REND. SEM. MAT. UNIV. PADOVA, Vol. 117 (2007)

Cell Decomposition for Two Dimensional Local Fields.

ALI BLEYBEL (*)

ABSTRACT - We prove a cell decomposition theorem for the two-dimensional local field $\boldsymbol{Q}_p((t))$.

Introduction.

The purpose of this paper is to give a cell decomposition for the field of Laurent series on p-adic fields. Originally, cell decomposition theorems were used to prove rationality of Igusa Zeta function, and of Poincaré series, avoiding resolution of singularities. Also, using cell decomposition for a p-adic field (for p a fixed prime), Denef (1986) gave a new proof of elimination of quantifiers in $\boldsymbol{Q}_p$. Later, Pas (1989) proved a uniform (in p) cell decomposition for p-adic fields, thus obtaining a *uniform* quantifier elimination.

In this paper we prove a cell decomposition for $\boldsymbol{Q}_p((t))$ (or for $K((t))$ where K is a finite algebraic extension of $\boldsymbol{Q}_p$), using both the t-adic and p-adic valuations. Our proof uses a mixture of Denef and Pas results, and may be useful for the development of motivic integration on the two-dimensional local field $\boldsymbol{Q}_p((t))$, in a similar way to the work of Cluckers & Loeser (2004) (hereafter CL 2004). Note that integration over higher dimensional local fields was also addressed by Hrushovski & Kazhdan (2005).

In section 1 we recall the language of 2-valued fields used in the paper, as well of the definitions for definable sets and cells. In section 2 we state Cell Decomposition I together with its proof. This section also contains a lemma needed in the proofs of theorems I and II. Theorem II (Cell decomposition II) is stated and proved in section 3. Finally a generalized cell decomposition theorem is stated and proved in section 4.

(*) Indirizzo dell'A.: Lebanese University, Faculty of Sciences, Beirut, Lebanon
E-mail: bleybel@etu.upmc.fr

Acknowledgments. I gratefully thank Francois Loeser who suggested the problem and pointed out to me the basic idea for the solution. I also thank Raf Cluckers for fruitful discussions, and an anonymous referee for constructive comments.

1. Language of 2-valued fields.

An n-dimensional local field is a complete discrete valuation field F whose residue field is $n-1$-dimensional local field (Fesenko 2003). Let $\bar{K}$ be a 2-dimensional local field, also named 2-valued field. $\bar{K}$ is equipped with a valuation $\mathrm{ord}_1 : \bar{K}^\times \to \Gamma$ for some ordered abelian group Γ, $\bar{R}$ its valuation ring with residue field K. K is also a valued field, with valuation $\mathrm{ord}_2 : K \to \Sigma$ with Σ an ordered abelian group, R its valuation ring and k the residue field. There is a projection $\mathrm{res} : \bar{R} \to K = \bar{R}/\bar{P}$ of the valuation ring onto the residue field, where $\bar{P}$ is the maximal ideal of $\bar{R}$. We also define an angular component map $\overline{\mathrm{ac}} : \bar{K}^\times \to K^\times$. $\overline{\mathrm{ac}}$ is a multiplicative map which can be extended by putting $\overline{\mathrm{ac}}(0) = 0$. The language adopted is a multi-sorted language

$$\mathcal{L} = (\boldsymbol{L}_{\mathrm{Val}}, \boldsymbol{L}_{\mathrm{RV}}, \boldsymbol{L}_{\mathrm{Ord}_1}, \boldsymbol{L}_{\mathrm{Ord}_2}, \mathrm{ord}_1, \mathrm{ord}_2, \overline{\mathrm{ac}})$$

with 4-sorts

(i) a Val-sort for the 2-valued field sort,
(ii) an RV-sort for the 1-valued field sort,
(iii) an Ord_1-sort for the value group (with respect to the first valuation map) sort, and
(iv) an Ord_2-sort for the value group (with respect to the second valuation map) sort,

where $\boldsymbol{L}_{\mathrm{Val}}$ is the language of rings $\boldsymbol{L}_{\mathrm{Rings}} = (+, -, ., 0, 1)$ and $\boldsymbol{L}_{\mathrm{RV}}$ is an expansion of Macintyre's language $\boldsymbol{L}_{\mathrm{Mac}} \equiv \boldsymbol{L}_{\mathrm{Rings}} \cup \{P_n | \, n \in \boldsymbol{N}, n > 1\}$ where P_n are predicates whose interpretations are the set of nonzero n^{th}-power) and $\boldsymbol{L}_{\mathrm{Ord}_1}$ (and $\boldsymbol{L}_{\mathrm{Ord}_2}$) is an expansion of the language of ordered groups, for instance $\mathcal{L}_{PR\infty}$, a variant of the Presburger language $\mathcal{L}_{PR} \equiv \{+, 0, 1, \leq\} \cup \{\equiv_n \mid n \in \boldsymbol{N}, n > 1\}$ defined by $\mathcal{L}_{PR\infty} \equiv \mathcal{L}_{PR} \cup \{\infty\}$. A structure for this language consists of a tuple $(\bar{K}, K, \Gamma, \Sigma)$ where $\bar{K}$ is a 2-valued field with residue field K, value group Γ valuation map ord_1 and an angular component map $\overline{\mathrm{ac}}$, and K is a valued field with a value group Σ and a valuation map ord_2, together with an interpretation of $\boldsymbol{L}_{\mathrm{RV}}$ in K, and an interpretation of $\boldsymbol{L}_{\mathrm{Ord}_1}$ and $\boldsymbol{L}_{\mathrm{Ord}_2}$ in Γ and Σ respectively.

A formula in this language is built up from symbols of $\mathcal{L}$ together with variables, the logical connectives $\wedge$ (and), $\vee$ (or), $\neg$ (not), the quantifiers $\exists$, $\forall$, the equality symbol $=$ and parameters.

When $\bar{K} = K((t))$ for some valued field K, there exists a natural valuation map $\mathrm{ord}_1 := \mathrm{ord}_t : K((t))^{\times} \to \boldsymbol{Z}$ (extended by putting $\mathrm{ord}_t(0) = \infty$) and a natural angular component map $\overline{\mathrm{ac}} : \sum_{i \geq l} a_i t^i \mapsto a_l \neq 0$. The valued field K is assumed Henselian of zero characteristic.

If we interpret $\equiv_n$ in $\boldsymbol{L}_{\mathrm{Ord}_1}$ and $\boldsymbol{L}_{\mathrm{Ord}_2}$ as "congruent modulo n" in $\boldsymbol{Z}$ then $(\boldsymbol{Q}_p((t)), \boldsymbol{Q}_p, \boldsymbol{Z}, \boldsymbol{Z})$ is a structure for the language $\mathcal{L}$, with the natural p-adic valuation ord_p as an interpretation for ord_2 (where we also extend ord_p by $\mathrm{ord}_p(0) = \infty$).

Also, note that the map res is definable in this language (in a non-canonical way) (see the remark under definition 2.2 and the lemma 3.4 in (Pas 1989)).

