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On the Dimension of an Irrigable Measure.

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ABSTRACT - In this paper the problem of determining if a given measure is irrigable, in the sense of [4], or not is addressed. A notion of irrigability dimension of a measure is given and lower and upper bounds are proved in terms of the minimal Hausdorff and respectively Minkowski dimension of a set on which the measure is concentrated.

A notion of resolution dimension of a measure based on its discrete approximations is also introduced and its relation with the irrigation dimension is studied.

Introduction.

In the paper [4] the authors have introduced a cost functional to the aim of modeling ramified structures, such as trees, root systems, lungs and cardiovascular systems. A very similar functional (even if the variable employed has a different form) has been introduced in [10]. The aim of the functional is to force the fibers to keep themselves together penalizing, in this way, their branching. The necessity of keeping the functional low competes with a boundary condition which, on the other hand, forces the fibers to bifurcate prescribing that the fluid they carry must reach a given measure spread out on a volume. The result of this competition is that the fibers take advantage in keeping themselves together as long as possible and then branching, always into a finite number of branches, while approaching the terminal points, giving rise to the ramified structure. In [10] the problem consisting in determining the cases, depending on an index, in which all the probability measures can be reached by a system of fibers (an irrigation pattern) of finite cost, i.e. are irrigable measures, is formulated and solved in a very close setting.

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In this work we shall investigate a more general question consisting in characterizing, for a given value of the index, what probability measures are irrigable or not. The answer to this question will clearly show, in particular, what are the cases in which all the probability measures will turn out to be irrigable, giving in this way a different proof of the already mentioned result in [10]. The fact that a measure spread on a set of high dimension forces the fibers to a more frequent branching, and therefore needs a higher cost, seems to suggest that the higher it is the dimension on which a measure is spread the more difficult it becomes to irrigate it. For a better formalization, we introduce the notion of irrigability dimension of a measure and then we equivalently express the above stated problem in terms of giving some estimates on the irrigability dimension of a given positive measure which is always supposed to be Borel regular, with a bounded support and a finite mass (by normalization we shall suppose it to be a probability measure). We shall show, with some examples, that the intuitive and conjecturable idea that the irrigability dimension of a measure coincides with the Hausdorff dimension of its support is groundless in spite of the fact that both the two values express how much the measure is spread out. On the other hand, we shall give some lower and upper bounds for the irrigability dimension $d(\mu)$ of a probability measure μ by means of the minimal Hausdorff and respectively Minkowski dimension of a set on which the measure is concentrated.

This result will be overproved. Indeed, we shall prove it directly, getting some further meaningful information and introducing some tools which will be also used in other parts of the paper but we shall also be able to deduce it from a deeper estimate of $d(\mu)$ which will need the introduction of new notions. More precisely, it will need the notion of *resolution dimension* of a measure which, affected by an index, expresses the possibility to describe the measure by means of discrete approximations. When the measure is suitably regular, the value of the resolution dimension does not depend on the index, while for a generic measure, as will be explained by some examples, the resolution dimension is “out of focus” in the sense that different indexes give different values. We shall show that, in any case, it is always possible to find an index, suitably characterized, which gives a resolution dimension which coincides with the irrigability dimension.

The paper is organized as follows: In Section 1 we shall introduce the notion of irrigability dimension and we shall state the main results which do not make use of the notion of resolution dimension of a measure. Sections 2 and 3 are respectively dedicated to the lower and upper estimate given for $d(\mu)$ by means of the minimum among the Hausdorff and the Minkowski dimension of the sets on which the measure is concentrated. In Section 4 remarks and examples, mainly based on the compactness results stated in [4], which show that the estimates are, in a certain sense, sharp are collected. In Section 5 we shall introduce the notion of resolution dimension of a measure and we shall state some fundamental properties. The proof of the irrigability and nonirrigability results which can be deduced from conditions on the resolution dimension will be respectively shown in sections 6 and 7. In Section 8 we shall show how the irrigability dimension of a measure can be seen as a resolution dimension with respect to some index $p \geq 1$ and how to chose such a suitable value of p. Then we shall give another proof of the main result in Section A (Theorem 1.1).

Since we are dealing with notions introduced for the first time in [4] and [3], which will be used without any explanation, in order to help the reader we have gathered up in Appendix A the notation and the results in [4] and [3] which are essential for the understanding of this paper. In Appendix B we give the proof of the propositions stated in Section 5 with some examples which justify the required assumptions. Finally, in Appendix C we give the index of the main notation.

1. Dimensions of a measure and irrigability results.

We just recall the definition of irrigation pattern while, as said in the introduction, we have gathered up in Appendix A the notation and the results in [4] and [3] which will help the reader for the understanding of this paper.

Let $(\Omega, \ |\cdot|)$ be a nonatomic probability space which we interpret as the reference configuration of a fluid material body. We can also interpret it as the trunk section of a tree, this trunk being thought of as a set of fibers which can bifurcate into branches. A *set of Ω with source point $S \in \mathbb{R}^N$* is a mapping

$$\chi : \Omega \times \mathbb{R}_+ \to \mathbb{R}^N$$

such that:

C1) For a.e. *material point* $p \in \Omega$, $\chi_p(t) : t \mapsto \chi(p,t)$ is a Lipschitz continuous map with a Lipschitz constant less than or equal to one.

C2) For a.e. $p \in \Omega$: $\chi_p(0) = S$.

The condition $|\Omega| = 1$ is of course assumed by normalization in order to simplify the exposition. In some cases this normalization will be impossible (we can, for instance, work with two different spaces and assume an inclusion), then we shall consider all the notions trivially extended to the case $|\Omega| < +\infty$. We shall consider the source point $S \in \mathbb{R}^N$ as given and we shall denote by $\mathbf{C}_S(\Omega)$ and $\mathbf{P}_S(\Omega)$ the set of all the set of fibers of Ω and respectively the set of all the measurable set of fibers of Ω and we shall call the elements of $\mathbf{P}_S(\Omega)$ *irrigation patterns.*

We shall introduce some definitions which will be used to formalize the irrigability problem.

DEFINITION 1.1. *For a fixed real number* $\alpha \in]0, 1[$ *we shall call critical dimension of the exponent* α *the constant* $d_\alpha = \dfrac{1}{1-\alpha} > 1$, *i.e. the conjugate index* $\left(\dfrac{1}{\alpha}\right)'$ *of the index* $\dfrac{1}{\alpha} > 1$.

DEFINITION 1.2. *Let* $\alpha \in]0, 1[$ *be given and let* μ *be a probability measure on* $\mathbb{R}^N$. *We shall say that* μ *is an irrigable measure with respect to* α *(or that* μ *is* α*-irrigable) if there exists a pattern* $\chi \in \mathbf{P}_S(\Omega)$ *of finite cost* $I_\alpha(\chi) < +\infty$ *such that* $\mu_\chi = \mu$.

It is clear that two approaches are possible and equivalent: one can fix a constant $\alpha \in]0, 1[$ and investigate the irrigable measures with respect to this constant or fix a measure μ and find out the constants $\alpha \in]0, 1[$ with respect to which μ is irrigable. This second point of view leads us to introduce the following definition.

DEFINITION 1.3. *Let* μ *be a positive Borel measure on* $\mathbb{R}^N$, *then we shall call irrigability dimension of* μ *the number*

$$d(\mu) = \inf\{d_\alpha \mid \mu \quad \text{is irrigable with respect to } \alpha\}.$$

REMARK 1.1. *For any probability measure* μ, *by definition, the irrigability dimension* $d(\mu)$ *of* μ *is greater or equal to* 1.

REMARK 1.2. *If* μ *is an irrigable measure with respect to* α, *then* μ *is also irrigable with respect to every constant* $\beta \in [\alpha, 1[$. *Indeed, let* $\chi \in P_S(\Omega)$ *such that* $I_\alpha(\chi) < +\infty$ *and* $\mu_\chi = \mu$, *then for all* $\beta \geq \alpha$, $I_\beta(\chi) \leq I_\alpha(\chi)$.

REMARK 1.3. *By the definition of* $d(\mu)$ *and by* Remark 1.2 *it follows that for a given* $\alpha \in\,]0, 1[$ *and for a given measure* μ*:*

1) *if* $d(\mu) < d_\alpha$ *then* μ *is* α*-irrigable;*
2) *if* $d(\mu) > d_\alpha$ *then* μ *is not* α*-irrigable.*

As we shall show in Section 4, both cases can occur when $d_\alpha = d(\mu)$, see examples 4.4 and 4.5.

The aim of the first part of this paper is to give operative estimates of $d(\mu)$ in terms of geometrical properties of the measure μ. So we introduce the following two definitions.

DEFINITION 1.4. *We shall say that a positive Borel measure* μ *on* $\mathbb{R}^N$ *is concentrated on a Borel set* B *if* $\mu(\mathbb{R}^N \setminus B) = 0$ *and we shall call \cd of* μ *the smallest Hausdorff dimension* $d(B)$ *of a set* B *on which* μ *is concentrated i.e. the number*

$$d_c(\mu) = inf\,\{d(B) \,|\, \mu \text{ is concentrated on } B\}\,.$$

DEFINITION 1.5. *We shall denote by* $\operatorname{supp}(\mu)$ *the support of* μ *in the sense of distributions and shall call* support dimension *of* μ, $d_s(\mu)$, *its Hausdorff dimension.*

REMARK 1.4. *The support of a measure can be characterized as the smallest closed set on which* μ *is concentrated and the existence of such a set a priori follows by the separability of* $\mathbb{R}^N$, *precisely by the Lindelöf property. While, as stated above, the existence of the smallest closed set on which* μ *is concentrated is granted, it is clear that the smallest set on which* μ *is concentrated, in general, does not exist. This is the reason for which the infimum is taken in Definition 1.4, even if a set* B *of minimal dimension on which* μ *is concentrated can always be fixed. Moreover being* $\operatorname{supp}(\mu)$ *a set on which* μ *is concentrated, it follows that*

$$d_c(\mu) \leq d_s(\mu)\,.$$

These two geometrical dimensions are not sufficient to study the irrigability of a measure, as we shall show later in examples 4.1 and 4.3.

