F\boldsymbol{v} = \boldsymbol{w}$. By Remark 3.3.3, $M^\vee \otimes M \otimes \Omega^1$ can be written as a twist $N(1)$ for some F-module N over Γ (resp. over $\tilde{\Gamma}$); hence the series $\boldsymbol{w} + F\boldsymbol{w} + F^2\boldsymbol{w} + \cdots$ converges in $M^\vee \otimes M \otimes \Omega^1$ to the unique solution of $\boldsymbol{v} - F\boldsymbol{v} = \boldsymbol{w}$.

It remains to prove that the unique connection ∇ compatible with F is in fact integrable. It is enough to check this over $\tilde{\Gamma}$; moreover, it is enough to exhibit a single integrable connection compatible with F, as this must then coincide with the connection constructed above. To do this, we apply Proposition 4.2.3 to produce an F-invariant basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ of M, then set

$$\nabla(c_1\boldsymbol{e}_1 + \cdots + c_n\boldsymbol{e}_n) = \boldsymbol{e}_1 \otimes dc_1 + \cdots + \boldsymbol{e}_n \otimes dc_n;$$

this map is easily seen to be an integrable connection compatible with F. □

REMARK 4.2.6. For true annuli, the construction of Definition 4.2.4 is due to Fontaine [7, 1.2]. In general, one consequence of the construction is that the categories of unit-root F-modules over Γ relative to two different Frobenius lifts are canonically equivalent, since the category of finite free $\tilde{\mathfrak{o}}_q$-modules equipped with continuous G-action does not depend on the choice of the Frobenius lift. In fact, one may even change the choice of the underlying Frobenius lift σ_K, as long as it does not change what $\tilde{\mathfrak{o}}_q$ is. Note that the same is true of unit-root (F, ∇)-modules; more precisely, if a ∇-module M over Γ admits a unit-root Frobenius structure for one Frobenius lift, it admits a unit-root Frobenius structure for any Frobenius lift. That is because the connection on M can be recovered from $D_\Gamma(M)$, by specifying that elements of $D_\Gamma(M)$ are horizontal.

4.3 – *Positioning Frobenius.*

It will be useful to prove a positioning lemma for elements of $k((L))_\lambda$.

LEMMA 4.3.1. *Assume that the field k is algebraically closed. Suppose $x \in k((L))_\lambda$ cannot be written as $a^q - a$ for any $a \in k((L))_\lambda$. Then there exists $c > 0$ such that for any nonnegative integer i and any $y \in k((L))_\lambda$ with $x - y + y^q \in k((q^i L))_\lambda$, we have $v_\lambda(x - y + y^q) \leq -cq^i$.*

PROOF. Clearly there is no harm in replacing x by $x - y_0 + y_0^q$ for any $y_0 \in k((L))_\lambda$. In particular, write $x = \sum_{z \in L} c_z\{z\}$, and let x_-, x_0, x_+ be the sum of $c_z\{z\}$ over those z with $\lambda(z)$ negative, zero, positive, respectively. Since k is algebraically closed, we have $x_0 = y - y^q$ for some $y \in k$. Since $v_\lambda(x_+) > 0$, we have $x_+ = y - y^q$ for $y = x_+ + x_+^q + x_+^{q^2} + \cdots$. We may thus reduce to the case $x = x_-$; in particular, x has finite support.

For $z \in L$ nonzero, let $i(z)$ denote the largest integer i such that $z/q^i \in L$. Set

$$y_1 = \sum_{z \in L, \lambda(z) < 0} \sum_{i=1}^{i(z)} (c_z\{z\})^{q^{-i}},$$

so that $x_1 = x + y_1 - y_1^q$ is supported on $L \setminus qL$. We cannot have $x_1 = 0$, or else we could have written $x = a^q - a$ for some $a \in k((L))_\lambda$. There must thus be a smallest (under λ) element z of the support of x_1. For any nonnegative integer i and any $y \in k((L))_\lambda$ with $x - y + y^q \in k((q^i L))_\lambda$, the support of $x - y + y^q$ must contain $q^{i+j}z$ for some nonnegative integer j, and so $v_\lambda(x - y + y^q) \leq -\lambda(z)q^i$, as desired. □

4.4 – *Successive decimation.*

We now give a version of Tsuzuki's construction for solving p-adic differential equations.

CONVENTION 4.4.1. Throughout this subsection, assume that k is algebraically closed and that E is a monomial field over k, and fix a coordinate system on Γ. Also assume σ is a standard Frobenius lift; we may then safely confound the naïve and Frobenius-based partial valuations.

DEFINITION 4.4.2. For $\mu \in L^\vee$ nonzero, write L_μ for the sublattice of $z \in L$ for which $\mu(z) \in p\mathbb{Z}$.

LEMMA 4.4.3. *Suppose that A is an invertible $n \times n$ matrix over Γ with $w(A - I_n) > 0$, that N_μ is an $n \times n$ matrix over Γ supported on L_μ, and that $N_\mu A + \partial_\mu(A) = qAN_\mu^\sigma$. Then A is supported on L_μ.*

PROOF. Suppose the contrary; write $A = B + C$ with B supported on L_μ and $h = w(C)$ as large as possible, so in particular $h > 0$. Write $C = \sum_{z \in L} C_z\{z\}$. Since h is as large as possible, there exists $z \in L \setminus L_\mu$ such that $w(C_z) = h$; the coefficient of $\{z\}$ in $\partial_\mu(C)$ then also has valuation h. However, in the equality

$$\partial_\mu(C) = (qBN_\mu^\sigma - N_\mu B - \partial_\mu(B)) + (qCN_\mu^\sigma - N_\mu C),$$

the first term in parentheses is supported on L_μ while the second term has valuation strictly greater than h (since $w(A - I_n) > 0$ forces $w(N_\mu) > 0$). This contradiction yields the desired result. □

LEMMA 4.4.4. *Pick $\mu \in L^\vee$ nonzero. Given $r > 0$, let N_μ be an $n \times n$ matrix over Γ_r such that $w(N_\mu) > 0$ and $w_r(N_\mu) > 0$. Then for any $s \in (0, r)$, there exists an invertible $n \times n$ matrix U over Γ_s such that $w_s(U - I_n) > 0$, $w(U - I_n) \geq w(N_\mu)$, and $U^{-1}N_\mu U + U^{-1}\partial_\mu(U)$ is supported on L_μ.*

PROOF. Define a sequence $U_0, U_1,$ of invertible matrices over Γ_r with $w(U_j - I_n) \geq w(N_\mu)$ and $w_r(U_j - I_n) \geq w_r(N_\mu)$, as follows. Start with $U_0 = I_n$. Given U_j, put $N_j = U_j^{-1}N_\mu U_j + U_j^{-1}\partial_\mu(U_j)$. Write $N_j = \sum_{z \in L} N_{j,z}\{z\}$, let X_j be the sum of $\mu(z)^{-1}N_{j,z}\{z\}$ over all $z \in L \setminus L_\mu$, and put $U_{j+1} = U_j(I_n - X_j)$.