Notice the analogy of the language $\mathcal{L}$ with the three-sorted Denef-Pas language $\mathcal{L}_{\mathrm{DP}}$ (Pas 1989).

Finally, we will use the fact that if $(\bar{K}, K, \Gamma, \Sigma)$ is a structure for $\mathcal{L}$ then (K, Σ) is a structure for the 2-sorted language $\mathcal{L}_{\mathrm{Mac}} = (\boldsymbol{L}_{\mathrm{RV}}, \boldsymbol{L}_{\mathrm{Ord}_2})$.

Consider now the $\mathcal{L}$-theory $\boldsymbol{T}_2$ of 2-valued Henselian fields of zero characteristic and having surjective valuation maps, surjective angular component map, and 1-valued Henselian field of characteristic zero and bounded ramification ($\mathrm{ord}_2(p) = 1$ for some prime p).

In this context we have the following variant of Denef-Pas Theorem on elimination of 2-valued field quantifiers

THEOREM 1.1. *The theory $\boldsymbol{T}_2$ admits elimination of quantifiers in the 2-valued field sort. More precisely, every $\mathcal{L}$ formula $\phi(x, \xi, \alpha, \beta)$, with x variables in the* Val-*sort, ξ variables in the* RV-*sort, α variables in the* Ord_1-*sort and β variables in the* Ord_2-*sort, is $\boldsymbol{T}_2$ equivalent to a finite disjunction of formulas of the form*

$$\psi(\overline{\mathrm{ac}}\, f_1(x), \ldots, \overline{\mathrm{ac}}\, f_k(x), \xi, \beta) \wedge \theta(\mathrm{ord}_1 f_1(x), \ldots, \mathrm{ord}_1 f_k(x), \alpha, \beta),$$

with ψ a $(\boldsymbol{L}_{\mathrm{RV}} \cup \boldsymbol{L}_{\mathrm{Ord}_2})$-formula, θ a $(\boldsymbol{L}_{\mathrm{Ord}_1} \cup \boldsymbol{L}_{\mathrm{Ord}_2})$-formula and $f_1, \ldots, f_k$ polynomials in $\boldsymbol{Z}[X]$.

Note the analogy of our statement with theorem 2.1.1 of (CL 2004).

PROOF. Direct application of theorem 4.1 and lemma 5.3 of Pas (1989). □

A subset C of $\bar{K}^m \times K^n \times \Gamma^{r_1} \times \Sigma^{r_2}$(where m, n, r_1 and r_2 are positive integers) is called definable if it is definable by an $\mathcal{L}$-formula.

A function f is definable if its graph is a definable subset.

A subset D of $K^n \times \Sigma^r$ is $\mathcal{L}_{\mathrm{Mac}}$-definable if it is definable by an $\mathcal{L}_{\mathrm{Mac}}$-formula.

An $\mathcal{L}_{\mathrm{Mac}}$-definable subset of K^n (when K is a p-adic field, that is a finite algebraic extension of $\boldsymbol{Q}_p$) is also a semi-algebraic set in the standard terminology, e.g. Denef (1984).

REMARK 1.2. It should be noted that even if $\Gamma = \Sigma = \boldsymbol{Z}$, it is not allowed to mix variables of Γ-sort with variables of Σ-sort.

DEFINITION 1.3. Let $x = (x_1, \dots, x_m)$ be Val-variables, $\xi = (\xi_1, \dots, \xi_n)$ RV-variables. Let C be a definable subset of $\bar{K}^m \times K^n$. Let b_1, b_2, c be definable functions from C to $\bar{K}$, λ_1, λ_2 positive integers, d_1, d_2, e definable functions from $\mathrm{Proj}_{K^n} C$ (the image of C by the projection of $\bar{K}^m \times K^n$ onto K^n) to K, and let $\diamondsuit_1$, $\diamondsuit_2$, $\square_1$ and $\square_2$ be $<$, $\leq$, or no condition.

For each $\xi \in K^n$, let $A(\xi)$ be the set of $(x, T) \in \bar{K}^m \times \bar{K}$ subject to the definable conditions

$$\Big\{(x,T) \in \bar{K}^m \times K | (x,\xi) \in C,\ \mathrm{ord}_1 b_1(x,\xi) \diamondsuit_1 \lambda_1 \cdot \mathrm{ord}_1(T - c(x,\xi)) \diamondsuit_2 \mathrm{ord}_1 b_2(x,\xi), \\ \mathrm{ord}_2\, d_1(\xi) \square_1 \lambda_2 \cdot \mathrm{ord}_2(\overline{\mathrm{ac}}(T - c(x,\xi)) - e(\xi)) \square_2 \mathrm{ord}_2\, d_2(\xi)\Big\}$$

and suppose that for all $\xi, \xi' \in K^n$ with $\xi \neq \xi'$ $A(\xi) \cap A(\xi') = \oslash$, then

$$A = \bigcup_{\xi} A(\xi)$$

is a *cell* in $\bar{K}^m \times K^n$ with parameters $(\xi_1, \dots, \xi_n)$, primary center $c(x, \xi)$ and secondary center $e(\xi)$; $A(\xi)$ is a *fiber* of the cell A.

2. Cell decomposition I.

Now consider a model for the theory $\boldsymbol{T}_2$ such that $\bar{K} = K((t))$ and $\Gamma = \Sigma = \boldsymbol{Z}$.

THEOREM 2.1. *Let $f(x, T)$ be a polynomial in T of degree d whose coefficients are definable functions in $x \in K((t))^m$, $\xi \in K^n$.*

Then there exists a partition of $K((t))^m \times K((t))$ in a finite number of cells A with parameters $(\xi_1, \dots, \xi_n, \xi'_1, \dots, \xi'_r) \in K^{n+r}$, each cell has primary and *secondary centers* $c(x,\xi)$ *and* $e(\xi,\xi')$ *respectively such that if we write*

$$f(x,T)=\sum_{i=0}^{d} a_i(x,\xi)(T-c(x,\xi))^i \tag{2.1.1}$$

then for all $(\xi,\xi')\in K^{n+r}$ *and for all* $(x,T)\in A(\xi,\xi')$ *we have*

$$\operatorname{ord}_t f(x,T)=\operatorname{ord}_t a_{i_0}(x,\xi)(T-c(x,\xi))^{i_0}=\min_{0\le i\le d}\operatorname{ord}_t a_i(x,\xi)(T-c(x,\xi))^i \tag{2.1.2}$$

where i_0 *does not depend on* (x,ξ,ξ',T)*, and*

$$\operatorname{ord}_p(\overline{\operatorname{ac}}\, f(x,T))\le\min_{0\le i\le d}\operatorname{ord}_p\Big(b_i(\xi,\xi')\big(\overline{\operatorname{ac}}(T-c(x,\xi))-e(\xi,\xi')\big)^i\Big)+l \tag{2.1.3}$$

for some $l\in \boldsymbol{N}$*;* $b_i(\xi,\xi')$ *are (partial)* $\mathcal{L}_{\mathrm{Mac}}$*-definable functions* $K^{n+r}\to K$ *(to be defined below) and* $\xi'_1,\dots,\xi'_r\in K$ *such that* $\xi'_1=\overline{\operatorname{ac}}\, f_1(x,\xi),\dots,\xi'_r=$ $=\overline{\operatorname{ac}}\, f_r(x,\xi)$ *where* $f_1(x,\xi),\dots,f_r(x,\xi)$ *are polynomials in* x *with integer coefficients.*