DEFINITION 1.6. *Let* $X \subset \mathbb{R}^N$ *be a bounded set. We shall call Minkowski dimension of the set* X *(see* [8]*) the constant*

$$d_M(X) = N - \liminf_{\delta\to 0} \log_\delta |N_\delta(X)| \tag{1.1}$$

where, for all $\delta > 0$,

$$N_\delta(X) = \{y \in \mathbb{R}^N | d(y, X) < \delta\}\,.$$

It is useful to remark that

$$0 \le d_M(X) \le N \qquad \forall X \neq \emptyset\,. \tag{1.2}$$

Moreover the Minkowski dimension of a set $X \subset \mathbb{R}^N$ can be characterized by the following two properties:

$$\forall \beta < d_M(X) \quad \limsup_{\delta \to 0} |N_\delta(X)| \delta^{\beta - N} = +\infty \tag{1.3}$$

and

$$\forall \beta > d_M(X) \quad \lim_{\delta \to 0} |N_\delta(X)| \delta^{\beta - N} = 0\,. \tag{1.4}$$

LEMMA 1.1. *Let $X \subset \mathbb{R}^N$ and $\beta > d_M(X)$. Then we can cover X by using $\delta^{-\beta}$ balls of radius δ for all δ sufficiently small.*

PROOF. Being $\beta > d_M(X)$, we have, by (1.4), that for all $C > 0$ and for $\delta > 0$ sufficiently small

$$|N_{\frac{\delta}{2}}(X)| \le C \delta^{N - \beta}\,.$$

We consider any family of disjoint balls $(B_i)_{i \in I}$ of radius $\frac{\delta}{2}$ contained in $N_{\frac{\delta}{2}}(X)$. We know that, being, for all $i \in I$, $|B_i| = b_N \left(\frac{\delta}{2}\right)^N$ (b_N stands for the measure of the unitary ball of $\mathbb{R}^N$), $\operatorname{card}(I) b_N \left(\frac{\delta}{2}\right)^N \le |N_{\frac{\delta}{2}}(X)| \le C\delta^{N-\beta}$ so, taking as C the constant $\frac{b_N}{2^N}$,

$$\operatorname{card}(I) \le C \frac{2^N}{b_N} \delta^{-\beta} = \delta^{-\beta}\,. \tag{1.5}$$

We have shown that the number of elements of any family consisting of disjoint balls contained in $N_{\frac{\delta}{2}}(X)$ is bounded by $\delta^{-\beta}$. This allows us to find a family of such balls which is maximal by inclusion. The corresponding family of balls with the same centers but with double radius, by maximality, turns out to be a covering of X. Inequality (1.5) gives the thesis. □

LEMMA 1.2. *Let $X \subset \mathbb{R}^N$ and $\beta < d_M(X)$. It is not possible to find a constant $C > 0$ such that one can cover X with only $C\delta^{-\beta}$ balls of radius δ for all δ sufficiently small.*

PROOF. We shall proceed by contradiction assuming that there exists a constant $C > 0$ such that for δ sufficiently small it is possible to cover X using $C\delta^{-\beta}$ balls of radius δ. It is useful to remark that doubling the radius of these balls we get a covering of $N_\delta(X)$, so we have

$$|N_\delta(X)| \le C\delta^{-\beta} b_N (2\delta)^N \le cost\ \delta^{N-\beta}\ ,$$

which gives $\beta \ge d_M(X)$ by (1.3). □

REMARK 1.5. *Collecting the last two lemmas, we can say that for a set* $X \subset \mathbb{R}^N$

$$d_M(X) = \inf\{\beta \ge 0 | X \textit{ can be covered by } C_\beta \delta^{-\beta} \textit{ balls of radius } \delta \textit{ for all } \delta \le 1\}\,. \tag{1.6}$$

DEFINITION 1.7. *Let μ be a probability measure, we shall use the notation*

$$d_M(\mu) = \inf\{d_M(X) \mid \mu \textit{ is concentrated on } X\} \tag{1.7}$$

and we shall call it Minkowski dimension of μ.

REMARK 1.6. *For any subset X of $\mathbb{R}^N$ the Minkowski dimensions of X and of its closure $\overline{X}$ are the same. Therefore*

$$d_M(\mu) = d_M(\operatorname{supp}(\mu))\,.$$

Moreover the Hausdorff dimension $d(X)$ of a set X is less or equal to $d_M(X)$. So for any probability measure μ

$$d_s(\mu) \le d_M(\mu)\,. \tag{1.8}$$

REMARK 1.7. *Let μ be a probability measure, then collecting Remark* 1.4 *and* (1.8) *we have that the following inequalities hold for $d_s(\mu)$*

$$d_c(\mu) \le d_s(\mu) \le d_M(\mu)\,. \tag{1.9}$$

A similar estimate is enjoyed by $d(\mu)$. Indeed, we shall prove the following statement.

THEOREM 1.1. (Lower and Upper bound on $d(\mu)$). *Let μ be a probability measure then the following bounds hold for $d(\mu)$*

$$d_c(\mu) \le d(\mu) \le \max\{d_M(\mu), 1\}\,. \tag{1.10}$$

The first inequality in (1.10) is a straightforward consequence of a deeper and more precise result stated in the following theorem, whose proof is in Section 2.

THEOREM 1.2. *Let $\alpha \in]0, 1[$ and let μ be an α-irrigable probability measure, then μ is concentrated on a d_α-negligible set, in particular,*

$$d_c(\mu) \leq d_\alpha \,. \tag{1.11}$$

Theorems 1.1 and 1.2 widely answer the question considered in [10] about the values of α which make every measure of bounded support irrigable. Indeed, we can deduce the following corollaries.

COROLLARY 1.1. *Let $\alpha \in]0, 1[$, $\alpha > \frac{1}{N'}$. Then any probability measure μ with a bounded support is α-irrigable.*

PROOF. Remarking that $\alpha > \frac{1}{N'}$ is equivalent to $d_\alpha = \left(\frac{1}{\alpha}\right)' > N$, combining (1.2) with (1.10), we have, for every μ,

$$d_\alpha > N \geq \max\{d_M(\mu), 1\} \geq d(\mu) \,,$$

so every probability measure μ with a bounded support is α-irrigable by Remark 1.3,1. □

COROLLARY 1.2. *Let $\alpha \in]0, 1[$ be such that any probability measure μ with a bounded support is α-irrigable, then $\alpha > \frac{1}{N'}$.*

PROOF. From Theorem 1.2 we have that any probability measure μ with a bounded support is concentrated on a d_α-negligible set. So, $N < d_\alpha$, namely $\alpha > \frac{1}{N'}$. □

In spite of inequalities (1.10) and (1.9) it is not possible to establish some general inequality between $d(\mu)$ and $d_s(\mu)$, as shown in Section 4 by examples 4.1 and 4.3.

By the following lemmas we shall make the estimates on the dimension $d(\mu)$ more precise in the case in which the probability measure μ enjoys some regularity properties.

DEFINITION 1.8. *Let μ be a probability measure and $\beta \geq 0$. We shall say that μ is Ahlfors regular in dimension β if*

$$\text{(AR)} \quad \exists C_1, C_2 > 0 \text{ s.t. } \forall r \in [0,1],\ \forall x \in \operatorname{supp}(\mu) : \overline{C}_1 r^\beta \leq \mu(B(x,r)) \leq C_2 r^\beta .$$

We shall separately consider the two bounds in (AR). So for a probability measure μ and a real number $\beta \geq 0$ we shall consider the two conditions

$$\text{(LAR)} \qquad \exists C > 0 \text{ s.t. } \forall r \in [0,1],\ \forall x \in \operatorname{supp}(\mu): \ \ Cr^{\beta} \leq \mu(B(x,r))\,.$$

and

$$\text{(UAR)} \qquad \exists C > 0 \text{ s.t. } \forall r \in [0,1],\ \forall x \in \operatorname{supp}(\mu): \ \ \mu(B(x,r)) \leq Cr^{\beta}\,.$$

In (UAR) the restriction $x \in \operatorname{supp}(\mu)$ can be removed, this could make the value of C_2 increase at most of 2^{β}. It is useful to recall the following definition.

DEFINITION 1.9. *A probability measure $\nu : \mathbb{R}^N \to \mathbb{R}_+$ satisfies the uniform density property (in short u.d.p.) in dimension $\beta \geq 0$ on a set M if*

$$\exists C_1 > 0 \text{ s.t. } \forall x \in M,\ \forall r \in [0,\ 1]: \ C_1 r^{\beta} \leq \nu(B(x,r))\,.$$

LEMMA 1.3. *Let ν be a probability measure which satisfies the u.d.p. in dimension $\beta \geq 0$ on a subset B. Then*

$$d_M(B) \leq \beta\,. \tag{1.12}$$

PROOF. Let us fix $\delta > 0$ and let us consider any family $(B_i)_{i\in I}$ of disjoint balls of radius $\frac{\delta}{2}$ with centers on B. By hypotheses, $\nu(B_i) \geq C2^{-\beta}\delta^{\beta}$ and $\nu(B) \leq 1$, therefore $\operatorname{card}(I) \leq 2^{\beta}C^{-1}\delta^{-\beta}$. So we can consider a family $(B_i)_{i\in I}$ as above maximal by inclusion. The maximality of $(B_i)_{i\in I}$ guarantees that, for any other point $x \in B$, $d\big(x, \bigcup_{i\in I} B_i\big) < \frac{\delta}{2}$ holds. Therefore the family $(\tilde{B}_i)_{i\in I}$ which is obtained by doubling the radius of the balls B_i is a covering of B. So we have proved that B can be covered by const $\delta^{-\beta}$ balls of radius δ arbitrarily small and so by Remark 1.5 $d_M(B) \leq \beta$. □

COROLLARY 1.3. *Let μ be a probability measure. Let $\beta \geq 0$ such that μ satisfies (LAR) (i.e. μ satisfies the uniform density property in dimension β on* supp(μ)*). Then*

$$d_M(\mu) \leq \beta\,. \tag{1.13}$$

REMARK 1.8. *The thesis of Corollary 1.3 still holds true if one assumes the existence of a probability measure ν which satisfies the uniform density property in dimension β on a set B on which μ is concentrated.*

LEMMA 1.4. *Let μ be a probability measure concentrated on a set $A \subset \mathbb{R}^N$. Let $\beta \geq 0$ such that μ satisfies (UAR). Then*

$$\mathcal{H}^\beta(A) > 0. \tag{1.14}$$

PROOF. Let $(X_i)_{i\in I}$ be any countable covering of A. Every X_i is contained in a ball B_i with a radius equal to (X_i). So, by (UAR)

$$1 = \mu(\mathbb{R}^N) = \mu(A) \leq \sum_{i\in I} \mu(B_i) \leq C \sum_{i\in I} (X_i)^\beta ,$$

from which we have

$$\sum_{i\in I} (X_i)^\beta \geq C^{-1} > 0 .$$

□

COROLLARY 1.4. *Let μ be a probability measure. Let $\beta \geq 0$ such that μ satisfies (UAR). Then*

$$d_c(\mu) \geq \beta . \tag{1.15}$$

COROLLARY 1.5. *Let μ be an Ahlfors regular probability measure in dimension $\beta \geq 1$. By* Corollary 1.3 *and* Corollary 1.4, *being $\beta = \max\{\beta, 1\} \geq \max\{d_M(\mu), 1\}$, the lower and upper bounds stated in* Remark 1.7 *and* Theorem 1.1 *for $d_s(\mu)$ and $d(\mu)$ respectively, give*

$$d_c(\mu) = d_s(\mu) = d(\mu) = d_M(\mu) = \beta .$$

This guarantees that, in the case of an Ahlfors regular probability measure, all the geometrical dimensions $d_c(\mu)$, $d_s(\mu)$ and $d_M(\mu)$ and the irrigability dimension $d(\mu)$ are equal to the Ahlfors dimension β.