For $j > 0$, if $w(U_j - I_n) < \infty$, one sees that $w(U_{j+1} - I_n) > w(U_j - I_n)$. Hence the U_j converge π-adically; since they all satisfy $w_r(U_j - I_n) \geq w_r(N_\mu) > 0$, the U_j converge under w_s to a limit U satisfying $w_s(U - I_n) \geq w_s(N_\mu)$. In particular, U is invertible and $U^{-1}N_\mu U + U^{-1}\partial_\mu(U)$ is supported on L_μ. (Compare [24, Lemma 6.1.4] and [10, Lemma 5.1.3].) □

LEMMA 4.4.5. *Let $N_1, \dots, N_m$ be $n \times n$ matrices over Γ_r, such that $w(N_i) > 0$ and $w_r(N_i) > 0$ for $i = 1, \dots, m$. Suppose that there exists an invertible matrix A over Γ with $w(A - I_n) > 0$, such that*

$$N_iA + \partial_i(A) = qAN_i^\sigma \qquad (i = 1, \dots, m).$$

Then for any $s \in (0, r)$, there exists an invertible matrix U over Γ_s, such that $w(U - I_n) \geq \min_i\{w(N_i)\}$, $w_s(U - I_n) \geq \min_i\{w_s(N_i)\}$, and $U^{-1}AU^\sigma$ is supported on pL.

Proof. For $i=0,\dots,m$, let S_i be the sublattice of $z \in L$ such that $\mu_j(z) \in p\mathbb{Z}$ for $j=1,\dots,i$. Pick $s_1,\dots,s_{m-1}$ with $0<s<$ $<s_{m-1}<\cdots<s_1<r$, and put $s_0=r$ and $s_m=s$. Put $U_0=I_n$. Given U_i invertible over Γ_{s_i} such that $A_i=U_i^{-1}AU_i^\sigma$ is supported on S_i, note that $M_i=U_i^{-1}N_iU_i+U_i^{-1}\partial_i(U_i)$ satisfies the equation $M_iA_i+\partial_i(A_i)=$ $=qA_iM_i^\sigma$. We may then argue (as in the proof of Proposition 4.2.5) that M_i is congruent to a matrix supported on S_i modulo successively larger powers of π.

Since M_i is supported on S_i, we may apply Lemma 4.4.4 to produce $U_{i+1}=U_iV_i$, with V_i supported on S_i, such that $U_{i+1}^{-1}N_iU_{i+1}+U_{i+1}^{-1}\partial_i(U_{i+1})$ is supported on S_{i+1}. Then $U_{i+1}^{-1}AU_{i+1}^\sigma=V_i^{-1}A_iV_i^\sigma$ is supported on S_i; by Lemma 4.4.3, $U_{i+1}^{-1}AU_{i+1}^\sigma$ is also supported on S_{i+1}. Thus the iteration continues, and we may set $U=U_m$. □

We are now ready for the decisive step, analogous to [24, Lemma 5.2.4].

Proposition 4.4.6. *Let $N_1,\dots,N_m$ be $n\times n$ matrices over Γ_{con} with $w(N_i)>0$ for $i=1,\dots,m$. Suppose that there exists an invertible $n\times n$ matrix A over Γ such that $w(A-I_n)>w(p)/(p-1)$ and*

$$N_iA+\partial_i(A)=qAN_i^\sigma \qquad (i=1,\dots,m).$$

Then there exists an invertible $n\times n$ matrix U over Γ such that $AU^\sigma=U$.

Proof. Suppose the contrary; then there exists some smallest integer $h>w(p)/(p-1)$ such that the equation $U^{-1}AU^\sigma\equiv I_n \pmod{\pi^{h+1}}$ cannot be solved for U invertible over Γ. Since Γ_{con} is π-adically dense in Γ, we may change basis over Γ_{con} to reduce to the case where $h=w(A-I_n)$ and we cannot write the reduction of $\pi^{-h}(A-I_n)$ modulo π in the form $B-B^\sigma$.

Choose $r_0>0$ such that for $i=1,\dots,m$, N_i has entries in Γ_{r_0} and $w_{r_0}(N_i)>0$. Since $h>w(p)/(p-1)$, we have $hp/(h+w(p))>1$; we can thus choose c with $1<c<hp/(h+w(p))$. Write $r_j=r_0p^{-j}c^j$.

Define a sequence $U_0,U_1,\dots$ of invertible matrices over Γ_{con} as follows. Start with $U_0=I_n$. For $j\ge 0$, suppose that we have constructed an invertible matrix U_j over Γ_{r_j} with the following properties:

(a) $w(U_j-I_n)\ge h$ and $w_{r_j}(U_j-I_n)>0$;

(b) $A_j=U_j^{-1}AU_j^\sigma$ and $N_{i,j}=U_j^{-1}N_iU_j+U_j^{-1}\partial_i(U_j)$ are supported on p^jL for $i=1,\dots,m$;

(c) $w_{r_j}(N_{i,j})>jw(p)$.

Write A_j' and $N_{i,j}'$ for the matrices A_j and $p^{-j}N_{i,j}$ viewed in Γ^{E_j} for

$E_j = k((p^j L))_\lambda$, put $\mu_i' = p^{-j}\mu_i$, and let ∂_i' be the derivation on Γ^{E_j} corresponding to μ_i'. Then $w_{r_j}(N_{i,j}') > 0$, and

$$N_{i,j}' A_j' + \partial_i'(A_j') = q A_j' (N_{i,j}')^\sigma.$$

Put $s = r_j(h + w(p))c/(hp) < r_j$, and apply Lemma 4.4.5 to produce U_{j+1} over Γ_s supported on $p^j L$, with $w(U_{j+1} - I_n) \geq h$ and $w_s(U_{j+1} - I_n) > 0$, such that $A_{j+1} = U_{j+1}^{-1} A_j U_{j+1}^\sigma$ is supported on $p^{j+1}L$.

Since $r_{j+1} = sh/(h + w(p)) < s$, (a) is satisfied again. From the equation

$$N_{i,j+1} A_{j+1} + \partial_i(A_{j+1}) = q A_{j+1} N_{i,j+1}^\sigma \tag{4.4.6.1}$$

(a consequence of Remark 4.1.3), we see that each $N_{i,j+1}$ is also supported on $p^{j+1}L$ (the argument is as in the proof of Lemma 4.4.5). Hence (b) is satisfied again.