Proof. We assume the result for all polynomials of degree $<d$. The theorem then holds for $f'(x,T)$ (derivative of $f(x,T)$ with respect to T), and so there exists a partition of $K((t))^m\times K((t))$ in cells $A=\bigcup_{\xi,\xi'}A(\xi)$ (of centers $c(x,\xi)$ and $e(\xi,\xi')$) as above such that

$$\operatorname{ord}_t f'(x,T)=\min_{1\le i\le d}\operatorname{ord}_t\, i a_i(x,\xi)(T-c(x,\xi))^{i-1} \tag{2.1.4}$$

and

$$\operatorname{ord}_p(\overline{\operatorname{ac}}\, f'(x,T))\le \le\min_{0\le i\le d-1}\operatorname{ord}_p\Big(b'_i(\xi,\xi')\times\big(\overline{\operatorname{ac}}(T-c(x,\xi))-e(\xi,\xi')\big)^i\Big)+l' \tag{2.1.5}$$

where $b'_i(\xi,\xi')$ are partial definable functions $K^{n+r}\to K$, $l'\in\boldsymbol{N}$, $\xi'_j=f'_j(x,\xi)$ and $f'_j(x,\xi)$ are polynomials in x with integer coefficients.

For each $\xi\in K^n$, let $A(\xi)$ be defined by

$$\Big\{(x,T)\in K((t))^m\times K\,|\,(x,\xi)\in C,\ \operatorname{ord}_t b_1(x,\xi)\diamondsuit_1\lambda_1\cdot\operatorname{ord}_t(T-c(x,\xi))\diamondsuit_2\operatorname{ord}_t b_2(x,\xi),$$
$$\operatorname{ord}_p d_1(\xi)\,\square_1\,\lambda_2\cdot\operatorname{ord}_p(\overline{\operatorname{ac}}(T-c(x,\xi))-e(\xi))\,\square_2\operatorname{ord}_p d_2(\xi)\Big\}$$

where $\lambda_1,\lambda_2,C,b_1,b_2,d_1,d_2,c$ and e are as in (1.3) and where the RV-variables ξ' were added to the parameters ξ (by defining e.g. $c'(x,\xi,\xi')\equiv c(x,\xi)$ for all $(x,\xi,\xi')\in K((t))^m\times K^n\times K^r$ and replacing $n+r$ by n', and then renaming c' as c and n' as n).

We will further partition A into subcells on which the theorem holds for $f(x,T)$.

Consider only the nontrivial case, where the set $\tilde{I}$ of j such that

$$\mathrm{ord}_t a_j(x,T)(T-c(x,\xi))^j = \min_{0\le i\le d} \mathrm{ord}_t a_i(x,\xi)(T-c(x,\xi))^i \tag{2.1.6}$$

has a cardinality > 1.

Let $i_0 \in \tilde{I}$; then each fiber $A(\xi)$ is a disjoint union of two sets

$$A_1(\xi) = \left\{(x,T) \in A(\xi) \mid \mathrm{ord}_t f(x,T) = \mathrm{ord}_t a_{i_0}(x,\xi)(T-c(x,\xi))^{i_0}\right\} \tag{2.1.7}$$

and

$$A_2(\xi) = \left\{(x,T) \in A(\xi) \mid \mathrm{ord}_t f(x,T) > \mathrm{ord}_t a_{i_0}(x,\xi)(T-c(x,\xi))^{i_0}\right\}. \tag{2.1.8}$$

Consider the polynomials $F_j(\zeta)$ given by

$$F_j(\zeta) = \sum_{i\in\tilde{I}} g_{ij}(\xi,\xi')\zeta^i \tag{2.1.9}$$

where g_{ij} are $\mathcal{L}_{\mathrm{Mac}}$-definable functions $K^n \times K^r \to K$, such that

$$\overline{\mathrm{ac}}(a_i(x,\xi)) = g_{ij}(\xi, \overline{\mathrm{ac}}\, f_1(x,\xi), \ldots, \overline{\mathrm{ac}}\, f_r(x,\xi))$$

for $(x,\xi) \in X_j$, where $(X_j)_j$ form a partition of $K((t))^m \times K^n$, and $f_1, \ldots, f_r$ are polynomials in x with integer coefficients, as in lemma 2.2 below.

As the functions g_{ij} are definable in $\mathcal{L}_{\mathrm{Mac}}$, we can apply the cell decomposition theorem I of Denef (1986) to $F_j(\zeta)$; it follows that there exists a finite partition of K^{n+r+1} in Denef-type cells B_j (of center $e_j(\xi,\xi')$, $(\xi,\xi') \in K^{n+r}$) defined by

$$B_j = \left\{(\xi,\xi',\zeta) \in K^{n+r} \times K \mid (\xi,\xi') \in D_j, \mathrm{ord}_p c_{1j}(\xi,\xi') \square_{1j} \mathrm{ord}_p(\zeta - e_j(\xi,\xi')) \square_{2j} \mathrm{ord}_p c_{2j}(\xi,\xi')\right\},$$

where $D_j \subset K^{n+r}$ is a $\mathcal{L}_{\mathrm{Mac}}$-definable set, and $c_{1j}(\xi,\xi')$, $c_{2j}(\xi,\xi')$ and $e_j(\xi,\xi')$ are

$\mathcal{L}_{\mathrm{Mac}}$-definable functions $K^{n+r} \to K$, such that

$$\mathrm{ord}_p F_j(\zeta) \le \min_{0\le i\le d} \mathrm{ord}_p(b_{ij}(\xi,\xi')(\zeta - e_j(\xi,\xi'))^i) + l_j \tag{2.1.10}$$

with $l_j \in \boldsymbol{N}$; $b_{ij}(\xi,\xi')$ are the coefficients of $F_j(\zeta)$ written in the form

$$F_j(\zeta) = \sum_i b_{ij}(\xi,\xi')(\zeta - e_j(\xi,\xi'))^i, \tag{2.1.11}$$

and $\square_{1j}, \square_{2j}$ (for each j) is $<, \le$ or no condition.

As $\zeta = e_j(\xi, \xi')$ if and only if $\operatorname{ord}_p(\zeta - e_j(\xi, \xi')) = \infty$, it follows that by the above observation, $(x, T) \in A_2$ if and only if

$$F_j(\overline{\mathrm{ac}}(T - c(x, \xi)) = 0$$

for some j, if and only if

$$\overline{\mathrm{ac}}(T - c(x, \xi)) = e_j(\xi, \overline{\mathrm{ac}}\, f_1(x, \xi), \ldots, \overline{\mathrm{ac}}\, f_r(x, \xi))$$

and $b_{0j}(\xi, \overline{\mathrm{ac}}\, f_1(x, \xi), \ldots, \overline{\mathrm{ac}}\, f_r(x, \xi)) = 0$.