COROLLARY 1.6. *An Ahlfors regular probability measure μ of dimension $\beta \geq 1$, is α-irrigable for all $\alpha \in]0, 1[$ s.t. $d_\alpha > \beta$ i.e. for all $\alpha \in]\frac{1}{\beta'}, 1[$ and is not irrigable for all $\alpha \in]0, 1[$ s.t. $d_\alpha \leq \beta$ i.e. for all $\alpha \in]0, \frac{1}{\beta'}]$.*

PROOF. Let $\alpha \in]0, 1[$. If $d_\alpha \neq \beta = d(\mu)$, the thesis follows from Remark 1.3. Moreover, when $d_\alpha = \beta$, by Theorem 1.2 it is clear that an Ahlfors regular probability measure of dimension $\beta = d_\alpha$, is not α-irrigable. Indeed by Lemma 1.4 it cannot be concentrated on a d_α-negligible set. □

We shall make use of this last argument when in Section 4 we shall show that, in general, $d(\mu) = \inf\{d_\alpha \mid \mu \text{ is } \alpha\text{-irrigable}\}$ is not a minimum, see Example 4.4.

2. Lower bound on $d(\mu)$.

This section is devoted to the proof of Theorem 1.2 from which $d_c(\mu) \leq d(\mu)$ trivially follows.

LEMMA 2.1. *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω and $r > 0$, then*

$$\mu_\chi(\mathbb{R}^N \setminus B_r(S)) \leq \left(\frac{I_\alpha(\chi)}{r}\right)^{\frac{1}{\alpha}}. \tag{2.1}$$

PROOF. Taking into account that the less expensive way to carry some part of the fluid out of $B_r(S)$ is to move it in a unique tube in the radial direction and to leave the other part in the source point, we have

$$[\mu_\chi(\mathbb{R}^N \setminus B_r(S))]^\alpha r \leq I_\alpha(\chi) \, ,$$

from which the thesis follows. □

COROLLARY 2.1. *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω. If $r \geq (I_\alpha(\chi))^{1-\alpha}$, then*

$$\mu_\chi(\mathbb{R}^N \setminus B_r(S)) \leq I_\alpha(\chi) \, . \tag{2.2}$$

In [3] the following lemma has been proved.

LEMMA 2.2. *Let $\chi \in P_S(\Omega)$ be a simple irrigation pattern of Ω without dispersion and $\varepsilon > 0$, then there exists a finite number $k \in \mathbb{N}$ of points $x_i \in F_\chi$ such that, denoting by χ_i the branch of χ with source point x_i,*

$$\sum_{i=1}^{k} I_\alpha(\chi_i) < \varepsilon \tag{2.3}$$

$$\Big(\mu_\chi - \sum_{i=1}^{k} \mu_{\chi_i}\Big)(\mathbb{R}^N) < \varepsilon \, . \tag{2.4}$$

LEMMA 2.3. *Let $\chi \in P_S(\Omega)$ be a simple irrigation pattern of Ω without dispersion and $\varepsilon > 0$, then $\exists A \subset \mathbb{R}^N$ such that*

1) *A can be covered by a finite number of balls $B_i = B_{r_i}(x_i)$, s.t. $\sum_i (r_i)^{d_\alpha} < \varepsilon$;*
2) *$\mu_\chi(\mathbb{R}^N \setminus A) \leq \varepsilon$.*

PROOF. By Lemma 2.2 we can find a finite number $k \in \mathbb{N}$ of points $x_i \in \mathbb{R}^N$ such that, by denoting by χ_i the branch of χ starting from x_i and by $\varepsilon_i = I_\alpha(\chi_i)$, we have

$$\sum_{i=1}^{k} \varepsilon_i < \varepsilon . \tag{2.5}$$

Calling, for all $i \in \{1, \dots, k\}$, as suggested by Corollary 2.1, $r_i = (I_\alpha(\chi_i))^{1-\alpha} = (\varepsilon_i)^{1-\alpha}$ we have,

$$\sum_i r_i^{d_\alpha} = \sum_i \varepsilon_i < \varepsilon .$$

Moreover, from (2.2), we can deduce that

$$\mu_{\chi_i}(\mathbb{R}^N \setminus B_{r_i}(x_i)) \le \varepsilon_i .$$

Applying (2.4), (2.3) and (2.5) we get, for $A = \bigcup_{i=1}^{k} B_{r_i}(x_i)$, that

$$\mu_\chi(\mathbb{R}^N \setminus A) \le (\mu_\chi - \sum_{i=1}^{k} \mu_{\chi_i})(\mathbb{R}^N) + \sum_{i=1}^{k} \mu_{\chi_i}(\mathbb{R}^N \setminus B_{r_i}(x_i)) < \varepsilon + \sum_{i=1}^{k} \varepsilon_i < 2\varepsilon .$$

Replacing ε by $\frac{\varepsilon}{2}$ we complete the proof. □

PROOF OF THEOREM 1.2. By hypotheses there exists an irrigation pattern $\chi \in P_S(\Omega)$ of finite cost $I_\alpha(\chi) < +\infty$, such that $\mu = \mu_\chi$. By Lemma A.2 we know that $d(F_\chi) = 1 < d_\alpha$, therefore we can reduce ourselves, as Remark A.3 suggests, to a pattern χ without dispersion. Moreover, if one considers a pattern which is optimal with respect to the cost functional, the pattern can also be supposed to be simple (see Definition A.7), see [3, Theorem 6.1].

So, for every $n \in \mathbb{N}$, we can apply Lemma 2.3 to the pattern χ and to $\varepsilon = 2^{-n} > 0$. Therefore for all $n \in \mathbb{N}$ there exists $A_n \subset \mathbb{R}^N$ which satisfies 1) and 2) of Lemma 2.3 for $\varepsilon = 2^{-n}$. For a fixed $h \in \mathbb{N}$ we shall denote by $D_h = \bigcap_{n>h} A_n$

Then

$$\mu_\chi(\mathbb{R}^N \setminus D_h) = \mu_\chi\Big(\bigcup_{n>h} \mathbb{R}^N \setminus A_n\Big) \le \sum_{n>h} \mu_\chi(\mathbb{R}^N \setminus A_n) \le \sum_{n>h} \frac{1}{2^n} = \frac{1}{2^h} . \tag{2.6}$$

Moreover, being $D_h \subset A_n$ for all $n > h$, by Lemma 2.3,1), D_h is covered by a finite number k of balls of radius r_i verifying $\sum_{i=1}^{k} r_i^{d_\alpha} < 2^{-n}$, from which

$\mathcal{H}^{d_\alpha}(D_h) = 0$ follows by the definition of Hausdorff outer measure. For all $i \in \mathbb{N}$

$$\mu_\chi\Big(\mathbb{R}^N \setminus \bigcup_{h\in\mathbb{N}} D_h\Big) \le \mu_\chi(\mathbb{R}^N \setminus D_i) \le \frac{1}{2^i} \ ,$$

therefore we have

$$\mu_\chi\Big(\mathbb{R}^N \setminus \bigcup_{h\in\mathbb{N}} D_h\Big) = 0$$

and so μ is concentrated on $\bigcup_{h\in\mathbb{N}} D_h$. Since, for all $h \in \mathbb{N}$, $\mathcal{H}^{d_\alpha}(D_h) = 0$ we get that μ is concentrated on a d_α-negligible set. □

PROOF OF THEOREM 1.1 (lower bound $d_c(\mu) \le d(\mu)$). By Theorem 1.2 we have proved in particular that, for every $\alpha \in \,]0, 1[$, if μ is α-irrigable then $d_c(\mu) \le d_\alpha$. By the definition of $d(\mu)$, taking the infimum on d_α in the above inequality, the thesis follows. □

3. Upper bound on $d(\mu)$.

The main goal of this section is the proof of the following theorem, from which the upper bound on $d(\mu)$ stated in Theorem 1.1 easily follows.

THEOREM 3.1. *Let μ be a probability measure and $\alpha \in \,]0, 1[$, then μ is α-irrigable provided $d_M(\mu) < d_\alpha$.*

To this aim, we need to introduce some definitions and to establish some preliminary lemmas.

DEFINITION 3.1. *Let $I = \{1, 2, \dots, n\} \subset \mathbb{N}$ be a finite set of indexes. We shall say that $(P_i, \gamma_i)_{i\in I}$ is a hierarchy of collectors if*

- *$\forall i \in I \ : \ P_i$ is a finite subset of $\mathbb{R}^N$ with k_i elements x^i_j, $1 \le j \le k_i$;*
- *$\forall i \in I, i \ne n$, γ_i maps P_i in P_{i+1} while γ_n is a map on P_n of constant value S (which is the "head" of the hierarchy and will be the source S in the applications).*

In the following we shall call each map γ_i the "dependence" map of the points $x^i_j \in P_i$ from those $x^{i+1}_j \in P_{i+1}$.

REMARK 3.1. *For a given hierarchy of collectors $(P_i, \gamma_i)_{i\in I}$, every time we fix a point $x = x_j^1 \in P_1$, we find, using the dependence maps, a chain of points $\{x, \gamma_1(x), \gamma_2(\gamma_1(x)), \dots, S\}$ which allows us to reach the source S "in a hierarchical way". We can consider the elements of such a chain as the vertices of a polygonal which runs with unitary speed. Reversing the time, we get a path which starts from the source S and arrives in x. We shall call by $g_x : \mathbb{R}_+ \to \mathbb{R}^N$ this path, parameterized in the whole of $\mathbb{R}_+$, by considering it constant after reaching x.*

In what follows, let us set, $\forall x \in P_1$, $\gamma^1(x) = \gamma_1(x)$, $\gamma^2(x) = \gamma_2(\gamma^1(x))$ and recoursively

$$\gamma^j(x) = \gamma_j(\gamma^{j-1}(x)) \in P_{j+1}\,.$$

For a given hierarchy of collectors $(P_i, \gamma_i)_{i\in I}$, we shall deal with a probability measure $\overline{\mu}_1$ concentrated on P_1, namely $\overline{\mu}_1 = \sum_{i=1}^{k_1} m_j^1 \delta_{x_j^1}$ is the sum of a finite number of Dirac masses centered on the points x_j^1 of P_1.

Being Ω a non atomic probability space, by Lyapunov Theorem, we can split Ω into $k_1 (= \mathrm{card}(P_1))$ sets Ω_j such that $|\Omega_j| = m_j^1$, i.e. we can split Ω into k_1 sets whose measures are just the masses m_j^1 we find in the points $(x_j^1)_{j\in k_1}$ at the base of the hierarchy.

DEFINITION 3.2. *Let $(P_i, \gamma_i)_{i\in I}$ be a hierarchy of collectors and $\overline{\mu}_1 = \sum_{j=1}^{k_1} m_j^1 \delta_{x_j^1}$ a probability measure concentrated on the base P_1 of the hierarchy.*

We shall say that $\chi : \Omega \times \mathbb{R}_+ \to \mathbb{R}^N$ is a distribution pattern relative to $\overline{\mu}_1$ and to the hierarchy $(P_i, \gamma_i)_{i\in I}$ if $\forall p \in \Omega$, and $\forall t \geq 0$:

$$\chi(p, t) = g_{x_j^1}(t) \qquad \text{for } p \in \Omega_j\,,$$

where the paths $g_{x_j^1}$ and the partition $(\Omega_j)_{1\leq j\leq k_1}$ are as above.