To check (c), note that on one hand, (4.4.6.1) and the fact that $w(\partial_i(A_{j+1})) \geq h + (j+1)w(p)$ imply that $w(N_{i,j+1}) \geq h + (j+1)w(p)$. On the other hand, the facts that $w_s(N_{i,j}) > w_{r_j}(N_{i,j}) > jw(p)$, U_{j+1} is supported on $p^j L$, and $w_s(U_{j+1} - I_n) > 0$ imply that $w_s(N_{i,j+1}) > jw(p)$, and so

$$\begin{aligned}
w_{r_{j+1}}(N_{i,j+1}) &= \min_{m \geq h + (j+1)w(p)} \{ r_{j+1} v_m(N_{i,j+1}) + m \} \\
&\geq \frac{r_{j+1}}{s} w_s(N_{i,j+1}) + (h + (j+1)w(p))\Big(1 - \frac{r_{j+1}}{s}\Big) \\
&> \frac{r_{j+1}}{s}(jw(p)) + (h + (j+1)w(p))\Big(1 - \frac{r_{j+1}}{s}\Big) \\
&= \frac{h}{h + w(p)}(jw(p)) + (h + (j+1)w(p)) \frac{w(p)}{h + w(p)} \\
&= (j+1)w(p).
\end{aligned}$$

Hence (c) is satisfied again, and the iteration may continue.

Note that if we take $X = U_j - I_n$, then $A - X + X^\sigma \equiv A_j (\operatorname{mod} \pi^h)$. This means that on one hand, $A - X + X^\sigma$ is congruent modulo π^h to a matrix supported on $p^j L$, and on the other hand,

$$\begin{aligned}
v_h(A - X + X^\sigma) &\geq \min\{v_h(A), v_h(X), v_h(X^\sigma)\} \\
&\geq \min\{v_h(A), -h r_0^{-1} p^{j+1} c^{-j}\}
\end{aligned}$$

since $w_{r_j}(X) > 0$. However, since $c > 1$, this last inequality contradicts Lemma 4.3.1 for j large. This contradiction means that our original assumption was incorrect, i.e., the desired matrix U does exist, as desired. □

REMARK 4.4.7. In his setting, Tsuzuki actually proves a stronger result [24, Proposition 6.1.10] that produces solutions of p-adic differential equations even without a Frobenius structure. As noted in Remark 3.1.6, one cannot hope to do likewise in our setting.

4.5 – *Trivialization.*

We now begin reaping the fruits of our labors, first in a restricted setting. Note that Convention 4.4.1 is no longer in force.

PROPOSITION 4.5.1. *Suppose that the field k is algebraically closed. Let M be a ∇-module over Γ_{con} which becomes an (F, ∇)-module over Γ. Then as a representation of $G = \mathrm{Gal}(E^{\mathrm{sep}}/E)$, $D_\Gamma(M)$ has finite image; moreover, if $D_\Gamma(M)$ is trivial modulo π^m for some integer $m > w(p)/(p-1)$, then $D_\Gamma(M)$ is trivial.*

PROOF. We treat the second assertion first. Suppose that $D_\Gamma(M)$ is trivial modulo π^m for some integer $m > w(p)/(p-1)$. By Remark 4.2.6, we can change the choice of the Frobenius lift without affecting the fact that ∇ admits a compatible Frobenius structure over Γ, or that $D_\Gamma(M)$ is trivial modulo π^m. In particular, we may assume that σ is a standard Frobenius lift; we may then choose a coordinate system for Γ to drop back into the purview of Convention 4.4.1.

Given a basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ of M, define the $n \times n$ matrices $A, N_1, \dots, N_m$ by

$$F(\boldsymbol{e}_l) = \sum_j A_{jl} \boldsymbol{e}_j$$

$$\Delta_i(\boldsymbol{e}_l) = \sum_j (N_i)_{jl} \boldsymbol{e}_j;$$

as in Remark 4.1.2, we then have $N_i A + \partial_i(A) = qAN_i^\sigma$ for $i = 1, \dots, m$. By hypothesis, we can arrange to have $w(A - I_n) > w(p)/(p-1)$; we may thus apply Proposition 4.4.6 to produce an invertible $n \times n$ matrix U over Γ with $AU^\sigma = U$. Writing $\boldsymbol{v}_l = \sum_j U_{jl} \boldsymbol{e}_j$, we then have $F\boldsymbol{v}_j = \boldsymbol{v}_j$ for $j = 1, \dots, n$; that is, $M \otimes_{\Gamma_{\mathrm{con}}} \Gamma$ admits an F-invariant basis, so $D_\Gamma(M)$ is trivial.

We now proceed to the first assertion. Pick an integer m with $m > w(p)/(p-1)$. Let E' be the fixed field of the kernel of the action of G on $D_\Gamma(M)/\pi^m D_\Gamma(M)$. Then by what we have just shown, the restriction of

$D_\Gamma(M)$ to $\mathrm{Gal}(E^{\mathrm{sep}}/E')$ is trivial; hence $D_\Gamma(M)$, as a representation of G, has finite image. $\square$

Finally, we give the analogue of Tsuzuki's unit-root monodromy theorem [24, Theorems 4.2.6 and 5.1.1].

THEOREM 4.5.2. *Let M be a unit-root (F, ∇)-module over Γ_{con}. Then there exists a finite separable extension E' of E such that $M \otimes_{\Gamma_{\mathrm{con}}} \Gamma_{\mathrm{con}}^{E'}$ admits a basis of elements which are horizontal, and also F-invariant in case k is algebraically closed.*

PROOF. Suppose k is algebraically closed; by Proposition 4.5.1, for some finite separable extension E' of E, there exists a basis of $M \otimes_{\Gamma_{\mathrm{con}}} \Gamma'$ (for $\Gamma' = \Gamma^{E'}$) consisting of horizontal F-invariant elements. Since M is unit-root, we may apply [14, Lemma 5.4.1] to deduce that any F-invariant element of $M \otimes_{\Gamma_{\mathrm{con}}} \Gamma'$ actually belongs to $M \otimes_{\Gamma_{\mathrm{con}}} \Gamma'_{\mathrm{con}}$; this yields the claim.

For k general, let E' denote the completion of the compositum of E and k^{alg} over k. Then the restriction of $D_\Gamma(M)$ to $\mathrm{Gal}((E')^{\mathrm{sep}}/E')$ has finite image by Proposition 4.5.1. By a standard approximation argument, we can replace E by a finite separable extension in such a way as to trivialize the action of the resulting $\mathrm{Gal}((E')^{\mathrm{sep}}/E')$; the resulting action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ is trivial by Hilbert 90. This yields the claim. (Alternatively, one may proceed as in [11, Proposition 6.11] to reduce the case of k general to the case of k algebraically closed.) $\square$

5. Monodromy of (F, ∇)-modules.

In this section, we recall the slope filtration theorem of [11] (in the form presented in [14]), and combine it with the unit-root monodromy theorem (Theorem 4.5.2) to obtain the p-adic local monodromy theorem for fake annuli (Theorem 5.2.4).