It follows that $A_1(\xi)$ can be written as a finite union of sets of the following form

$$\bigcup_{\xi'} \Big\{(x, T) \in A | (x, \xi, \xi') \in Y_j, \overline{\mathrm{ac}}(T - c(x, \xi)) \neq e_j(\xi, \xi'),$$
$$\operatorname{ord}_p c_{1j}(\xi, \xi') \square_{1j} \operatorname{ord}_p(\overline{\mathrm{ac}}(T - c(x, \xi)) - e_j(\xi, \xi')) \square_{2j} \operatorname{ord}_p c_{2j}(\xi, \xi')\Big\}$$

or

$$\bigcup_{\xi'} \Big\{(x, T) \in A |\ (x, \xi, \xi') \in Z_j, b_{0j}(\xi, \xi') \neq 0, \overline{\mathrm{ac}}(T - c(x, \xi)) = e_j(\xi, \xi')\Big\}.$$

Also it follows that $A_2(\xi)$ is a finite union of sets of the form

$$\bigcup_{\xi'} \Big\{(x, T) \in A |\ (x, \xi, \xi') \in Z_j, b_{0j}(\xi, \xi') = 0, \overline{\mathrm{ac}}(T - c(x, \xi)) = e_j(\xi, \xi')\Big\},$$

where

$$Y_j = \{(x, \xi, \xi') \in K((t))^m \times K^n \times K^r | (x, \xi) \in X_j, (\xi, \xi') \in D_j,$$
$$\overline{\mathrm{ac}}\, f_1(x, \xi) = \xi'_1, \ldots, \overline{\mathrm{ac}}\, f_r(x, \xi) = \xi'_r\},$$

and

$$Z_j = \{(x, \xi, \xi') \in K((t))^m \times K^n \times K^r | (x, \xi) \in X_j, (\xi, \xi') \in D_j,$$
$$\overline{\mathrm{ac}}\, f_1(x, \xi) = \xi'_1, \ldots, \overline{\mathrm{ac}}\, f_r(x, \xi) = \xi'_r, c_{1j}(\xi, \xi') \diamond_{1j} 0, c_{2j}(\xi, \xi') \diamond_{2j} 0\}$$

with $\diamond_{1j}$ is $\neq$ or no condition and $\diamond_{2j}$ is $=$ or no condition.

Note that one of the centers $e(\xi, \xi')$ or $e_j(\xi, \xi')$ in the above description can be eliminated, (or both of them and a new center introduced, see the proof of theorem II below), whence each of the sets A_1 and A_2 is a finite union of cells.

On A_1 the theorem is easily seen to hold, as

$$\overline{\mathrm{ac}} f(x, T) = F_j(\overline{\mathrm{ac}}(T - c(x, \xi))). \tag{2.1.12}$$

On A_2, notice that as the condition

$$\overline{\mathrm{ac}}(T - c(x,\xi)) = e_j(\xi, \overline{\mathrm{ac}}\, f_1(x,\xi), \ldots, \overline{\mathrm{ac}}\, f_r(x,\xi))$$

holds, thus we may follow the steps of Pas' proof on pages (148-154).

The crucial point in these steps is to find a new center $d(x,\xi)$ for the cell A_2 such that

$$f(x,T) = f'(x, d(x,\xi))(T - d(x,\xi)) + \sum_{j=2}^{d} \frac{f^{(j)}(x, d(x,\xi))}{j!}(T - d(x,\xi))^j.$$

This entails that

$$\begin{aligned}\mathrm{ord}_t\, f(x,T) =& \mathrm{ord}_t\, f'(x, d(x,\xi))(T - d(x,\xi)) \\ &= \min_{1\le j\le d} \mathrm{ord}_t \Big(\frac{f^{(j)}(x, d(x,\xi))}{j!}(T - d(x,\xi))^j\Big)\end{aligned}$$

on A_2, and

$$\overline{\mathrm{ac}}\, f(x,T) = \overline{\mathrm{ac}}\, f'(x, d(x,\xi))\overline{\mathrm{ac}}\,(T - d(x,\xi)),$$

so,

$$\mathrm{ord}_p\, \overline{\mathrm{ac}}\, f(x,T) = \mathrm{ord}_p\, \overline{\mathrm{ac}}\, f'(x, d(x,\xi)) + \mathrm{ord}_p\, \overline{\mathrm{ac}}(T - d(x,\xi))$$

and the second statement of the theorem holds on A_2 (by eliminating one of the (secondary) centers e or 0, and by observing that $\mathrm{ord}_p\, \overline{\mathrm{ac}}\, f'(x,T)$ is bounded on A, as $\overline{\mathrm{ac}}\, f'(x,T) \neq 0$ on A_2 and using 2.1.5). □

The following statement should be folklore; nevertheless we provide a detailed proof.

LEMMA 2.2. *Let f_i $(i = 1, \ldots, l)$, be definable functions*

(2.2.1) $$f_i : C \to K((t))$$

where C is a definable subset of $K((t))^m \times K^n$. Then there exists a partition of $K((t))^m \times K^n$ into definable subsets X_j, such that

(2.2.2) $$\overline{\mathrm{ac}} \circ f_i(x,\xi) = g_{ij}(\xi, \overline{\mathrm{ac}}\, h_1(x,\xi), \ldots, \overline{\mathrm{ac}}\, h_r(x,\xi))$$

for all $(x,\xi) \in X_j$, and where $h_1, \ldots, h_r$ are polynomials with integer coefficients in x and g_{ij} is a $\mathcal{L}_{\mathrm{Mac}}$-definable function from a $\mathcal{L}_{\mathrm{Mac}}$-definable subset $C' \subset K^{n+r}$ into K, with r is a positive integer.

PROOF. Let us consider first the case $l = 1, f := f_1$. As f is a definable function, $\overline{\mathrm{ac}} \circ f$ is also definable and its graph is defined by an $\mathcal{L}$-formula

$\psi(x_1,\dots,x_m,\xi_1,\dots,\xi_n,\zeta)$ in m Val-variables $(x_1,\dots,x_m)$ and $n+1$ RV-variables $\xi_1,\dots,\xi_n,\zeta$;

$$\psi(x_1,\dots,x_m,\xi_1,\dots,\xi_n,\zeta)\equiv\Big(\overline{\mathrm{ac}}\circ f(x_1,\dots,x_m,\xi_1,\dots,\xi_n)=\zeta\Big). \tag{2.2.3}$$