REMARK 3.2. *Let $(P_i, \gamma_i)_{i\in I}$ be a hierarchy of collectors and $\overline{\mu}_1 = \sum_{j=1}^{k_1} m_j^1 \delta_{x_j^1}$ a probability measure concentrated on the base P_1 of the hierarchy. Let χ be a distribution pattern relative to $\overline{\mu}_1$ and to the hierarchy $(P_i, \gamma_i)_{i\in I}$. Then, by construction, being $i_\chi(\Omega_j) = \{x_j^1\}$, we have*

$$\mu_\chi = \sum_{j=1}^{k_1} m_j^1 \delta_{x_j^1} = \overline{\mu}_1\,.$$

DEFINITION 3.3. *Let* $(P_i, \gamma_i)_{1\le i\le n}$ *be a hierarchy of collectors. For any discrete probability measure* $\overline{\mu}_1$ *concentrated on* P_1 *we shall recoursively call for all* $i \in \{2, \dots, n\}$, $\overline{\mu}_i$ *the image measure of* $\overline{\mu}_{i-1}$ *through the function* γ_{i-1}.

Each one of these measures can be considered as a discrete measure defined on the whole of the space and concentrated on P_i.

LEMMA 3.1. *Let* $(P_i, \gamma_i)_{i\in I}$ *be a hierarchy of collectors and* $\overline{\mu}_1$ *be a discrete probability measure concentrated on the base level* P_1 *of the hierarchy. Let* χ *be a distribution pattern relative to* $\overline{\mu}_1$ *and to the hierarchy, then*

$$I_\alpha(\chi) = \sum_{i\in I} \sum_{x\in P_i} (m_i(x))^\alpha |x - \gamma_i(x)| \,, \tag{3.1}$$

where, for all $i \in I$, and for all $x \in P_i$

$$m_i(x) = \overline{\mu}_i(\{x\})\,.$$

PROOF. We shall proceed by induction on $n = \mathrm{card}\,(I)$. The thesis is obvious in the case $n = 1$. Let us suppose that the statement is true for $\mathrm{card}\,(I) = n-1$ and let us prove that the statement is also true for $\mathrm{card}\,(I) = n$. Let us remark that each one of the $k_n = \mathrm{card}\,(P_n)$ elements of the last level set P_n can be seen as the head of a hierarchy of $n-1$ levels, given by the sets $P_i(x)$, where for all $1 \le i \le n-1$:

$$P_i(x) = \{y \in P_i \mid \gamma_{n-1}(\gamma_{n-2}((\gamma_i(y)))) = x\}\,.$$

and by the suitable restrictions of the maps γ_i, $i = 1, \dots, n-1$.

Therefore we can apply the induction hypotheses to the k_n branches $\overline{\chi}_x$ of χ which start from the point $x \in P_n$. So each one of these patterns has a cost which can be estimated by

$$I_\alpha(\overline{\chi}_x) = \sum_{i=1}^{n-1} \sum_{y\in P_i(x)} (m_i(y))^\alpha |x - \gamma_i(y)|\,.$$

To bring $\overline{\chi}_x$ back to the source S obtaining the pattern χ_x, restriction of χ to $\bigcup_{x_j^1 \in P_1(x)} \Omega_j$, we must add to $I_\alpha(\overline{\chi}_x)$ the cost necessary for the connection of x to the source S. Therefore we have

$$\begin{aligned} I_\alpha(\chi_x) &= (m_n(x))^\alpha |x - S| + I_\alpha(\overline{\chi}_x) \\ &= (m_n(x))^\alpha |x - \gamma_n(x)| + \sum_{i=1}^{n-1} \sum_{y\in P_i(x)} (m_i(y))^\alpha |x - \gamma_i(y)|\,. \end{aligned}$$

Since the whole χ can be regarded as a multiple branch starting from the source S which has the patterns χ_x as the corresponding single branches, by additivity we have

$$I_\alpha(\chi) = \sum_{x\in P_n} I_\alpha(\chi_x) = \sum_{x\in P_n} (m_n(x))^\alpha |x-\gamma_n(x)| + \sum_{x\in P_n}\sum_{i=1}^{n-1}\sum_{y\in P_i(x)} (m_i(y))^\alpha |x-\gamma_i(y)|$$
$$= \sum_{i=1}^{n}\sum_{x\in P_i} (m_i(x))^\alpha |x-\gamma_i(x)|\,.$$

□

Lemma 3.1 admits the following corollary.

Corollary 3.1. *Let $(P_i,\gamma_i)_{i\in I}$ be a hierarchy of collectors and $\overline{\mu}_1$ be a probability measure concentrated on the base level P_1 of the hierarchy. Let χ be a distribution pattern relative to $\overline{\mu}_1$ and to the hierarchy, then*

$$I_\alpha(\chi) \le \sum_{i\in I} k_i^{1-\alpha} l_i\ ,$$

where for all $i\in\{1,\ldots,n\}$

$$l_i = \max_{x\in P_i} |x-\gamma_i(x)|\,.$$

Proof. The thesis follows because for all $i\in I$:

$$\sum_{x\in P_i} (m_i(x))^\alpha \le (k_i)^{1-\alpha}\,.$$

Indeed, by Hölder inequality, being, for all $i\in I$, $\sum_{x\in P_i} m_i(x) = 1$, we have:

$$\sum_{x\in P_i} (m_i(x))^\alpha \le \left(\sum_{x\in P_i} m_i(x)\right)^\alpha \left(\sum_{x\in P_i} 1\right)^{1-\alpha} = k_i^{1-\alpha}\,.$$

□

Proof of Theorem 3.1. If $d_M(\mu) < d_\alpha$, we can fix a constant β such that $d_M(\mu) < \beta < d_\alpha$.

Given $n\in\mathbb{N}$, $n\ge 1$, let us consider a covering of $\operatorname{supp}(\mu)$ consisting of balls with radius 2^{-n}. Let us call X_n the set made of the centers of such balls and let us set $X_0=\{S\}$. We introduce for $n\ge 1$ the map $\varphi_n : X_n \to X_{n-1}$ which chooses, for every point $x\in X_n$, one of the closest points $\varphi_n(x)\in X_{n-1}$. It is easy to see that for $n\ge 2$ (and for $n\ge 1$, with a suitable choice of S and a normalization of the diameter of the support of μ)

$$\forall x \in X_n \;:\; |x - \varphi_n(x)| \le 3 \cdot 2^{-n} \,. \tag{3.2}$$

Moreover, by Lemma 1.1, being $d_M(\mu) < \beta$, we can choose X_n and a constant $C > 0$ so that

$$\operatorname{card}(X_n) \le C\,(2^{-n})^{-\beta} = C\,2^{n\beta} \,. \tag{3.3}$$

Let us now put a total order on X_n. On each center $x \in X_n$ we shall put the mass

$$m_x^n = \mu(B_{2^{-n}}(x) \setminus \bigcup_{y<x} B_{2^{-n}}(y)) \,.$$

In this way we get a probability measure $\mu_n = \sum_{x\in X_n} m_x^n \delta_x$ such that $\mu_n \rightharpoonup \mu$. Now, for a fixed $n \in \mathbb{N}$, all $1 \le i \le n$, let us call $P_i = X_{n-i+1}$ and $\gamma_i = \varphi_{n-i+1}$. By (3.3) we have:

$$\forall i \in \{1, \dots, n\} : k_i = \operatorname{card}(P_i) = \operatorname{card}(X_{n-i+1}) \le C\,(2^{-(n-i+1)})^{-\beta} \,, \tag{3.4}$$

while, by (3.2),

$$\forall i \in \{1, \dots, n\} : l_i = \max_{x\in P_i} |x - \gamma_i(x)| = \max_{x\in X_{n-i+1}} |x - \varphi_{n-i+1}(x)| \le 3(2^{-(n-i+1)}). \tag{3.5}$$

If we denote by χ_n a distribution pattern relative to the hierarchy of collectors $(P_i,\, \gamma_i)_{1\le i\le n}$ and to $\overline{\mu}_1 = \mu_n$, by Corollary 3.1, using also (3.4) and (3.5), we have

$$\begin{aligned} I_\alpha(\chi_n) &\le \sum_{i=1}^{n} (k_i)^{1-\alpha} l_i \le C^{1-\alpha} \sum_{i=1}^{n} [(2^{-(n-i+1)})^{-\beta}]^{1-\alpha} 3(2^{-(n-i+1)}) \\ &= 3C^{1-\alpha} \sum_{i=1}^{n} 2^{-(n-i+1)(-\beta(1-\alpha)+1)} = 3C^{1-\alpha} \sum_{j=1}^{n} 2^{-jb} \le \frac{3C^{1-\alpha}}{2^b - 1} \,, \end{aligned}$$

where, being $\beta < d_\alpha$, is

$$b = -\beta(1-\alpha) + 1 > 0 \,.$$

The independence on n of the above bound allows us to build a sequence of patterns $(\chi_n)_{n\in\mathbb{N}}$ to which we can apply the compactness theorem [4, Theorem 8.1] and to get, in this way, the existence of a limit pattern χ of finite cost such that $\mu_\chi = \mu$. □

It is worth remarking that the measure μ_n taken in the proof of Theorem 3.1 could be replaced by any probability measure centered on the points of X_n such that the Kantorovitch-Wasserstein distance between μ_n and μ (see Definition 5.3) is less or equal to 2^{-n}.

PROOF OF THEOREM 1.1 (upper bound $d(\mu) \leq \max\{d_M(\mu), 1\}$). Arguing by contradiction, let us suppose $d(\mu) > \max\{d_M(\mu), 1\}$. Then there exists a constant $\alpha \in]0, 1[$ such that $d_M(\mu) < d_\alpha < d(\mu)$. From one side $d_M(\mu) < d_\alpha$, so we have from Theorem 3.1 that μ is α-irrigable; on the other side $d_\alpha < d(\mu)$, so we get from Remark 1.3 (2) that μ cannot be α-irrigable. □

4. Remarks and examples.

DEFINITION 4.1. *Let $\alpha \in]0, 1[$ and let μ be a finite measure on $\mathbb{R}^N$. We shall call α-cost of the measure μ the value of the functional I_α on the optimal patterns χ which irrigate the measure μ.*

LEMMA 4.1. *Let $\alpha \in]0, 1[$, ν and μ be two finite measure on $\mathbb{R}^N$ such that $\nu \leq \mu$. If μ is α-irrigable then also ν is α-irrigable, moreover the α-cost to irrigate ν is less expensive than the α-cost for μ.*

PROOF. For any $n \in \mathbb{N}$, let us consider a countable borel partition $\mathcal{A}_n = (A_i^n)_{i \in I}$ of $\mathbb{R}^N$ made of sets of diameter less or equal to $\frac{1}{n}$ for all $i \in I$. By hypotheses there exists an irrigation pattern χ, defined on $\Omega \times \mathbb{R}_+$, where Ω is a probability space, s.t. $I_\alpha(\chi) < +\infty$ and $\mu_\chi = \mu$. Let us call, for all i, $\Omega_{i,n} = i_\chi^{-1}(A_i^n)$. By construction we get

$$|\Omega_{i,n}| = |i_\chi^{-1}(A_i^n)| = \mu_\chi(A_i^n) = \mu(A_i^n) \geq \nu(A_i^n)\,.$$

Therefore, being any $\Omega_{i,n}$ a non atomic set, by Lyapunov Theorem, $\Omega_{i,n}$ admits a subset $\Omega'_{i,n}$ such that $|\Omega'_{i,n}| = \nu(A_i^n)$. Let us consider $\Omega'_n = \bigcup_i \Omega'_{i,n}$ and let us denote $\chi_n = \chi_{|\Omega'_n}$.