Throughout this section, we retain Convention 4.0.1.

5.1 – *Isoclinicity.*

DEFINITION 5.1.1. Let M be an F-module of rank 1 over $\Gamma_{\mathrm{an,con}}$, and let $\boldsymbol{v}$ be a generator of M. Then $F\boldsymbol{v} = r\boldsymbol{v}$ for some $r \in \Gamma_{\mathrm{an,con}}$ which is a unit, that is, $r \in \Gamma_{\mathrm{con}}[\pi^{-1}]$. In particular, $w(r)$ is well-defined; it also does not depend on r, since changing the choice of generator multiplies r by u^σ/u for some $u \in \Gamma_{\mathrm{con}}[\pi^{-1}]$, whereas $w(u^\sigma/u) = 0$. We call the integer $w(r)$ the *degree* of M, and denote it by $\deg(M)$; if M has rank $n > 1$, we define the degree of M as $\deg(\wedge^n M)$. We write $\mu(M) = \deg(M)/\mathrm{rank}(M)$ and call it the *slope* of M.

DEFINITION 5.1.2. An F-module M over $\Gamma_{\mathrm{con}}[\pi^{-1}]$ is *unit-root* (or *étale*) if it contains an F-stable Γ_{con}-lattice which forms a unit-root F-module over Γ_{con}. Note that if M is a unit-root (F, ∇)-module over $\Gamma_{\mathrm{con}}[\pi^{-1}]$, then any unit-root Γ_{con}-lattice is stable under ∇ (as can be seen by applying Frobenius repeatedly).

DEFINITION 5.1.3. An F-module M over $\Gamma_{\mathrm{con}}[\pi^{-1}]$ is *isoclinic of slope s* if there exist integers c and d with $c/d = s$ such that $([d]_*M)(-c)$ is unit-root; note that necessarily $s = \mu(M)$. An F-module M over $\Gamma_{\mathrm{an,con}}$ is *isoclinic of slope s* if it is the base extension of an isoclinic F-module of slope s over $\Gamma_{\mathrm{con}}[\pi^{-1}]$; the base extension from isoclinic F-modules of a given slope over $\Gamma_{\mathrm{con}}[\pi^{-1}]$ to isoclinic F-modules of that slope over $\Gamma_{\mathrm{an,con}}$ is an equivalence of categories [14, Theorem 6.3.3].

REMARK 5.1.4. Note that this is not the definition of isoclinicity used in [14], but it is equivalent to it thanks to [14, Proposition 6.3.5].

The base extension functor mentioned above also behaves nicely with respect to connections; see [14, Proposition 7.1.7].

PROPOSITION 5.1.5. *Let M be an isoclinic F-module over $\Gamma_{\mathrm{con}}[\pi^{-1}]$. Suppose that $M \otimes \Gamma_{\mathrm{an,con}}$, with its given Frobenius, admits the structure of an (F, ∇)-module. Then M, with its given Frobenius, already admits the structure of an (F, ∇)-module.*

5.2 – *Slope filtrations and local monodromy.*

The slope filtration theorem can be stated as follows; see Remark 5.2.5 for a precise citation.

THEOREM 5.2.1. *Let M be an F-module over $\Gamma_{\mathrm{an,con}}$. Then there exists a unique filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ by saturated F-submodules with the following properties.*

(a) *For* $i=1,\dots,l$, *the quotient* M_i/M_{i-1} *is isoclinic of some slope* s_i.

(b) $s_1 < \cdots < s_l$.

Definition 5.2.2. In Theorem 5.2.1, we refer to the numbers $s_1,\dots,s_l$ as the *Harder-Narasimhan slopes* (or *HN slopes* for short) of M, viewed as a multiset in which s_i occurs with multiplicity $\mathrm{rank}(M_i/M_{i-1})$. See [14, § 4.6] for more on the calculus of the HN slopes.

The relevance of the slope filtration theorem to (F,∇)-modules comes via the following fact [14, Proposition 7.1.6].

Proposition 5.2.3. *Let* M *be an* (F,∇)*-module over* $\Gamma_{\mathrm{an,con}}$. *Then each step of the filtration of Theorem* 5.2.1 *is an* (F,∇)*-submodule.*

Using the slope filtration theorem, we easily obtain the p-adic local monodromy theorem.

Theorem 5.2.4 (p-adic local monodromy theorem). *Let* M *be an* (F,∇)*-module over* $\Gamma_{\mathrm{an,con}}$. *Then* M *is quasi-unipotent; moreover, if* M *is isoclinic, then* M *is quasi-constant.*

Proof. Let $0=M_0\subset M_1\subset\cdots\subset M_l=M$ be the filtration of the underlying F-module of M given by Theorem 5.2.1; by Proposition 5.2.3, this is also a filtration of (F,∇)-submodules. From the definition of isoclinicity plus Proposition 5.1.5, each successive quotient M_i/M_{i-1} can be written as $N_i\otimes\Gamma_{\mathrm{an,con}}$, where N_i is an (F,∇)-module over $\Gamma_{\mathrm{con}}[\pi^{-1}]$ whose underlying F-module is isoclinic.

It suffices to check that each N_i is quasi-constant; since that condition does not depend on the Frobenius structure, we may check after applying $[d]_*$ and twisting. We may thus reduce to the case where N_i is unit-root; in that case, Theorem 4.5.2 asserts that indeed N_i is quasi-constant, as desired. $\square$

Remark 5.2.5. For true annuli, Theorem 5.2.4 is what is normally called the "p-adic (local) monodromy theorem". The proof here, restricted to that case, is essentially the same as in [11, Theorem 6.12], except that the invocation of the slope filtration theorem [11, Theorem 6.10] is replaced with the more refined form [14, Theorem 6.4.1]. Proofs in the true annuli case have also been given by André [1, Théorème 7.1.1] and

Mebkhout [21, Corollaire 5.0-23]; these rely not on a close analysis of Frobenius (as in the slope filtration theorem), but on the close analysis of connections on annuli given by the p-adic index theorem of Christol-Mebkhout [3]. As per Remark 3.1.6, it seems unlikely that such an approach can be made to work in the fake annuli setting, at least without integrating Frobenius structures into the analysis.

REMARK 5.2.6. In case k is algebraically closed, one can refine the conclusion of Theorem 5.2.4. Namely, given an constant (F, ∇)-module, the K-span of the horizontal sections form an F-module over K, to which we may apply the classical Dieudonné-Manin theorem; the result is a decomposition of the given (F, ∇)-module into pieces of the form $[d]^* \Gamma_{\mathrm{an,con}}(c)$. Such pieces are called *standard* in [11] and [14].

6. Complements.

In this section, we gather some consequences of the p-adic local monodromy theorem for fake annuli. These generalize known consequences of the ordinary pLMT: calculation of some extension groups, local duality, and full faithfulness of overconvergent-to-convergent restriction.