In ψ atomic formulas of the form $h(x_1,\dots,x_m)=0$ (where h is a polynomial in $(x_1,\dots,x_m)$ with integer coefficients) can be replaced by $\overline{\mathrm{ac}}\,h(x_1,\dots,x_m)=0$. We may suppose then that the variables $x_1,\dots,x_m$ appear in ψ only through the RV-terms $\overline{\mathrm{ac}}f_i(x,\xi)$ and the Ord_1-terms $\mathrm{ord}_t\,h_j(x,\xi)$, $(i=1,\dots,r;j=1,\dots,s)$. Let ϕ be the formula obtained by replacing in ψ $\overline{\mathrm{ac}}\,f_i(x,\xi)$ by a RV-variable ρ_i $(i=1,\dots,r)$, and $\mathrm{ord}_t\,h_j(x,\xi)$ by a Ord_1-variable l_j, $(j=1,\dots,s)$. Then 2.2.3 is equivalent to

$$(\exists\rho)(\exists l)\Bigg[\phi(\xi,\zeta,\rho_1,\dots,\rho_r,l_1,\dots,l_s)\wedge\Big(\bigwedge_{i=1}^{r}\overline{\mathrm{ac}}\,f_i(x,\xi)=\rho_i\Big)\wedge \wedge\Big(\bigwedge_{j=1}^{s}\mathrm{ord}_t\,h_j(x,\xi)=l_j\Big)\Bigg]$$

Notice that ϕ defines the graph of a (partial) function

$$g:K^n\times K^r\times\Gamma^s\to K \tag{2.2.4}$$

whose domain is given by

$$D=\Bigg\{(\xi,\rho_1,\dots,\rho_r,l_1,\dots,l_s)\in K^{n+r}\times\Gamma^s|(\exists x\,(x,\xi)\in C)\ \Big(\bigwedge_{i=1}^{r}\overline{\mathrm{ac}}\,f_i(x,\xi)=\rho_i\Big)\wedge\Big(\bigwedge_{j=1}^{s}\mathrm{ord}_t\,h_j(x,\xi)=l_j\Big)\Bigg\}$$

The 2-valued-field quantifiers in the above description can be eliminated, whence the domain of g is definable in the language $\mathcal{L}_{\mathrm{Mac}}$.

Let now $\zeta,\zeta'\in K$ be such that $\phi(\xi,\zeta,\rho_1,\dots,\rho_r,l_1,\dots,l_s)\wedge \wedge\,\phi(\xi,\zeta',\rho_1,\dots,\rho_r,l_1,\dots,l_s)$ holds for some $(\xi,\rho_1,\dots,\rho_r,l_1,\dots,l_s)\in D$, then we have

$$\exists x\big[\psi(x,\xi,\zeta)\wedge\psi(x,\xi,\zeta')\big]$$

and so $\zeta=\zeta'$.

Apply now theorem (1.1) to ϕ, to get

$$\phi(\xi,\zeta,\rho_1,\dots,\rho_r,l_1,\dots,l_s)\Leftrightarrow\bigvee_{j=1}^{N}\Big(\chi_j\wedge\theta_j\Big) \tag{2.2.5}$$

where χ_j is an $\mathcal{L}_{\mathrm{Mac}}$-formula, and θ_j is an $\boldsymbol{L}_{\mathrm{Ord}_1}$-formula. We can verify then that χ_j defines the graph of a partial function $g_j : K^n \times k^r \to K$. Assume that $\chi_j(\xi, \zeta, \rho_1, \ldots, \rho_r)$ and $\chi_j(\xi, \zeta', \rho_1, \ldots, \rho_r)$ hold, then $\chi_j(\xi, \zeta, \rho_1, \ldots, \rho_r) \wedge \theta_j(l_1, \ldots, l_s)$ and $\chi_j(\xi, \zeta', \rho_1, \ldots, \rho_r) \wedge \theta_j(l_1, \ldots, l_s)$ also hold and then by the above result we should have $\zeta = \zeta'$.

Finally let

$$X_j = \big\{(x, \xi) \in C \mid \theta_j(\mathrm{ord}_t(h_1(x, \xi)), \ldots, \mathrm{ord}_t(h_s(x, \xi)))\big\}, \tag{2.2.6}$$

then, for all $(x, \xi) \in X_j$

$$\overline{\mathrm{ac}}\, f(x, \xi) = g_j(\xi_1, \ldots, \xi_n, \overline{\mathrm{ac}}\, f_1(x, \xi), \ldots, \overline{\mathrm{ac}}\, f_r(x, \xi)). \tag{2.2.7}$$

The case $l > 1$ is trivially proved by simultaneous applications of the lemma to each function f_i separately and taking intersections.

3. Cell decomposition II.

Analogously to Denef (1986) and Pas (1989), we prove the cell decomposition theorem II, which relies on theorem I.

THEOREM 3.1. *Let $f_1(x, T), \ldots, f_r(x, T)$ be polynomials as in theorem I, and $n_1 \in \boldsymbol{N}^*$. Then there exists a finite partition of $K((t))^m \times K((t))$ in cells. Each such cell A has parameters $(\xi_1, \ldots, \xi_l)$ and primary and secondary centers $c(x, \xi)$ and $e(\xi)$ respectively such that, for all $\xi \in K^n$ and for all $(x, T) \in A(\xi)$*

$$\mathrm{ord}_t\, f_i(x, T) = \mathrm{ord}_t(h_i(x, \xi)(T - c(x, \xi))^{\mu_i})$$

and

$$\overline{\mathrm{ac}} f_i(x, T) = u_i(\overline{\mathrm{ac}}(T - c(x, \xi)), \xi)^{n_1} g_i(\xi)(\overline{\mathrm{ac}}(T - c(x, \xi)) - e(\xi))^{\upsilon_i}$$

for $i = 1, \ldots, r$, and where $h_i(x, \xi), g_i(\xi)$ are definable functions to $K((t))$ and K respectively; $\mathrm{ord}_p u = 0$ and μ_i, υ_i are non-negative integers that do not depend on (x, ξ, T).

PROOF. Consider first the case $r = 1; f(x, T) := f_1(x, T)$. In the proof of theorem I, we realize that we can partition $K((t))^m \times K((t))$ in cells A on which we have

$$\overline{\mathrm{ac}} f(x, T) = \overline{\mathrm{ac}}\, a_{i_0}(x, \xi) \overline{\mathrm{ac}}\,(T - c(x, \xi))^{i_0}$$

or

$$\overline{\mathrm{ac}}\, f(x,T)=\sum_{i\in\tilde{I}}\overline{\mathrm{ac}}\, a_i(x,\xi)\overline{\mathrm{ac}}\,(T-c(x,\xi))^i$$

where $\tilde{I}$ is as in the proof of theorem 2.1. Clearly we need only to consider the case where the cardinality of $\tilde{I}$ is greater than 1; consider then the polynomials

$$F_j(\xi,\xi',\zeta)=\sum_{i\in\tilde{I}} g'_{ij}(\xi,\xi')\zeta^i$$

where

$$g'_{ij}(\xi,\overline{\mathrm{ac}}\, f_1(x,\xi),\dots,\overline{\mathrm{ac}}\, f_r(x,\xi))=\overline{\mathrm{ac}}\, a_i(x,\xi)$$

for $(x,\xi)\in X_j$, $i=1\in\tilde{I}$ and where X_j $(j=1,\dots,s)$ is a definable subset of $K((t))^m\times K^n$, and $f_1,\dots,f_r$ are polynomials in $x\in K((t))$ with integer coefficients as in lemma (3.2) above. We can apply theorem II of Denef (1986) separately to each of the polynomials $F_j(\xi,\xi',\zeta)$ and then substitute $\overline{\mathrm{ac}}(T-c(x,\xi))$ for ζ to get the desired result.