By construction $\mu_{\chi_n} \rightharpoonup \nu$ and

$$I_\alpha(\chi_n) \leq I_\alpha(\chi) < +\infty\,. \tag{4.1}$$

Therefore, by compactness, we get a limit pattern $\overline{\chi}$ such that $\mu_{\overline{\chi}} = \nu$. Moreover, being I_α a lower semicontinuous functional, (4.1) gives $I_\alpha(\overline{\chi}) \leq \liminf_{n\to+\infty} I_\alpha(\chi_n) \leq I_\alpha(\chi)$. □

The following corollaries easily follow.

COROLLARY 4.1. *Let $\alpha \in]0, 1[$, $c \in \mathbb{R}$ and ν and μ two finite Radon measures on $\mathbb{R}^N$ such that $\nu \leq c\mu$. Then*

$$d(\nu) \leq d(\mu)\,.$$

COROLLARY 4.2. *Let μ and ν be finite Radon measures such that $c_1\mu \leq \nu \leq c_2\mu$ for some positive constants c_1, c_2. Then we have*

$$d(\nu) = d(\mu)\,.$$

REMARK 4.1. *The pattern $\overline{\chi}$, found in the proof of Lemma* 4.1, *is the limit pattern, modulo equivalence, of a sequence of subpatterns of χ but it is not a subpattern in general. So one could wonder if it is always possible to find a subpattern of χ which irrigates ν.*

The answer to this question is negative. For instance, one can consider $\Omega = [0, 1]$ and, for a.e. $p \in [0, 1]$ and for all $t \geq 0$,

$$\chi(p, t) = \min(p, t)\,. \tag{4.2}$$

It is clear that χ irrigates the Lebesgue measure μ_L on $[0, 1]$. *On the other side, it is not possible to find a subpattern of χ which can irrigate $\nu = \frac{1}{2}\mu_L$. Indeed, in such a case, one should find a subset $A \subset [0, 1]$ of density $\frac{1}{2}$ everywhere and this is not possible. A pattern $\overline{\chi} : \left[0, \frac{1}{2}\right] \times \mathbb{R}_+ \to \mathbb{R}$, provided by the proof of* Lemma 4.1 *is, for instance,*

$$\chi(p, t) = \min(2p, t)\,, \tag{4.3}$$

which irrigates $\frac{1}{2}\mu_L$.

The simple idea that the irrigability of a probability measure μ depends only on the dimension of the support is false. Indeed, $d_s(\mu)$ and $d(\mu)$ are not comparable in general, even if the dimensions $d_c(\mu)$ and $d_M(\mu)$ which respectively give a lower and an upper bound on $d_s(\mu)$ are also bounds for $d(\mu)$, as stated in Remark 1.7 and Theorem 1.1.

It is easy to see that, in general, $d_s(\mu) \not\leq d(\mu)$, as stated in the following example.

EXAMPLE 4.1. *There exist probability measures μ such that $d_s(\mu) = N$ (maximum possible value) and $d(\mu) = 1$ (minimum possible value) i.e. which are α-irrigable for all $\alpha \in]0, 1[$.*

PROOF. Let us call B the unit ball of $\mathbb{R}^N$, $S = 0$ and let $\tilde{B} = \{x_1, x_2, \ldots, x_n, \ldots\}$ be the countable set consisting in the points of B with rational coordinates.

Let us consider $\mu = \sum_{n\geq 1} \left(\frac{1}{2}\right)^n \delta_n$ where, $\forall n \in \mathbb{N}$, δ_n is a Dirac mass centered in x_n. By construction, $d_s(\mu) = N$. Moreover we shall prove that μ is α - irrigable for all $\alpha \in]0, 1[$, i.e. $d(\mu) = 1$. Let χ be the pattern which at unitary speed carries from S in the n-th point of $\tilde{B}$ the mass $\frac{1}{2^n}$. Then for any $\alpha \in]0, 1[$ we have: $I_\alpha(\chi) \leq \sum_{n\geq 1} \left(\frac{1}{2}\right)^{\alpha n} = \frac{1}{2^\alpha - 1} < +\infty$. Therefore, being by construction $\mu_\chi = \mu$, μ is α-irrigable. □

In order to show that also the converse inequality is, in general, not true, we shall point out the following property.

PROPOSITION 4.1. *Let $\alpha \in]0, 1[$ and μ be a probability measure which is not α-irrigable. Then for all $n \in \mathbb{N}$ it is possible to find a discrete approximation $\tilde{\mu}$ of μ, of sufficiently high resolution* (*see Definition* 5.1), *such that any pattern $\tilde{\chi}$ which irrigates $\tilde{\mu}$ has a cost $I_\alpha(\tilde{\chi}) \geq n$.*

PROOF. Assume by contradiction that we can find a sequence $(\tilde{\mu}_n)_{n\in\mathbb{N}}$ of discrete approximations of μ weakly converging to μ and a sequence $(\tilde{\chi}_n)_{n\in\mathbb{N}}$ of patterns, where, $\forall n \in \mathbb{N}$, $\tilde{\chi}_n$ irrigates $\tilde{\mu}_n$, such that, $\forall n \in \mathbb{N}$: $I_\alpha(\tilde{\chi}_n) < c$. Then we could apply the compactness theorem [4, Theorem 8.1] obtaining a limit pattern $\overline{\chi}$ of finite cost which irrigates μ. □

EXAMPLE 4.2. *There exists a probability μ with a countable support which is not α-irrigable for $\alpha = \frac{1}{N'}$.*

PROOF. Let B be the unit ball of $\mathbb{R}^N$, let μ_L be the Lebesgue measure on B and $\alpha = \frac{1}{N'}$. Being, by Theorem 1.2, μ_L not α-irrigable, by Proposition 4.1, we can consider a discretization μ_1 of μ_L such that for any pattern $\tilde{\chi}_1$ which irrigates μ_1: $I_\alpha(\chi_1) \geq 1$. Analogously, let μ_2 be a discretization of $\frac{1}{2}\mu_L$ distributed on $\frac{1}{2}B$ (ball of radius $\frac{1}{2}$) such that for any pattern χ_2 which irrigates μ_2: $I_\alpha(\chi_2) \geq 2$. Recoursively, for any $n \in \mathbb{N}$ let μ_n be a discretization of $\frac{1}{2^n}\mu_L$ restricted to $\frac{1}{n}B$ such that for any pattern χ_n which irrigates it,

$$I_\alpha(\chi_n) \geq n \tag{4.4}$$

holds true. Let $\overline{\mu} = \sum_{n\geq 1} \mu_n$ (normalized, if we really want to produce a probability measure) and let us remark that $\operatorname{supp}(\overline{\mu}) = \bigcup_{n\geq 1} \operatorname{supp}(\mu_n) \cup \{0\}$ and therefore, being for all $n \geq 1$ $\operatorname{supp}(\mu_n)$ a finite set, $\operatorname{supp}(\overline{\mu})$ is countable.

Let us show that $\overline{\mu}$ is not α-irrigable. Indeed, the α-irrigability of $\overline{\mu}$ would imply, by Lemma 4.1 (being $\mu_n \leq \overline{\mu}$ for all $n \geq 1$), that any μ_n is α-irrigable with a bounded cost and this is in contradiction with (4.4). □

A measure as in the above statement satisfies, in particular, the condition in the following one and shows that, in general, $d(\mu) \not\leq d_s(\mu)$.

EXAMPLE 4.3. *There exist probability measures μ such that $d_s(\mu) = 0$ (minimum possible value) and $d(\mu) = N$ (maximum possible value).*

We have stated in Section 1 that the information that, for a probability measure μ and a real number $\alpha \in]0, 1[$, the critical dimension d_α coincides with the irrigability dimension $d(\mu)$, (i.e. $\alpha = \frac{1}{(d(\mu))'}$) does not allow to decide whether the measure is irrigable or not. Examples 4.4 and 4.5 will motivate this claim.

EXAMPLE 4.4. *Let μ be an Ahlfors probability measure in dimension $\beta \geq 0$. Then μ is not α-irrigable if $d_\alpha = \beta = d(\mu)$.*

PROOF. The thesis follows from Corollary 1.6. □

REMARK 4.2. *One has Ahlfors regular measures for every dimension $\beta < N$. Indeed, let $\mathcal{C}$ be a selfsimilar (Cantor) set of $\mathbb{R}^N$ with dimension $\beta > 0$. Let us call $\mathcal{H}^\beta_{\lfloor \mathcal{C}}$ the Hausdorff measure distributed on $\mathcal{C}$, i.e. the measure on $\mathbb{R}^N$ defined setting $\forall X \subset \mathbb{R}^N$*

$$\mathcal{H}^\beta_{\lfloor \mathcal{C}}(X) = \mathcal{H}^\beta(X \cap \mathcal{C}) .$$

Then $\mathcal{H}^\beta_{\lfloor \mathcal{C}}$ is Ahlfors regular with dimension β.

EXAMPLE 4.5. *There exist some measures μ for which $d(\mu)$ is a minimum, i.e. there exist some measures μ and some exponents $\alpha \in]0, 1[$ such that $d(\mu) = d_\alpha$ and μ is α-irrigable.*

PROOF. Indeed, let us fix $\alpha \in]0, 1[$ and let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a sequence of self similar (Cantor) sets in $\mathbb{R}^N$ with dimension d_{α_n} where $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence converging to α from below, by Corollary 1.5 and Remark 4.2 we know that $d(\mathcal{H}^{d_{\alpha_n}}_{\lfloor\mathcal{C}_n}) = d_{\alpha_n} < d_\alpha$ and, by Remark 1.3 (1), we get that $\mathcal{H}^{d_{\alpha_n}}_{\lfloor\mathcal{C}_n}$ is α-irrigable. Let us consider a suitable sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers, sufficiently small to allow us to consider $\mu = \sum_{n\in\mathbb{N}} \varepsilon_n \mathcal{H}^{d_{\alpha_n}}_{\lfloor\mathcal{C}_n}$. We know, by Corollary 4.2 that also $\varepsilon_n\mathcal{H}^{d_{\alpha_n}}_{\lfloor\mathcal{C}_n}$ are irrigable and we call χ_n an irrigation pattern which irrigates $\varepsilon_n\mathcal{H}^{d_{\alpha_n}}_{\lfloor\mathcal{C}_n}$ (i.e. such that $I_\alpha(\chi_n) < +\infty$ and $\mu_{\chi_n} = \varepsilon_n\mathcal{H}^{d_{\alpha_n}}_{\lfloor\mathcal{C}_n}$). Under the choice of a sufficiently infinitesimal sequence of coefficients $(\varepsilon_n)_{n\in\mathbb{N}}$, we have $\sum_{n\in\mathbb{N}} I_\alpha(\chi_n) < +\infty$.