Throughout this section, retain Convention 4.0.1.

6.1 – *Kernels and cokernels.*

We calculate some Hom and Ext groups in the category of (F, ∇)-modules.

DEFINITION 6.1.1. For M an F-module over some ring, let $H^0_F(M)$ and $H^1_F(M)$ denote the kernel and cokernel of $F - 1$ on M. Note that $\mathrm{Hom}_F(M_1, M_2) = H^0_F(M_1^\vee \otimes M_2)$ and $\mathrm{Ext}^1_F(M_1, M_2) = H^1_F(M_1^\vee \otimes M_2)$.

DEFINITION 6.1.2. For M an (F, ∇)-module over some ring, let $H^0_{F,\nabla}(M)$ be the subgroup of $\boldsymbol{v} \in M$ with $F(\boldsymbol{v}) = \boldsymbol{v}$ and $\nabla(\boldsymbol{v}) = 0$. Let $H^1_{F,\nabla}(M)$ be the set of pairs $(\boldsymbol{v}, \omega) \in M \times (M \otimes \Omega^1)$ with

$$\omega + \nabla(\boldsymbol{v}) = (F \otimes d\sigma)(\omega), \qquad \nabla_1(\omega) = 0, \tag{6.1.2.1}$$

modulo pairs of the form $(F(\boldsymbol{w}) - \boldsymbol{w}, \nabla(\boldsymbol{w}))$ for some $\boldsymbol{w} \in M$. Note that

$\mathrm{Hom}_{F,\nabla}(M_1, M_2) = H^0_{F,\nabla}(M_1^\vee \otimes M_2)$ and $\mathrm{Ext}^1_{F,\nabla}(M_1, M_2) = H^1_{F,\nabla}(M_1^\vee \otimes M_2)$. In particular, by Proposition 3.4.7, we can use any Frobenius lift σ to compute $H^i_{F,\nabla}(M)$.

LEMMA 6.1.3. *For d an integer, we have*

$$H^0_{F,\nabla}(\Gamma_{\mathrm{an,con}}(d)) = \begin{cases} K_q & d = 0 \\ 0 & d \neq 0. \end{cases}$$

PROOF. Note that $\ker(d : \Gamma_{\mathrm{an,con}} \to \Omega^1_{\Gamma_{\mathrm{an,con}}/\mathfrak{o}}) = K$. Then note that for $x \in K$ nonzero, $w(x^\sigma \pi^d) = w(x) + d$ can only equal $w(x)$ for $d = 0$. □

PROPOSITION 6.1.4. *Let M be an (F, ∇)-module over $\Gamma_{\mathrm{an,con}}$ whose HN slopes are all positive. Then $H^1_{F,\nabla}(M) = 0$.*

PROOF. Consider a short exact sequence $0 \to M \to N \to \Gamma_{\mathrm{an,con}} \to 0$; by [14, Proposition 7.4.4], the exact sequence splits if and only if N has smallest HN slope zero. In particular, this may be checked after enlarging k, applying $[a]_*$, and passing from $k((L))_\lambda$ to a finite separable extension. By Theorem 5.2.4, this allows us to reduce to the case where N is a successive extension of twists of trivial (F, ∇)-modules whose slopes are the HN slopes of N. If these slopes are all positive, then repeated application of Lemma 6.1.3 implies that the map $N \to \Gamma_{\mathrm{an,con}}$ is zero, which it isn't; hence N has smallest HN slope zero. It follows that $H^1_{F,\nabla}(M) = 0$, as desired. □

PROPOSITION 6.1.5. *Assume that k is algebraically closed. Put $n = w(q)$, let J be the subgroup of $x \in K$ satisfying $qx^\sigma = \pi^n x$, and let $z_1, \dots, z_m$ be a basis of L. Then for d an integer, we have*

$$H^1_{F,\nabla}(\Gamma_{\mathrm{an,con}}(d)) = \begin{cases} J\, d\log\{z_1\} \oplus \cdots \oplus J\, d\log\{z_m\} & d = -n \\ 0 & d \neq -n\,. \end{cases}$$

PROOF. For $d > 0$, Proposition 6.1.4 implies that $H^1_{F,\nabla}(\Gamma_{\mathrm{an,con}}(d)) = 0$; we may thus focus on $d \leq 0$. First suppose $d = 0$. Let $0 \to \Gamma_{\mathrm{an,con}} \to M \to \Gamma_{\mathrm{an,con}} \to 0$ be a short exact sequence of (F, ∇)-modules. Then by Theorem 5.2.1 and Lemma 6.1.3, M cannot have any nonzero slopes, so M is isoclinic of slope 0; as in Definition 5.1.3, we thus obtain a short exact sequence $0 \to \Gamma_{\mathrm{con}}[\pi^{-1}] \to M_0 \to \Gamma_{\mathrm{con}}[\pi^{-1}] \to 0$ of (F, ∇)-modules over $\Gamma_{\mathrm{con}}[\pi^{-1}]$ from which the original sequence is obtained by tensoring up to $\Gamma_{\mathrm{an,con}}$. Choose a basis $\boldsymbol{v}, \boldsymbol{w}$ of M_0 such that $\boldsymbol{v}$ is an F-stable element of

$\Gamma_{\mathrm{con}}[\pi^{-1}]$ within M_0, $\boldsymbol{w}$ maps to an F-stable element under the map $M_0 \to \Gamma_{\mathrm{con}}[\pi^{-1}]$, and $F(\boldsymbol{w}) - \boldsymbol{w} = c\boldsymbol{v}$ with $w(c) > w(p)/(p-1)$. By Proposition 4.5.1, for some finite separable extension E' of E, $M_0 \otimes \Gamma^{E'}_{\mathrm{con}}[\pi^{-1}]$ admits a basis of F-stable elements $\boldsymbol{e}_1, \boldsymbol{e}_2$. By Lemma 6.1.3, $\boldsymbol{v}$ is a K_q-linear combination of $\boldsymbol{e}_1, \boldsymbol{e}_2$; we may thus assume that $\boldsymbol{e}_1 = \boldsymbol{v}$. Similarly, we may assume that $\boldsymbol{e}_2$ and $\boldsymbol{w}$ have the same image under $M \to \Gamma_{\mathrm{con}}[\pi^{-1}]$. Thus the original exact sequence splits, as desired.