Consider then the case $r>1$.

Let us now consider the following statement $P(A,s)$:

A is the intersection of s cells, with parameters $\xi=(\xi_1,\dots,\xi_n)$ and centers $c_1(x,\xi),\dots,c_s(x,\xi)$ and $e_1(\xi),\dots,e_s(\xi)$ respectively. Denote by $A(\xi)$ the intersection of the fibers of the cells of which A is the intersection. For all ξ, for all $(x,T)\in A(\xi)$, and for $i=1,\dots,r$ we have

$$\mathrm{ord}_t\, f_i(x,T)=\mathrm{ord}_t(h_i(x,\xi)(T-c_{\eta(i)}(x,\xi))^{\mu_i})$$

$$\overline{\mathrm{ac}} f_i(x,T)=\sum_{k\in I_i}\overline{\mathrm{ac}}(a_{ki}(x,\xi))\overline{\mathrm{ac}}\,((T-c_{\eta(i)}(x,\xi))^k)$$

where the $h_i(x,\xi)$ are definable functions, and the non-negative integers μ_i, the map $\eta:\{1,\dots,r\}\to\{1,\dots,s\}$ does not depend on (x,ξ,T).

Applying theorem I to each of the polynomials $f_1,\dots,f_r$ we get a finite partition of $K((t))^m\times K((t))$ in subsets A such that $P(A,r)$ holds. The next step is to show theorem II by descending induction.

Assuming we have a set A and an integer s, $1<s\le r$ such that $P(A,s)$ holds, we should partition A further in a finite number of sets B such that $P(B,s-1)$ holds.

Consider two different cells (which have respective centers $c_1(x,\xi)$,

$e_1(\xi)$ and $c_2(x,\xi)$, $e_2(\xi)$). By splitting A into

$$\{(x,T) \in A(\xi) |\, c_1(x,\xi) = c_2(x,\xi)\}$$

and its complement in A, so we will assume that $c_1(x,\xi) \neq c_2(x,\xi)$. Also, we will assume that $e_1(\xi) \neq e_2(\xi)$ by splitting C into

$$\{(x,\xi) \in C |\, e_1(\xi) = e_2(\xi)\}$$

and its complement in C.

There are four different cases:

(i) $\mathrm{ord}_t(T - c_1(x,\xi)) > \mathrm{ord}_t(c_2(x,\xi) - c_1(x,\xi))$ In this case we have

$$\begin{aligned} T - c_2(x,\xi) &= (T - c_1(x,\xi)) - (c_1(x,\xi) - c_2(x,\xi)) \\ &= (c_2(x,\xi) - c_1(x,\xi))\Big(1 - \frac{T - c_1(x,\xi)}{c_1(x,\xi) - c_2(x,\xi)}\Big). \end{aligned}$$

As $\mathrm{ord}_t \dfrac{T - c(x,\xi)}{c_1(x,\xi) - c_2(x,\xi)} > 0$ we get

$$\mathrm{ord}_t(T - c_2(x,\xi)) = \mathrm{ord}_t(c_2(x,\xi) - c_1(x,\xi))$$

and

(3.1.1) $$\overline{\mathrm{ac}}(T - c_2(x,\xi)) = \overline{\mathrm{ac}}((c_2(x,\xi) - c_1(x,\xi)).$$

Let

$$B_1 = \bigcup_{\xi} \big\{(x,T) \in A(\xi) | \mathrm{ord}_t(T - c_1(x,\xi)) > \mathrm{ord}_t(c_2(x,\xi) - c_1(x,\xi))\big\}$$

Then on B_1 the center c_2 is eliminated.

Note that it follows from 3.1.1 that

$$(\overline{\mathrm{ac}}(T - c_2(x,\xi)) - e_2(\xi)) = (\overline{\mathrm{ac}}(\,-(c_2(x,\xi) - c_1(x,\xi)) - e_2(\xi)))$$

and thus we can eliminate $e_2(\xi)$ too.

(ii) $\mathrm{ord}_t(T - c_1(x,\xi)) < \mathrm{ord}_t(c_2(x,\xi) - c_1(x,\xi))$ In this case we have $\mathrm{ord}_t(T - c_1(x,\xi)) = \mathrm{ord}_t(T - c_2(x,\xi))$ and $\overline{\mathrm{ac}}(T - c_1(x,\xi)) = \overline{\mathrm{ac}}(T - c_2(x,\xi))$. Let

$$B_2 = \bigcup_{\xi} \Big\{(x,T) \in A(\xi) | \mathrm{ord}_t(T - c_1(x,\xi)) < \mathrm{ord}_t(c_2(x,\xi) - c_1(x,\xi))\Big\}$$

and we can eliminate c_2.

Now note that as $\overline{\mathrm{ac}}(T - c_1(x,\xi)) = \overline{\mathrm{ac}}(T - c_2(x,\xi))$ we can repeat exactly the same arguments in Denef (1986) (page 163) to eliminate one of the centers $e_1(\xi)$ and $e_2(\xi)$.

(iii) $\operatorname{ord}_t(T - c_2(x,\xi)) > \operatorname{ord}_t(c_2(x,\xi) - c_1(x,\xi))$;

In this case we eliminate c_1 and e_2.

(iv) $\operatorname{ord}_t(T - c_1(x,\xi)) = \operatorname{ord}_t(c_2(x,\xi) - c_1(x,\xi)) = \operatorname{ord}_t(T - c_2(x,\xi))$.

In this case we have

$$\begin{aligned}\overline{\mathrm{ac}}(c_2(x,\xi) - c_1(x,\xi)) &= \overline{\mathrm{ac}}((T - c_1(x,\xi)) - (T - c_2(x,\xi))) \\ &= \overline{\mathrm{ac}}(T - c_1(x,\xi)) - \overline{\mathrm{ac}}(T - c_2(x,\xi))\end{aligned}$$

where we used the fact that $\operatorname{ord}_t\left\{1 - \dfrac{T - c_2(x,\xi)}{T - c_1(x,\xi)}\right\} = 0$. In this case we can eliminate either $c_1(x,\xi)$ or $c_2(x,\xi)$, and we can proceed as before for the elimination of $e_1(\xi)$ and $e_2(\xi)$.