Now let us consider the bunch χ of the sequence of patterns χ_n (see (A.1)) so that by Remark A.1 we have

$$\mu_\chi = \mu \tag{4.5}$$

and

$$I_\alpha(\chi) \le \sum_{n\in\mathbb{N}} I_\alpha(\chi_n) < +\infty\,. \tag{4.6}$$

Equality (4.5) and inequality (4.6) give the α-irrigability of μ and therefore $d(\mu) \le d_\alpha$. Moreover $d(\mu) \ge d_\alpha$. Indeed being, for all $n \in \mathbb{N}$, $\mu \ge \varepsilon_n\mathcal{H}^{d_{\alpha_n}}_{\lfloor\mathcal{C}_n}$ we get, by Corollary 4.1 and Remark 4.2 that $d(\mu) \ge d(\varepsilon_n\mathcal{H}^{d_{\alpha_n}}_{\lfloor\mathcal{C}_n}) = d(\mathcal{H}^{d_{\alpha_n}}_{\lfloor\mathcal{C}_n}) = d_{\alpha_n} \to d_\alpha$. □

5. Discretizations and resolution dimensions of a measure.

DEFINITION 5.1. *We shall say that a measure μ is a discrete measure if*

$$\mathrm{card}(\,\mathrm{supp}\,(\mu)) < \infty$$

and we shall call card(supp (μ)) *"resolution" of μ.*

DEFINITION 5.2. *For every $n \in \mathbb{N}$ we shall denote by D_n the set containing all the discrete probability measures whose resolution is less or equal to n. Equivalently, D_n is the set of all the convex combinations of n Dirac masses.*

For any $p \ge 1$ we recall the definition of Kantorovitch-Wasserstein distance of index p.

DEFINITION 5.3. *Let $p \ge 1$ and let μ, ν be two probability measures.*

We define the Kantorovitch-Wasserstein distance of index p between μ and ν by

$$d_p(\mu, \nu) = \left(\min_\sigma \int\limits_{\Omega\times\Omega} |x-y|^p d\sigma \right)^{\frac{1}{p}},$$

where the minimum is taken on all the transport plans σ which lead μ to ν, i.e. measures on $\Omega \times \Omega$ such that their push forward measures by the first and the second projection on Ω respectively are μ and ν ($\pi_{1\#}\sigma = \mu$ and $\pi_{2\#}\sigma = \nu$) (see [1] *for more details).*

DEFINITION 5.4. *Let μ be a probability measure. For every $n \in \mathbb{N}$, given $p \geq 1$, we shall denote by μ_n (or, when necessary, by μ_n^p) one of the elements of D_n of minimal distance with respect to the Kantorovitch-Wasserstein distance of index p from μ. We shall refer to μ_n as to a discretization of resolution n of μ (with respect to the index p).*

PROPOSITION 5.1. *Let $1 \leq p \leq q$ and let μ, ν be two probability measures. Then*

$$d_p(\mu, \nu) \leq d_q(\mu, \nu) \tag{5.1}$$

and

$$d_q(\mu, \nu) \leq d^{1-\frac{p}{q}} (d_p(\mu, \nu))^{\frac{p}{q}}, \tag{5.2}$$

where the constant d is the diameter of $\operatorname{supp}(\mu) \cup \operatorname{supp}(\nu)$.

PROOF. Let τ be an optimal transport plan from μ to ν with respect to the Kantorovitch-Wasserstein distance of index q. Then, by Hölder inequality,

$$[d_p(\mu, \nu)]^p \leq \int\limits_{\Omega\times\Omega} |x-y|^p d\tau \leq \left(\int\limits_{\Omega\times\Omega} |x-y|^q d\tau \right)^{\frac{p}{q}} \left(\int\limits_{\Omega\times\Omega} d\tau \right)^{1-\frac{p}{q}} = [d_q(\mu, \nu)]^p.$$

To prove (5.2) we shall consider an optimal transport plan τ from μ to ν with respect to the p distance. Let us call $d = (\operatorname{supp}(\mu) \cup \operatorname{supp}(\nu))$, then

$$[d_q(\mu, \nu)]^q \leq \int\limits_{\Omega\times\Omega} |x-y|^q d\tau \leq d^{q-p} \int\limits_{\Omega\times\Omega} |x-y|^p d\tau = d^{q-p} [d_p(\mu, \nu)]^p,$$

from which the thesis follows. □

In the following, for any $n \in \mathbb{N}$ and $p \geq 1$, we shall use the Kantorovitch-Wasserstein distance of index p of μ from D_n

$$\delta_n^p = d_p(\mu, \mu_n) = d_p(\mu, D_n) \,, \tag{5.3}$$

to test "how good" a discretization of resolution n can be. When we shall deal with more than one measure we shall use the more detailed notation $\delta_n^p(\mu) = d_p(\mu, D_n)$.

In [3] the following proposition, which gives a relation between the cost of an irrigation pattern χ and the Kantorovitch-Wasserstein distance $\delta_1^{\frac{1}{\alpha}}(\mu_\chi)$ of the irrigation measure μ_χ from a Dirac delta, has been proved.

PROPOSITION 5.2. *Let χ be an irrigation pattern, with a source point S, then*

$$\delta_1^{\frac{1}{\alpha}}(\mu_\chi) \leq d_{\frac{1}{\alpha}}(\mu_\chi, \delta_S) \leq I_\alpha(\chi) \,.$$

REMARK 5.1. *Let $1 \leq p \leq q$, $n \in \mathbb{N}$, $n \geq 1$ and let μ be a probability measure. Then from* (5.1) *and* (5.2), *applied for $\nu = \mu_n^p$ and $\nu = \mu_n^q$, we get*

$$\delta_n^p \leq \delta_n^q \tag{5.4}$$

and

$$\delta_n^q \leq d^{1-\frac{p}{q}} \left(\delta_n^p\right)^{\frac{p}{q}} \,, \tag{5.5}$$

where the constant d is the diameter of supp (μ).

It is clear that, increasing the number n, the discretizations μ_n become more accurate, therefore it is rather natural to make some decay hypothesis on δ_n^p.

DEFINITION 5.5. *Let μ be a probability measure and $p \geq 1$, then we shall call* resolution dimension *of μ of index p the constant $d_r^p(\mu)$ defined as follows*

$$d_r^p(\mu) = \left(- \limsup_{n \to +\infty} \log_n \delta_n^p \right)^{-1} \,. \tag{5.6}$$

REMARK 5.2. *Let $0 < a < (d_r^p(\mu))^{-1}$, then there exists $\overline{n}$ such that*

$$\forall n \geq \overline{n} \,:\, \delta_n^p \leq n^{-a} \,.$$

Conversely, if $a > (d_r^p(\mu))^{-1}$ *then for any* $C > 0$ *we have*

$$\delta_n^p > n^{-a}$$

for arbitrarily large values of $n \in \mathbb{N}$.

Proposition 5.1 allows us to state the corresponding properties of $d_r^p(\mu)$ in terms of the index p.

PROPOSITION 5.3. *Let* $1 \leq p \leq q$ *and* μ *a probability measure, then*

$$d_r^p(\mu) \leq d_r^q(\mu) \tag{5.7}$$

and

$$d_r^q(\mu) \leq \frac{q}{p} d_r^p(\mu)\,. \tag{5.8}$$

PROOF. Taking into account (5.6), both inequalities easily follow from (5.4) and (5.5). □

REMARK 5.3. *It is useful to remark that, by* (5.7) *and* (5.8), $d_r^p(\mu)$ *changes with continuity with respect to the index* p. *Moreover, if there exists a index* $p \geq 1$ *for which* $d_r^p(\mu) = 0$, *then for all* $q < +\infty$ $d_r^q(\mu) = 0$. *This means that, in such a case, we can not change the resolution dimension of* μ *acting on the index* p *as far as it is finite.*

In Appendix B we shall prove the following propositions.

PROPOSITION 5.4. *Let* μ *be a probability measure, then*

$$d_r^\infty(\mu) = d_M(\mu)\,.$$

PROPOSITION 5.5. *Let* μ *be a probability measure. Then*

$$d_c(\mu) \leq d_r^1(\mu) \leq d_r^p(\mu) \qquad \forall p \geq 1.$$

REMARK 5.4. *Since in the case* $p = +\infty$ *the dimension* $d_r^p(\mu)$ *agrees with* $d_M(\mu)$ *we shall use the notation* d_r^∞ *in order to denote a weaker case, according to* Proposition 5.3, of the dimension of index $+\infty$, defined as

$$d_r^\infty(\mu) = \sup_{p \geq 1} d_r^p(\mu)\,.$$

Example B.1 will show how for $p = +\infty$ the “strong” and the “weak” dimensions $d_M(\mu)$ and $d_r^\infty(\mu)$ are, in general, distinct.

By the following lemmas we shall estimate the resulution dimension $d_r^p(\mu)$ in the case in which the probability measure μ enjoys some regularity properties, beginning by considering a probability measure which satisfies the lower Ahlfors regularity (LAR). As a consequence of Corollary 1.3, taking into account that, by Proposition 5.4, $d_M(\mu) = d_r^\infty(\mu)$ we have the following corollary.

Corollary 5.1. *Let μ be a probability measure which satisfies (LAR) in dimension $\beta \geq 0$.*
Then

$$d_r^p(\mu) \leq \beta \qquad \forall p \geq 1\,. \tag{5.9}$$

Proof. Indeed, by Corollary 1.3, we have $d_r^\infty(\mu) = \sup_{p\geq 1} d_r^p(\mu) \leq d_M(\mu) \leq \beta$. □

In the case in which a probability measure μ satisfies the upper Ahlfors regularity (UAR), by Corollary 1.4 Proposition 5.5 admits the following corollary.

Corollary 5.2. *Let μ be a probability measure such that (UAR) holds true. Then*

$$\beta \leq d_r^1(\mu)\,. \tag{5.10}$$

From Corollary 5.1 and Corollary 5.2 we easily get the following proposition.

Proposition 5.6. *Let μ be an Ahlfors regular probability measure of dimension $\beta \geq 0$. Then*

$$d_r^p(\mu) = \beta \qquad \forall p \geq 1\,. \tag{5.11}$$

Remark 5.5. *So, when the measure is Ahlfors regular, the value of the resolution dimensions s does not depend on the index, while for a generic measure, as it will be shown in the Appendix B by some examples, the resolution dimension is "out of focus" in the sense that different indexes give different values. We shall show that, in any case, it is always possible to find an index, suitably characterized, which gives a resolution dimension which coincides with the irrigability dimension of the measure (see* Theorem 8.1 *below).*

6. Irrigability results via resolution dimension.

In this section we shall establish some irrigability estimates, in particular we shall prove the following proposition.