Now suppose $d < 0$; we may assume that σ is a standard Frobenius lift. Write Γ^-_{con} for the subring of Γ_{con} of series supported on the set $\{z \in L : \lambda(z) \leq 0\}$. We first check that given a pair (a, ω) representing an element of $H^1_{F,\nabla}(\Gamma_{\mathrm{an,con}}(d))$, if $a \in \Gamma^-_{\mathrm{con}}[\pi^{-1}]$ and a is supported on $(L \setminus qL) \cup \{0\}$, then we must have $a \in K$. Put $\omega = x_1 d\log\{z_1\} + + \cdots + x_m d\log\{z_m\}$, so that

$$x_i + \partial_i(a) = \pi^d q x_i^\sigma \qquad (i = 1, \dots, m).$$

For $z \in L \setminus qL$, suppose that the coefficient of $\{z\}$ in a is nonzero. Choose i with $\mu_i(z) \neq 0$, so that the coefficient of $\{z\}$ in $\partial_i(a)$ is nonzero, and for $j = 0, 1, \dots,$ let c_j be the coefficient of $\{z\}^{q^j}$ in x_i. Since σ is standard, the coefficient of $\{z\}$ in $q x_i^\sigma$ is zero; hence $c_0 \neq 0$. Moreover, $c_{j+1} = \pi^d q c_j^\sigma$ for $j = 0, 1,$. It follows that $w(c_j) = w(c_0) + j(w(q) + d)$ for all j; however, by the definition of $\Gamma_{\mathrm{an,con}}$, we must have $\liminf_j (w(c_j)/q^j) > 0$, contradiction. Thus the coefficient of $\{z\}$ in a is zero for each $z \in L \setminus qL$; since a is supported on $(L \setminus qL) \cup \{0\}$, we must have $a \in K$.

We next check that if $a \in \Gamma^-_{\mathrm{con}}[\pi^{-1}]$, then the pair (a, ω) represents the same class in $H^1_{F,\nabla}(\Gamma_{\mathrm{an,con}}(d))$ as another pair (a', ω') with $a' \in \Gamma^-_{\mathrm{con}}[\pi^{-1}]$ supported on $(L \setminus qL) \cup \{0\}$, and hence $a' \in K$ as above. Write $a = \sum_{z \in L} a_z \{z\}$, and for $j = 0, 1, \dots$, write $f_j(a)$ for the sum of $a_z\{z\}$ over all $z \in q^j L \setminus q^{j+1} L$. Then the sum

$$y = \sum_{j=0}^{\infty} \sum_{l=1}^{j} -(\pi^{\{-l\}})^d f_j(a)^{\sigma^{-l}},$$

where $\pi^{\{0\}} = 1$ and $\pi^{\{l+1\}} = (\pi^{\{l\}})^\sigma \pi$, converges in $\Gamma^-_{\mathrm{con}}[\pi^{-1}]$, so we can represent the same class in $H^1_{F,\nabla}(\Gamma_{\mathrm{an,con}}(d))$ by a pair with first member $a' = a - y + \pi^d y^\sigma$. Since

$$a' = a_0 + \sum_{j=0}^{\infty} (\pi^{\{-j\}})^d f_j(a)^{\sigma^{-j}},$$

a' is supported on $(L \setminus qL) \cup \{0\}$.

Next, we check that any pair (a, ω) represents the same class in $H^1_{F,\nabla}(\Gamma_{\mathrm{an,con}}(d))$ as another pair (a', ω') with $a' \in \Gamma^-_{\mathrm{con}}[\pi^{-1}]$. Write $a = \sum_{z \in L} a_z\{z\}$, and let a_+, a_0, a_- be the sum of $a_z\{z\}$ over those $z \in L$ with $\lambda(z)$ positive, zero, negative, respectively. We can then represent the same class in $H^1_{F,\nabla}(\Gamma_{\mathrm{an,con}}(d))$ by a pair with first member $a' = a - y + \pi^d y^\sigma$, for

$$y = \sum_{i=0}^{\infty} (\pi^{\{i\}})^d a_+^{\sigma^i}.$$

Then $a' = a_0 + a_- \in \Gamma^-_{\mathrm{con}}[\pi^{-1}]$.

Combining the previous paragraphs, we find that every element of $H^1_{F,\nabla}(\Gamma_{\mathrm{an,con}}(d))$ is represented by a pair (a, ω) with $a \in K$, and consequently $\pi^{-d} x_i = q x_i^\sigma$. Since k is algebraically closed, we can force $a = 0$; moreover, the resulting class representative is in fact unique. This yields the desired result. $\square$

Remark 6.1.6. Note that in the notation of Proposition 6.1.5, J is a one-dimensional vector space over K_q.

6.2 – *Duality and decompositions.*

Lemma 6.2.1. *Any irreducible (F, ∇)-module over $\Gamma_{\mathrm{an,con}}$ is isoclinic and quasi-constant.*

Proof. An irreducible (F, ∇)-module admits a slope filtration on its underlying F-module by Theorem 5.2.1, and the slope filtration is ∇-stable by Proposition 5.1.5. Hence it must have a single step, i.e., the module is isoclinic. Since any isoclinic (F, ∇)-module is quasi-constant (Theorem 5.2.1), the claims follow. $\square$

Definition 6.2.2. Let M, N be (F, ∇)-modules over a nearly admissible ring S. By the *cup product*, we will mean the natural bilinear map $H^0_{F,\nabla}(M) \times H^1_{F,\nabla}(N) \to H^1_{F,\nabla}(M \otimes N)$ sending $(x, (\boldsymbol{v}, \omega))$ to $(x \otimes \boldsymbol{v}, x \otimes \omega)$. Define the *Poincaré pairing* on M as the F-equivariant bilinear pairing obtained by composing the cup product map

$$H^0_{F,\nabla}(M) \times H^1_{F,\nabla}(M^\vee(-w(q))) \to H^1_{F,\nabla}(M \otimes M^\vee(-w(q)))$$

with the map

$$H^1_{F,\nabla}(M \otimes M^\vee(-w(q))) \to H^1_{F,\nabla}(S(-w(q)))$$

given by the trace map $M \otimes M^\vee \to S$.

PROPOSITION 6.2.3. *Assume that k is algebraically closed, and let M be an (F,∇)-module over $\Gamma_{\mathrm{an,con}}$. Then the Poincaré pairing $H^0_{F,\nabla}(M) \times \times H^1_{F,\nabla}(M^\vee(-w(q))) \to H^1_{F,\nabla}(\Gamma_{\mathrm{an,con}}(-w(q)))$ is perfect, i.e., it induces an isomorphism*

$$H^1_{F,\nabla}(M^\vee(-w(q))) \cong \mathrm{Hom}_K(H^0_{F,\nabla}(M), H^1_{F,\nabla}(\Gamma_{\mathrm{an,con}}(-w(q))).$$

PROOF. The argument consists of a series of reductions ending with an appeal to the calculation in Proposition 6.1.5. To begin with, by the snake and five lemmas, we may reduce to the case where M is irreducible; then M is isoclinic and quasi-constant by Lemma 6.2.1 (note that this relies on the full theory of slope filtrations). Let $s = c/d$ be the slope of M' written in lowest terms. Since k is algebraically closed, by Theorem 4.5.2, there exists a finite separable extension E' of E such that for $\Gamma'_{\mathrm{an,con}} = \Gamma^{E'}_{\mathrm{an,con}}$, $([d]_*M)(-c) \otimes \Gamma'_{\mathrm{an,con}}$ admits a basis of horizontal vectors.