Now we get a finite partition of $K((t))^m \times K((t))$ in cells A, such that

$$\operatorname{ord}_t f_i(x,T) = \operatorname{ord}_t(h_i(x,\xi)(T - c(x,\xi))^{\mu_i})$$

and

$$\overline{\mathrm{ac}} f_i(x,T) = \sum_{k \in I_i} \overline{\mathrm{ac}}(a_{ki}(x,\xi)) b_{ki}(\xi) \overline{\mathrm{ac}}((T - c(x,\xi))^{\upsilon'_{ik}})$$

for all $(x,T) \in A$ and $\mu_i, \upsilon'_{ik} \in \boldsymbol{N}$. Finally apply theorem II of Denef (1986) to the polynomials

$$G_{ij}(\xi,\xi',\zeta) = \sum_{k \in I_i} g'_{kij}(\xi,\xi') b_{ki}(\xi) \zeta^{\upsilon'_{ik}}$$

(where $g'_{kij}(\xi,\xi')$ are as in (4.1.6)) to get

$$G_{ij}(\xi,\xi',\zeta) = u_{ij}(\xi,\xi',\zeta)^n g_{ij}(\xi,\xi')(\zeta - e(\xi,\xi'))^{\upsilon_{ij}}$$

(after further partitioning of the cells A), $\upsilon_{ij} \in \boldsymbol{N}$. It suffices to substitute $\overline{\mathrm{ac}}(T - c(x,\xi))$ for ζ in the above to get the result. □

4. Generalized Cell Decomposition.

Let C be a definable subset of $\bar{K}^m \times K^n \times \Gamma^r \times \Sigma^s$.

We call the cells defined above *strict cells*. Now we define a generalized 1-cell (or 1-cell for short) by

$$A = \bigcup_{\xi,z,z'} A(\xi,z,z')$$

with

$$A(\xi,z,z') = \Big\{(x,T)\in \bar{K}^m \times \bar{K} \,|\, (x,\xi,z,z')\in C, \mathrm{ord}_t(T-c(x,\xi)) = \alpha(x,\xi,z),$$
$$\mathrm{ord}_p(\overline{\mathrm{ac}}(T-c(x,\xi))-e(\xi)) = \beta(x,\xi,z'), (\overline{\mathrm{ac}}(T-c(x,\xi))-e(\xi))\in \lambda P_{n_1}\Big\}$$

where $\xi=(\xi_1,\dots,\xi_n)$ are variables in the RV sort, $z=(z_1,\dots,z_r)$ are variables in the Ord_1 sort, $z'\in \Sigma^s$ are variables in the Ord_2 sort, and the definable functions, $c(x,\xi), e(\xi)$ are the centers of the cell, and such that the fibers $A(\xi,z,z')$ are disjoint for distinct (ξ,z,z'). We remind the reader that P_{n_1} is the set of nonzero n_1-powers of K, where n_1 is some positive integer (≥ 2) and $\lambda\in K$. The definable set C is called the parameter set of the cell A.

A 0-cell is defined by

$$A = \bigcup_{\xi,z,z'} A(\xi,z,z')$$
$$A(\xi,z,z') = \big\{(x,T)\in \bar{K}^m\times\bar{K} \,|\, (x,\xi,z,z')\in C, T = c(x,\xi)\big\}$$

for some definable function $c:\bar{K}^m\times K^n\to\bar{K}$.

Note that strict cells falls under the new definition, by adding one Ord_1-variable and one Ord_2-variable in the following way:

$$\mathrm{ord}_t(T-c(x,\xi)) = z$$
$$\mathrm{ord}_p(\overline{\mathrm{ac}}(T-c(x,\xi)) = z'$$

and then using the conditions

$$\mathrm{ord}_t\, b_1(x,\xi)\diamondsuit_1 \lambda_1\cdot \mathrm{ord}_t(T-c(x,\xi))\diamondsuit_2 \mathrm{ord}_t\, b_2(x,\xi)$$

and

$$\mathrm{ord}_p\, d_1(\xi)\,\Box_1\, \lambda_2\cdot\mathrm{ord}_p(\overline{\mathrm{ac}}(T-c(x,\xi))-e(\xi))\,\Box_2\, \mathrm{ord}_p\, d_2(\xi)$$

to constrain the variables z and z'.

Now fix a model $(\bar{K},K,\Gamma,\Sigma)$ for the theory $\boldsymbol{T}_2$, with $\bar{K}=K((t))$ and $\Gamma=\Sigma=\boldsymbol{Z}$.

Let us state theorem III (generalized cell decomposition theorem).

THEOREM 4.1. *Let X be a definable subset of $K((t))^m\times K((t))$, and f a definable function from X to $K((t))$. Then there is a finite partition of X in (generalized) cells A of centers $c(x,\xi)$ and $e(\xi)$ such that*

$$\overline{\mathrm{ac}}\,(f(x,T)) = g(\overline{\mathrm{ac}}(T-c(x,\xi)),x,\xi)$$
$$\mathrm{ord}_t(f(x,T)) = h(x,z)$$
$$\mathrm{ord}_p(\overline{\mathrm{ac}}(f(x,T))) = h'(\xi,z')$$

where g, h and h' are definable functions.

PROOF. As f and X are definable we have:

$$(x,T)\in X\equiv\psi(x,T)$$
$$\zeta=(\overline{\mathrm{ac}}\circ f)(x,T)\equiv\psi'(x,\zeta,T)$$
$$z=\mathrm{ord}_t(f(x,T))\equiv\phi(x,z,T)$$
$$z'=\mathrm{ord}_p(\overline{\mathrm{ac}}f(x,T))\equiv\phi'(x,z',T)$$

where $\zeta\in K$, $z\in\boldsymbol{Z}$, $z'\in\boldsymbol{Z}$, and $\psi,\psi'\phi,\phi'$ are $\mathcal{L}$–formulas.

We assume as usual that the occurrences of (x,T) in ψ,ψ',ϕ,ϕ' are uniquely through the RV-terms $\overline{\mathrm{ac}}f_i(x,T)$ and the Ord_1-terms $\mathrm{ord}_t\, g_j(x,T)$, $(i=1,\dots,r', j=1,\dots,s')$, where f_i and g_j are polynomials in (x,T) with integer coefficients.

Then, applying theorem (1.1), to the conjunction $\psi\wedge\psi'$ (for instance) we see that it is $\boldsymbol{T}_2$-equivalent to

$$\bigvee_{k=1}^{q}\Big(\chi_k(\zeta,\overline{\mathrm{ac}}\,f_1(x,T),\dots,\overline{\mathrm{ac}}\,f_r(x,T))\wedge\theta_k(\mathrm{ord}_t\,g_1(x,T),\dots,\mathrm{ord}_t\,g_s(x,T))\Big),$$

where χ_k is an $\mathcal{L}_{\mathrm{Mac}}$, and θ_k is an $\boldsymbol{L}_{\mathrm{Ord}_1}$-formula.

Applying theorem II to the polynomials f_i,g_j we can find a finite partition of $K((t))^{m+1}$ in cells, each cell A having parameter set C and parameters $(\xi_1,\dots,\xi_l)\in K^l,(z_1,\dots,z_r)\in\boldsymbol{Z}^r$, $(z'_1,\dots,z'_s)\in\boldsymbol{Z}^s$ and centers $c(x,\xi)$ and $e(\xi)$ such that

$$\overline{\mathrm{ac}}f_i(x,T)=u_i(\overline{\mathrm{ac}}(T-c(x,\xi)),\xi)^{n_1}d_i(\xi)(\overline{\mathrm{ac}}(T-c(x,\xi))-e(\xi))^{\upsilon_i}$$
$$\mathrm{ord}_t\,g_j(x,T)=\mathrm{ord}_t\,h_j(x,\xi)(T-c(x,\xi))^{\mu_j}$$

with $\mathrm{ord}_p\,u_i=0$, for all $i=1,\dots,r'$.