PROPOSITION 6.1. *Let μ be a probability measure and $p \geq 1$. If $p' > d_r^p(\mu)$, then μ is α-irrigable, with $\alpha = \frac{1}{p}$.*

The idea of the proof consists in fixing $n \in \mathbb{N}$ and in using the distribution pattern introduced in Section 2, induced by a hierarchy related to the measures μ_h, in order to irrigate μ_n. An estimate of the cost and a passage to the limit will then ensure the irrigability of μ.

More precisely, let us fix an integer $k > 1$ (we shall assume that k is large enough to have the estimate

$$2k^{-a} < 1 \,, \tag{6.1}$$

where we choose a number a such that $0 < 1 - \alpha = \frac{1}{p'} < a < (d_r^p(\mu))^{-1}$, such a bound will be useful later on). Then we shall denote by X_h the support of μ_{k^h} for any $h \leq n$. Let $X_{h+1} = \{x_1, x_2, \ldots, x_{k^{h+1}}\}$ and let $m_i = \mu_{k^{h+1}}(\{x_i\})$. We would like to have a map $\varphi_h : X_{h+1} \to X_h$ such that $d_p(\mu_{k^h}, \mu_{k^{h+1}}) = \left(\sum_{i=1}^{k^{h+1}} m_i l_i^p\right)^{\frac{1}{p}}$, where $l_i = |x_i - \varphi_h(x_i)|$. However such a formula would require the Kantorowich-Wassernstain distance between $\mu_{k^{h+1}}$ and μ_{k^h} to be achieved by a transport map, while the discrete nature of $\mu_{k^{h+1}}$ guarantees only the existence of an optimal transport plan, see [1]. Roughly speaking, if we want to carry in an optimal way the masses m_i, given on X_{h+1}, to the set X_h in order to reconstruct μ_{k^h}, we cannot bring each m_i to a unique point $\varphi_h(x_i)$ but we must split it in several parts and bring each piece to a different point. To avoid this problem, we shall replace the measures μ_{k^h} by other measures $\tilde{\mu}_{k^h}$, recoursively defined taking $\tilde{\mu}_{k^n} = \mu_{k^n}$ and proceeding backward by choosing, for $h < n$, a measure $\tilde{\mu}_{k^h}$ which gives an optimal approximation of $\tilde{\mu}_{k^{h+1}}$ on D_{k^h}.

For $h \leq n$, let $\tilde{X}_h = \operatorname{supp}(\tilde{\mu}_{k^h})$, $\tilde{X}_h = \{x_1^h, x_2^h, \ldots, x_{k^h}^h\}$ and $m_i^h = \tilde{\mu}_{k^h}(\{x_i^h\})$. Now, for $h < n$, it is not difficult to choose $\tilde{\mu}_{k^h}$ in such a way that an optimal transport plan for the Kantorowich-Wassernstain distance of index p between $\tilde{\mu}_{k^{h+1}}$ and $\tilde{\mu}_{k^h}$ can be induced by a transport map $\varphi_h : \tilde{X}_{h+1} \to \tilde{X}_h$.

Indeed, if an optimal transport plan splits a mass m_i^{h+1} bringing each piece to a different point of $\tilde{X}_h$, we just need to modify the values of the masses m_j^h in the points of $\tilde{X}_h$ of minimal distance from x_i^{h+1} in such a way to let the mass m_i^{h+1} fully carried to one of such points, arbitrarily chosen, which we shall chose as $\varphi_h(x_i^{h+1})$. Such a modification does not affect the minimality of $\tilde{\mu}_{k^n}$.

Therefore, by letting

$$\tilde{l}_i^{h+1} = |x_i^{h+1} - \varphi_h(x_i^{h+1})| ,$$

we can be sure that

$$d_p(\tilde{\mu}_{k^h}, \tilde{\mu}_{k^{h+1}}) = \left(\sum_{i=1}^{k^{h+1}} m_i^{h+1} (\tilde{l}_i^{h+1})^p \right)^{\frac{1}{p}} . \tag{6.2}$$

In the following, for $p \geq 1$ and for all $h \in \{0, 1, \ldots, n\}$, we shall set

$$\tilde{\delta}_{k^h}^p = d_p(\mu, \tilde{\mu}_{k^h}) ,$$

beside the already introduced notation

$$d_{k^h}^p = d_p(\mu, \mu_{k^h}) .$$

The following metric lemmas, which are only based on the optimality of $\tilde{\mu}_{k^h}$ asked in the recursive choice, will provide an estimate of the left hand side of (6.2).

LEMMA 6.1. *For all $p \geq 1$*

$$\forall h < n : \quad \tilde{\delta}_{k^h}^p \leq \sum_{i=0}^{n-h} 2^i \delta_{k^{h+i}}^p . \tag{6.3}$$

PROOF. In the following we will forget the index p. The result easily follows from an iteration of the following inequality

$$\begin{aligned} \forall h < n : \qquad \tilde{\delta}_{k^h} &\leq d(\tilde{\mu}_{k^h}, \tilde{\mu}_{k^{h+1}}) + d(\tilde{\mu}_{k^{h+1}}, \mu) \leq d(\mu_{k^h}, \tilde{\mu}_{k^{h+1}}) + \tilde{\delta}_{k^{h+1}} \\ &\leq d(\mu_{k^h}, \mu) + d(\mu, \tilde{\mu}_{k^{h+1}}) + \tilde{\delta}_{k^{h+1}} = \delta_{k^h} + 2\tilde{\delta}_{k^{h+1}} , \end{aligned} \tag{6.4}$$

which just uses the triangular inequality and the optimality of $\tilde{\mu}_{k^h}$. Then (6.3) follows by induction on $n-h$. It holds true for $h = n$ (being $\tilde{\mu}_{k^n} = \mu_{k^n}$) then, if we assume it true if h is replaced by $h+1$, taking into account (6.4) we have

$$\tilde{\delta}_{k^h} \leq \delta_{k^h} + 2\tilde{\delta}_{k^{h+1}} \leq 2 \sum_{i=0}^{n-h-1} 2^i \delta_{k^{h+i+1}} + \delta_k^h = \sum_{i=0}^{n-h-1} 2^{i+1} \delta_{k^{h+i+1}} + \delta_{k^h} = \sum_{i=0}^{n-h} 2^i \delta_{k^{h+i}} .$$

LEMMA 6.2. *Given* $p \geq 1$, $a < (d_r^p(\mu))^{-1}$. *Then* $\exists C > 0$ *such that*

$$\forall h \leq n : \qquad \tilde{\delta}^p_{k^h} \leq C(k^h)^{-a} \,. \tag{6.5}$$

PROOF. By Lemma 6.1 and Remark 5.2, we have

$$\begin{aligned}\delta^p_{k^h} &\leq \sum_{i=0}^{n-h} 2^i \delta^p_{k^{h+i}} \leq C \sum_{i=0}^{n-h} 2^i (k^{h+i})^{-a} = C(k^h)^{-a} \sum_{i=0}^{n-h} (2k^{-a})^i \\ &< C(k^h)^{-a} \sum_{i=0}^{+\infty} (2k^{-a})^i = C(k^h)^{-a} \,,\end{aligned}$$

where for the last inequality we have used (6.1). □

By the triangular inequality and (6.5) we get the following corollary.

COROLLARY 6.1. *Let* $p \geq 1$ *and let* $a < (d_r^p(\mu))^{-1}$. *Then* $\exists C > 0$ *such that*

$$\forall h < n : \quad d_p(\tilde{\mu}_{k^h}, \tilde{\mu}_{k^{h+1}}) \leq C(k^h)^{-a} \,. \tag{6.6}$$

Combining (6.2) *and* (6.6) *we have*

$$\left(\sum_{i=1}^{k^{h+1}} m_i^{h+1} (\tilde{l}_i^{h+1})^p \right)^{\frac{1}{p}} \leq C(k^h)^{-a} \,. \tag{6.7}$$

Now let us call χ_n a distribution pattern relative to the hierarchy $((P_i, \gamma_i))_{i \in I}$, $I = \{1, \ldots, n\}$, (where, for $1 \leq i \leq n$, $P_i = \tilde{X}_{n-i+1}$ and, for $1 \leq i < n$, $\gamma_i = \varphi_{n-i}$ while γ_n is the constant map on P_n of a constant value given by the source S) and to the measure $\overline{\mu}_1 = \mu_{k^n}$ defined on the basic level $P_1 = \tilde{X}_n$. Lemma 3.1 allows us to prove Proposition 6.1.

PROOF OF PROPOSITION 6.1. Since $p' > d_r^p(\mu)$, we can fix a such that $0 < 1 - \alpha = \dfrac{1}{p'} < a < (d_r^p(\mu))^{-1}$ and $k \in \mathbb{N}$ satisfying (6.1). Given $n \in \mathbb{N}$, we get the existence of a sequence $(\tilde{\mu}_{k^h})_{0 \leq h \leq n}$ such that Corollary 6.1 holds true. The cost $I_\alpha(\chi_n)$ needed to irrigate μ_{k^n} can be estimated with the use of (3.1) in Lemma 3.1. Taking into account (6.7) and calling $b = a - 1 + \alpha > 0$ we get by Hölder Inequality

$$
\begin{aligned}
I_\alpha(\chi_n) &= \sum_{i=1}^{n}\sum_{j=1}^{k^{n-i+1}} (m_j^{n-i+1})^\alpha |x_j^{n-i+1} - \gamma_i(x_j^{n-i+1})| = \sum_{i=1}^{n}\sum_{j=1}^{k^{n-i+1}} (m_j^{n-i+1})^\alpha \tilde{l}_j^{n-i+1} \\
&\leq \sum_{i=1}^{n}\left[\sum_{j=1}^{k^{n-i+1}} m_j^{n-i+1}(\tilde{l}_j^{n-i+1})^{\frac{1}{\alpha}}\right]^\alpha (k^{n-i+1})^{1-\alpha} \\
&\leq Ck^a \sum_{i=1}^{n} (k^{n-i+1})^{-(a-1+\alpha)} < Ck^a \sum_{i=1}^{+\infty} (k^{-b})^i < +\infty\,.
\end{aligned}
$$

Therefore we get a bound on the cost to irrigate μ_{k^n} which does not depend on n. Therefore there exist a constant $\tilde{C} = Ck^a > 0$ such that we can irrigate every discretization spending at most $\tilde{C}\sum_{i=1}^{+\infty}(k^{-b})^i$, where $b = a - 1 + \alpha > 0$. So, by the compactness theorem [4, Theorem 8.1], we have, passing to a subsequence, a limit pattern modulo equivalence χ such that $I_\alpha(\chi) < +\infty$. By construction, its irrigation measure μ_χ is just the measure μ which is therefore α-irrigable. □

Taking into account that $d_\alpha = \left(\frac{1}{\alpha}\right)'$, we can also restate Proposition 6.1 in the following way.

PROPOSITION 6.2. *Let μ be a probability measure for which there exists a constant $\alpha \in\,]0, 1[$ such that*

$$d_r^{\frac{1}{\alpha}}(\mu) < d_\alpha\,.$$

Then μ is α-irrigable.