Put $N = M \otimes \Gamma'_{\mathrm{an,con}}$ viewed as an (F,∇)-module over $\Gamma_{\mathrm{an,con}}$; then the trace from $\Gamma'_{\mathrm{an,con}}$ to $\Gamma_{\mathrm{an,con}}$ induces a projector on N with image M, and this map commutes with the Poincaré pairing. We may thus reduce to checking the perfectness of the Poincaré pairing for N instead of M.

In other words, we have reduced to the case where $([d]_*M)(-c)$ admits a basis of horizontal vectors. At this point, we may apply Proposition 3.4.7 to reduce to considering a standard Frobenius. Also, we may replace K by a Galois extension (since we can take traces down that extension); in particular, we can force K to contain the p^d-th roots of unity.

Since K contains the p^d-th roots of unity, we can form a trace for the morphism $\sigma^d : \Gamma_{\mathrm{an,con}} \to \Gamma_{\mathrm{an,con}}$, by averaging over automorphisms of $\Gamma_{\mathrm{an,con}}$ of the form $\{z_i\} \mapsto \zeta_i\{z_i\}$ for $\zeta_1, \dots, \zeta_m \in \mu_{p^d}$; this gives a trace map from $[d]^*[d]_*M$ to M. This means that to check perfectness of the pairing for M, it is enough to do so for $[d]^*[d]_*M$. Since the formation of H^0 and H^1 is insensitive to application of $[d]^*$, we are reduced to checking perfectness for $[d]_*M$.

However, Theorem 4.5.2 actually asserts that $([d]_*M)(-c)$ admits a basis of horizontal vectors *stable under* F^d. That is, as a (F^d,∇)-module, $[d]_*M$ splits up as a direct sum of copies of $\Gamma_{\mathrm{an,con}}(c)$. Once more by the snake lemma, we now reduce perfectness of the Poincaré pairing for $[d]_*M$

to perfectness for $\Gamma_{\mathrm{an,con}}(c)$. Since the latter follows from Proposition 6.1.5, we are done. $\square$

PROPOSITION 6.2.4. *Assume that k is algebraically closed. Let M_1, M_2 be irreducible (F,∇)-modules over $\Gamma_{\mathrm{an,con}}$, neither of which is isomorphic to a twist of the other. Then*

$$\mathrm{Hom}_{F,\nabla}(M_1, M_2) = \mathrm{Ext}^1_{F,\nabla}(M_1, M_2) = 0.$$

PROOF. If M_1 and M_2 are irreducible, then M_1 and M_2 are isomorphic if and only if $H^0_{F,\nabla}(M_1^\vee \otimes M_2) \neq 0$ if and only if $H^0_{F,\nabla}(M_2^\vee \otimes M_1) \neq 0$. Now Proposition 6.2.3 yields the desired result. $\square$

We may now refine the conclusion of Theorem 5.2.4 as follows.

DEFINITION 6.2.5. Let N be an irreducible (F,∇)-module. We say that another (F,∇)-module M is *N-typical* if M admits an exhaustive filtration by saturated (F,∇)-submodules, whose successive quotients are isomorphic to twists of N. If N is not to be specified, we say M is *isotypical*.

PROPOSITION 6.2.6. *Assume that k is algebraically closed. Let M be an (F,∇)-module over $\Gamma_{\mathrm{an,con}}$. Then M admits a unique direct sum decomposition $M_1 \oplus \cdots \oplus M_l$ into isotypical (F,∇)-submodules, such that each M_i is N_i-typical for some N_i, and no two N_i are twists of each other.*

PROOF. The uniqueness follows from the fact that there are no nonzero morphisms between (F,∇)-modules which are isotypical for irreducible modules which are not twists of each other; this follows by repeated application of Proposition 6.2.4.

We prove existence by induction on M. If M is irreducible, then M itself is isotypical. Otherwise, choose a short exact sequence $0 \to M_0 \to \to M \to N \to 0$ with M_0 irreducible. Decompose $N = \oplus N_i$ by the induction hypothesis, and let P_i be the preimage of N_i in M. For each i, if N_i is not M_0-typical, then the exact sequence $0 \to M_0 \to P_i \to N_i \to 0$ splits, again by repeated application of Proposition 6.2.4. This is true for all but possibly one i; we may thus split M as a direct sum of those N_i plus an M_0-typical factor. This completes the induction. $\square$

REMARK 6.2.7. One can doubtless refine Proposition 6.2.6 with more work. For instance, it should be possible to drop the restriction that k be algebraically closed. For another, it should be possible to show that an

M-typical (F,∇)-module is isomorphic to the tensor product of M with a unipotent (F,∇)-module; for true annuli, this amounts to a result of Matsuda [19, Theorem 7.8], which in turn mimics a result of Levelt [18] in the context of classical differential equations. (However, one must keep Remark 3.1.6 in mind: while Matsuda's result is purely about connections, one is compelled to use the Frobenius also when working on fake annuli.) This should in turn make it possible to construct a monodromy representation in this setting, as in [13, Theorem 4.45], and perhaps to relate it to some form of the Christol-Mebkhout construction, as in [13, Theorem 5.23]. The latter may be related to some conjectures of Matsuda; see [20].

6.3 – *Splitting exact sequences.*

LEMMA 6.3.1. *Let*

$$0 \to M_1 \to M \to M_2 \to 0 \tag{6.3.1.1}$$

be a short exact sequence of F-modules over $\Gamma_{\mathrm{con}}[\pi^{-1}]$ or $\Gamma_{\mathrm{an,con}}$, and put $d = \mathrm{rank}(M_1)$. Then the sequence splits if and only if the sequence

$$0 \to \wedge^d M_1 \to \wedge^d M \to (\wedge^d M)/(\wedge^d M_1) \to 0 \tag{6.3.1.2}$$

splits.