It is assumed that each polynomial f_i,g_j is either identically zero or nowhere vanishing on $A(\xi,z)$.

We will further partition the cell A on which the theorem holds .

On $A(\xi,z)$, $\psi\wedge\psi'$ is $\boldsymbol{T}_2$-equivalent to

$$(\exists\rho)(\exists l)(\exists\eta)\Bigg[\Big(\bigwedge_{i=1}^{r'}\mathrm{ord}_p\eta_i=0\wedge\eta_i^{n_1}d_i(\xi)(\overline{\mathrm{ac}}(T-c(x,\xi))-e(\xi))^{\upsilon_i}=\rho_i\Big)$$
$$\wedge\Big(\bigvee_{k=1}^{q}\Big(\chi_k(\zeta,\rho_1,\dots,\rho_{r'})\wedge\theta_k(l_1,\dots,l'_s)\Big)\Big)\wedge\Big(\bigwedge_{j=1}^{s'}\mathrm{ord}_t\,h_j(x,\xi)(T-c(x,\xi))^{\mu_j}=l_j\Big)$$
$$\wedge\,\theta(\xi,z,\overline{\mathrm{ac}}(T-c(x,\xi)),\mathrm{ord}_t(T-c(x,\xi)))\Bigg]$$

where $\rho_1, \ldots, \rho_{r'}, \eta_1, \ldots, \eta'_r$ are RV-variables and $l_1, \ldots, l'_s$ are Ord_1-variables, and θ is the formula that says that $(x, T) \in A(\xi, z, z')$.

We can verify that χ_k defines the graph of a partial function $g_k : K^{r'} \times \Sigma^{s'} \to K$ whose domain is defined by the formula

$$(\exists x)(\exists T)(\exists \xi)(\exists z)\Bigg[\theta_k \wedge \theta \wedge \Big(\bigwedge_{i=1}^{r'} \Big(u_i(\overline{\mathrm{ac}}(T - c(x, \xi)), \xi)^{n_1} d_i(\xi) \\ \times (\overline{\mathrm{ac}}(T - c(x, \xi)) - e(\xi))^{\upsilon_i} = \rho_i\Big)\Big) \wedge \Big(\bigwedge_{j=1}^{s'} \mathrm{ord}_t\, h_j(x, \xi)(T - c(x, \xi))^{\mu_j} = l_j\Big)\Bigg].$$

Call the formula in the brackets $\Psi_k(x, T, \xi, z, z')$. Then for all (x, T) such that $\exists \xi \exists z \exists z' \Psi_k(x, T, \xi, z, z')$ holds, we have $\zeta = g_k(\rho_1, \ldots, \rho'_r)$.

If for all $i = 1, \ldots, r'$, $\upsilon_i = 0$ and for all $j = 1, \ldots, s'$, $\mu_j = 0$, no further partitioning of the cell A is required, but we may need to constrain the parameter set C further to satisfy the requirements of the theorem.

For $i = 1, \ldots, r'$ such that $\upsilon_i \neq 0$ and $d_i(\xi) \neq 0$

$$(\exists \eta_i)\mathrm{ord}_p \eta_i = 0 \wedge \eta_i^{n_1} d_i(\xi)(\overline{\mathrm{ac}}(T - c(x, \xi)) - e(\xi))^{\upsilon_i} = \rho_i$$

$$\Leftrightarrow (\upsilon_i \mathrm{ord}_p(\overline{\mathrm{ac}}(T - c(x, \xi)) - e(\xi)) = \mathrm{ord}_p \rho_i - \mathrm{ord}_p d_i(\xi))$$

$$\wedge\, (\overline{\mathrm{ac}}(T - c(x, \xi)) - e(\xi)) \in (\rho_i / d_i(\xi))^{1/\upsilon_i} P_{n_1},$$

Also,

$$\mathrm{ord}_t\, h_j(x, \xi)(T - c(x, \xi))^{\mu_j} = l_j$$

$$\Leftrightarrow (\mu_j \mathrm{ord}_t(T - c(x, \xi)) = l_j - \mathrm{ord}_t\, h_j(x, \xi)),$$

for $j = 1, \ldots, s'$.

By theorem 1.1 of (Scowcroft & van den Dries 1988) there exists a partition of the $\mathcal{L}_{\mathrm{Mac}}$-definable set $\mathrm{Proj}_{K^n} C$ into $\mathcal{L}_{\mathrm{Mac}}$-definable subsets D, on each of which the definable function $d_i(\xi)$ is analytic. Thus, by partitioning the definable sets D further if necessary we can assume that $(\rho_i / d_i(\xi))^{1/\upsilon_i}$ have constant n_1-th power residue, hence $(\rho_i / d_i(\xi))^{1/\upsilon_i} P_{n_1} = \lambda_i P_{n_1}$ for some $\lambda_i \in K$.

In the above description we notify the reader that our cell decomposition may contain 0-cells (if we have $l_j = \infty$, $\mu_j \neq 0$ and $\mathrm{ord}_t\, h_j(x, \xi) < \infty$ for some j).

Also, using the observation that given two n^{th} power residues having non-empty intersection, one of them must contain the other, we deduce that $\overline{\mathrm{ac}}(T - c(x, \xi)) - e(\xi) \in \lambda_i P_{n_1}$ for some i.

Finally notice that a function given conjunctly by definable conditions is a function given by the conjunction of these conditions, hence the result follows.

The remaining statements of the theorem are left to the reader. □

REFERENCES

[1] R. CLUCKERS - F. LOESER, *Constructible motivic functions and motivic integration*, preprint, math. arxiv 2004.
[2] J. DENEF, *The rationality of the Poincaré series associated to the p-adic points on a variety*, Invent. Math. **77** (1984), pp. 1–23.
[3] J. DENEF, *p-adic semi-algebraic sets and cell decomposition*, J. reine angew. Math. **369** (1986), pp. 154–166.
[4] I. FESENKO, *Measure, integration and elements of harmonic analysis on generalized loop spaces*, www.maths.nott.ac.uk/personal/ibf/aoh.pdf, 2003.
[5] E. HRUSHOVSKI - D. KAZHDAN, *Integration in valued fields*, preprint, math.arxiv 2005.
[6] J. IGUSA, *An introduction to the theory of local zeta functions*, AMS/IP Studies in Advanced Mathematics, 14. International Press, Cambridge, MA, 2000.
[7] A. MACINTYRE *On definable subsets of p-adic fields*, J. Symb. Logic **41** (1976), pp. 605–610.
[8] J. PAS, *Uniform p-adic cell decomposition and local zeta functions*, J. Reine Angew. math., **399** (1989), pp. 137–172.
[9] P. SCOWCROFT - L. VAN DEN DRIES, *On the structure of semialgebraic sets over p-adic fields*, J. Symbolic Logic, **53** (1988), pp. 1138–1164.

Manoscritto pervenuto in redazione il 12 settembre 2005