PROOF. If (6.3.1.1) splits, then (6.3.1.2) splits by the Künneth decomposition. Conversely, if (6.3.1.2) splits, let N be the image of $(\wedge^{d-1} M_1) \otimes M$ in $\wedge^d M$ under $\wedge$; then the exact sequence

$$0 \to \wedge^d M_1 \to N \to (\wedge^{d-1} M_1 \otimes M_2) \to 0$$

splits. Tensor with M_1 to obtain another split exact sequence

$$0 \to (M_1 \otimes \wedge^d M_1) \to (M_1 \otimes N) \to (M_1 \otimes \wedge^{d-1} M_1 \otimes M_2) \to 0.$$

Twisting by $(\wedge^d M_1)^\vee$, we obtain a split exact sequence

$$0 \to M_1 \to P \to (M_1 \otimes M_1^\vee \otimes M_2) \to 0$$

for some P. Take the trace component within $M_1 \otimes M_1^\vee$, tensor with M_2, and let Q be the inverse image in P; we then obtain yet another split exact sequence

$$0 \to M_1 \to Q \to M_2 \to 0.$$

By backtracking through the definitions, we see that this is none other than (6.3.1.1). □

PROPOSITION 6.3.2. *Suppose that σ is standard and that k is algebraically closed. Let M be an (F, ∇)-module over $\Gamma_{\mathrm{an,con}}$. Then any exact sequence $0 \to M_1 \to M \to M_2 \to 0$ in the category of F-modules splits.*

PROOF. By Lemma 6.3.1, we may assume that $\operatorname{rank}(M_1) = 1$; by twisting, we may assume that $M_1 \cong \Gamma_{\mathrm{an,con}}$. Also, since k is algebraically closed, we may assume that $\pi^n = q$ for some integer n.

Choose a coordinate system for Γ. Let $\boldsymbol{v}$ be an F-stable element of M_1; then if $\boldsymbol{w} = \Delta_1^{i_1} \cdots \Delta_m^{i_m}(\boldsymbol{v})$, we have $F(\boldsymbol{w}) = q^{-i_1 - \cdots - i_m}\boldsymbol{w}$. We thus obtain a nonzero map $N \to M$ for some unipotent (F, ∇)-module N, whose image contains $\boldsymbol{v}$.

Since k is algebraically closed, by Proposition 6.2.6, we can write M as a direct sum of isotypical (F, ∇)-submodules $P_1 \oplus \cdots \oplus P_l$. At most one of the P_i is isotypical for the trivial (F, ∇)-module $\Gamma_{\mathrm{an,con}}$; if P_j is one of the others, then the map $N \to P_j$ obtained by composing the map $N \to M_1$ and the projection $M \to P_j$ is zero by Proposition 6.2.4. We may thus reduce to the case where M is unipotent.

In this case, by repeated application of Proposition 6.1.5, we deduce that M is isomorphic as an F-module to a direct sum of twists of the trivial F-module. Then $\boldsymbol{v}$ must be a K_q-linear combination of F-stable generators of summands in this decomposition; from this observation, we may construct an F-stable complement of M_1, yielding the desired splitting. □

REMARK 6.3.3. The proof of Proposition 6.3.2 depends heavily on the hypothesis that σ is a standard Frobenius lift. Whether the result should even hold otherwise is not entirely clear.

6.4 – *Descent of morphisms.*

Following the philosophy of [5] (as imitated in [12]), we now parlay our splitting results into statements that let us descend morphisms of (F, ∇)-modules from Γ to $\Gamma_{\mathrm{con}}[\pi^{-1}]$.

DEFINITION 6.4.1. If M is an F-module over $\Gamma_{\mathrm{con}}[\pi^{-1}]$, we may associate to M two sets of HN slopes, by passing to $\Gamma[\pi^{-1}]$ and invoking Theorem 5.2.1 for the trivial valuation on E, or by passing to $\Gamma_{\mathrm{an,con}}$; we refer to these as the *generic HN slopes* and *special HN slopes*, respectively. Note that the Newton polygon of the special slopes always lies on or above that of the generic slopes, with the same endpoint [14, Proposition 5.5.1].

PROPOSITION 6.4.2. *Suppose that k is algebraically closed. Let $0 \to M_1 \to M \to M_2 \to 0$ be a short exact sequence of F-modules over $\Gamma_{\mathrm{con}}[\pi^{-1}]$, for σ a standard Frobenius lift, such that M acquires the structure of an (F,∇)-module over $\Gamma_{\mathrm{an,con}}$. Suppose that each generic HN slope of M_1 is greater than each generic HN slope of M_2. Then the exact sequence splits.*

PROOF. By applying Lemma 6.3.1 and taking duals, we may assume that $\mathrm{rank}(M_2) = 1$; by twisting, we may assume that $M_2 \cong \Gamma_{\mathrm{con}}[\pi^{-1}]$. Then the slopes condition is that M_1 has all generic HN slopes positive. By Proposition 6.3.2, the exact sequence

$$0 \to M_1 \otimes \Gamma_{\mathrm{an,con}} \to M \otimes \Gamma_{\mathrm{an,con}} \to \Gamma_{\mathrm{an,con}} \to 0$$

of F-modules splits; by [14, Proposition 7.4.2], the original sequence also splits. □

THEOREM 6.4.3. (a) *Let M be an F-module over $\Gamma_{\mathrm{con}}[\pi^{-1}]$, for σ a standard Frobenius lift, which acquires the structure of an (F,∇)-module over $\Gamma_{\mathrm{an,con}}$. Then the natural map*

$$H^0_F(M) \to H^0_F(M \otimes \Gamma[\pi^{-1}])$$

is a bijection.

(b) *Let M be an (F,∇)-module over $\Gamma_{\mathrm{con}}[\pi^{-1}]$. Then the natural map*

$$H^0_{F,\nabla}(M) \to H^0_{F,\nabla}(M \otimes \Gamma[\pi^{-1}])$$

is a bijection.

PROOF. In either case, there is no harm in assuming that k is algebraically closed.

(a) Given $\boldsymbol{v} \in H^0_F(M \otimes \Gamma[\pi^{-1}])$, we obtain an F-equivariant dual map $\phi : M^\vee \to \Gamma[\pi^{-1}]$. Let N_0 be the kernel of ϕ; by [14, Proposition 7.5.1] (applicable because any monomial field admits a valuation p-basis, namely any coordinate system), the preimage $N_1 = \phi^{-1}(\Gamma_{\mathrm{con}}[\pi^{-1}])$ has the property that $N_1/N_0 \cong \Gamma_{\mathrm{con}}[\pi^{-1}]$, and $M^\vee/N_1$ has all generic slopes negative.

By Proposition 6.3.2, N_1 admits an F-stable complement in $M^\vee$, so N_1/N_0 admits an F-stable complement P in $M^\vee/N_0$. However, the generic slopes of P are the same as those of $M^\vee/N_1$, so they are all negative. Thus the map $P \to \Gamma[\pi^{-1}]$ obtained by composition with ϕ is forced to vanish by [14, Proposition 7.5.1], whereas $\phi : M^\vee/N_0 \to \Gamma[\pi^{-1}]$ is injective. We must then have $P = 0$ and $N_1 = M^\vee$, so ϕ maps into $\Gamma_{\mathrm{con}}[\pi^{-1}]$ and $\boldsymbol{v} \in M$, as desired.

(b) By Proposition 3.4.7, there is no harm in reducing to the case of σ standard. The result now follows from (a). □

REMARK 6.4.4. Note that even for true annuli, Theorem 6.4.3 makes an assertion (namely (a)) not covered by [12].

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