COROLLARY 6.2. *Let μ be a probability measure and let $p \geq 1$ be a solution to*

$$d_r^p(\mu) \leq p'\,. \tag{6.8}$$

Then

$$d(\mu) \leq p'\,. \tag{6.9}$$

PROOF. Let $\alpha \in\,]0, 1[$ be such that $d_\alpha > p'$. From $d_\alpha > p'$ we get $\frac{1}{\alpha} < p$, therefore, applying (5.7) and taking into account (6.8), we have $d_r^{\frac{1}{\alpha}}(\mu) \leq d_r^p(\mu) \leq p' < d_\alpha$. So, by Proposition 6.2, we know that μ is α-irrigable, therefore $d(\mu) \leq d_\alpha$. Letting $d_\alpha \to p'$ we get the thesis. □

7. Nonirrigability results via resolution dimension.

The aim of this section is to prove Proposition 7.1, or equivalently Proposition 7.2, which gives the counterpart of the results stated in Proposition 6.1 (and respectively of Proposition 6.2).

PROPOSITION 7.1. *Let μ be a probability measure and $p \geq 1$. If $p' < d_r^p(\mu)$, then μ is not α-irrigable, with $\alpha = \dfrac{1}{p}$.*

Taking into account that $d_\alpha = \left(\dfrac{1}{\alpha}\right)'$, Proposition 7.1 can be also restated in the following way.

PROPOSITION 7.2. *Let μ be a probability measure for which there exists a constant $\alpha \in]0, 1[$ such that*

$$d_r^{\frac{1}{\alpha}}(\mu) > d_\alpha .$$

Then μ is not α-irrigable.

Proposition 7.2 admits the following corollary whose proof is similar to the proof of Corollary 6.2.

COROLLARY 7.1. *Let μ be a probability measure and let $p \geq 1$ be a solution to*

$$d_r^p(\mu) \geq p' . \tag{7.1}$$

Then

$$d(\mu) \geq p' . \tag{7.2}$$

The proof of Proposition 7.1 is based on the semicontinuity properties of two functions which we are going to introduce and which, time by time, give the maximum cost of the single or multiple branches (see Definition A.9).

DEFINITION 7.1. *Let χ be an irrigation pattern, then we shall consider the following two functions W_χ and S_χ defined on $\mathbb{R}_+$ by setting, for all $t \geq 0$,*

$$W_\chi(t) = \max\{I_\alpha(V) \,|\, V \in \mathcal{V}_t(\chi)\} \tag{7.3}$$

$$S_\chi(t) = \max\{I_\alpha(V) \,|\, V \in \mathcal{V}_t^s(\chi)\} . \tag{7.4}$$

We state some properties enjoyed by the above defined functions.

PROPOSITION 7.3. *Let χ be an irrigation pattern of finite cost. Then*

$$\forall t \geq 0 : \qquad S_\chi(t) \leq W_\chi(t) \tag{7.5}$$

and

$$\forall t_1 < t_2 : \qquad S_\chi(t_1) \geq W_\chi(t_2) . \tag{7.6}$$

So, in particular, S_χ and W_χ are decreasing functions.

PROOF. The proof of the statement relies on the definition of strict equivalence relation (see Definition A.4). Indeed, a strict vessel at time t contains a multiple vessel at a bigger time. □

PROPOSITION 7.4. *Let χ be an irrigation pattern of finite cost. Then S_χ is a lower semicontinuous function.*

PROOF. In view of Proposition 7.3, the statement is equivalent to the right continuity of S_χ.

Let $(t_n)_{n\in\mathbb{N}}$ be a decreasing sequence of real positive numbers such that $\lim_{n\to+\infty} t_n = \bar{t} \in \mathbb{R}_+$. We shall prove that $\lim_{n\to+\infty} S_\chi(t_n) = S_\chi(\bar{t})$. Let $\overline{V} \in \mathcal{V}^s_{\bar{t}}(\chi)$ such that $S_\chi(\bar{t}) = I_\alpha(\overline{V})$ and fix $\overline{p} \in \overline{V}$. Let us consider $V_n = [\overline{p}]^s_{t_n} \in \mathcal{V}^s_{t_n}(\chi)$. The sequence $(V_n)_{n\in\mathbb{N}}$ is monotone increasing under inclusion, moreover $\bigcup_{n\in\mathbb{N}} V_n = \overline{V}$.

Let us set, for any $n \in \mathbb{N}$, $A_n = V_n \times [t_n, +\infty[$, then using Remark A.11 we have

$$S_\chi(\bar{t}) = I_\alpha(\overline{V}) = I_\alpha\Big(\bigcup_{n\in\mathbb{N}} V_{t_n}, \bar{t}\Big) = \nu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \lim_{n\to+\infty} \nu(A_n) =$$

$$= \lim_{n\to+\infty} I_\alpha(V_n, t_n) \leq \lim_{n\to+\infty} S_\chi(t_n) . \qquad \square$$

PROPOSITION 7.5. *Let χ be an irrigation pattern of finite cost. Then W_χ is an upper semicontinuous function.*

PROOF. In view of Proposition 7.3, the statement is equivalent to the left continuity of W_χ. Let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence of real positive numbers less or equal to $\bar{t}$ such that $\lim_{n\to+\infty} t_n = \bar{t} \in \mathbb{R}_+$.

Let us consider, for any $n \in \mathbb{N}$, V_n a vessel at time t_n such that

$$I_\alpha(V_n) = W_\chi(t_n) .$$

Let us set, for any $n \in \mathbb{N}$, $A_n = V_n \times [t_n, +\infty[$, then using Remark A.11 we have

$$\nu(A_n) = I_\alpha(V_n, t_n) \geq I_\alpha(\overline{V}, \bar{t})\,. \tag{7.7}$$

We can assume that $\lim_{n\to+\infty} \nu(A_n) = \lim_{n\to+\infty} W_\chi(t_n) > 0$ (otherwise we have nothing to prove), therefore $(A_n)_{n\in\mathbb{N}}$ admits a subsequence, still denoted by $(A_n)_{n\in\mathbb{N}}$, with a nonempty intersection.

By using Lemma A.1 we get that the sequence of the vessels V_n is decreasing. Let us set $V = \bigcap_{n\in\mathbb{N}} V_n$. By construction, we have that V is a vessel at time $\bar{t}$. Consequently, being $(t_n)_{n\in\mathbb{N}}$ increasing, it follows that $(A_n)_{n\in\mathbb{N}}$, is decreasing. Therefore, by (7.7), we have

$$I_\alpha(V, \bar{t}) = \nu\Big(\bigcap_{n\in\mathbb{N}} A_n\Big) = \lim_{n\to+\infty} \nu(A_n) = \lim_{n\to+\infty} I_\alpha(V_n, t_n) = \lim_{n\to+\infty} W_\chi(t_n)$$

and so

$$\lim_{n\to+\infty} W_\chi(t_n) = I_\alpha(V) \leq W_\chi(\bar{t})\,.$$

□

Propositions 7.4 and 7.5 allow us to give the following proposition.

Proposition 7.6. *Let χ be an irrigation pattern, with a finite cost $I_\alpha(\chi) < +\infty$. Let $0 < a < I_\alpha(\chi)$. Then there exists a multiple branch χ' of χ such that $I_\alpha(\chi') \geq a$ which is the bunch of single branches $(\chi'_s)_j$ such that $I_\alpha((\chi'_s)_j) \leq a$ for all j.*

Proof. Let $\bar{t} = \sup\{t \,|\, W_\chi(t) \geq a\}$. Using Proposition 7.5 we get the existence of a vessel $V \in \mathcal{V}_{\bar{t}}(\chi)$ such that $I_\alpha(V) \geq a$. On the other hand, being $S_\chi(t) \leq W_\chi(t)$, by Proposition 7.4 we know that can not exist any vessel $V^s \in \mathcal{V}^s_{\bar{t}}(\chi)$ such that $I_\alpha(V^s) > a$. Indeed, on the contrary, by using the right continuity of S_χ, we would get a time $t > \bar{t}$ such that $W_\chi(t) \geq S_\chi(t) \geq a$, in contradiction to the maximality of $\bar{t}$. □

Theorem 7.1. *Let $\chi : \Omega \times [0, +\infty[\to \mathbb{R}^N$ be an irrigation pattern with a finite cost $c = I_\alpha(\chi)$. Then for $n \geq 1$ there exist n source points for a finite number of patterns χ_i such that*

1) $\forall i \; I_\alpha(\chi_i) \leq \dfrac{c}{n}$
2) $\mu_\chi = \sum_i \mu_{\chi_i}$.

PROOF. Let us fix $n \in \mathbb{N}$, $n \geq 1$, and let us apply Proposition 7.6 to the constant $a = \frac{c}{n} < c$, where $c = I_\alpha(\chi) < +\infty$. Then there exists $t_n \geq 0$ and a multiple branch χ' of χ such that $I_\alpha(\chi') \geq \frac{c}{n}$ and χ' is the bunch of single branches whose cost is less or equal to $\frac{c}{n}$. We can regard χ' as the union of a finite number of (not necessarily single) branches with a cost less or equal to $\frac{c}{n}$. If we consider the pattern $\chi \setminus \chi'$ of χ stumped of the branch χ' (see Definition A.8), according to Lemma A.3, we have that $I_\alpha(\chi \setminus \chi') \leq$ $\leq I_\alpha(\chi) - I_\alpha(\chi') \leq c - \frac{c}{n}$. Let us apply Proposition 7.6 with the same constant $\frac{c}{n}$ to the stumped pattern $\chi \setminus \chi'$ and proceed recoursively in this way. At every step, the cost of the iteratively stumped pattern loses at least $\frac{c}{n}$. So we can do at most $n-1$ stumps of this kind. At the end of this procedure we get at most n sources, which are the cut point in the stumping procedure, which globally give rise to a finite number of patterns each one with a cost which is less or equal to $\frac{c}{n}$. The second part of the statement is easily obtained by iterating (A.10). □

PROPOSITION 7.7. *Let μ be a probability measure which is α-irrigable. Then $\exists C > 0$ such that for all $n \in \mathbb{N}$*

$$\delta_n^{\frac{1}{\alpha}} \leq C n^{-(1-\alpha)}\,. \tag{7.8}$$

PROOF. Let χ be an irrigation pattern such that $I_\alpha(\chi) < +\infty$ and $\mu_\chi = \mu$. Let us apply the decomposition Theorem 7.1 to the pattern χ, so we get, for a fixed $n \in \mathbb{N}$, n source points S_i and a finite number of subpatterns χ_i verifying 1. and 2. of the thesis of the theorem.

Let $\mu_n = \sum_{i=1}^{n} \mu_{\chi_i}(\mathbb{R}^N)\delta_{S_i} \in \mathcal{D}_n$. By Proposition 5.2 we have

$$\left[\delta_n^{\frac{1}{\alpha}}\right]^{\frac{1}{\alpha}} \leq \left[d_{\frac{1}{\alpha}}(\mu, \mu_n)\right]^{\frac{1}{\alpha}} \leq \left[d_{\frac{1}{\alpha}}\Big(\sum_i \mu_{\chi_i}, \sum_i \mu_{\chi_i}(\mathbb{R}^N)\delta_{S_i}\Big)\right]^{\frac{1}{\alpha}}$$
$$\leq \sum_i (I_\alpha(\chi_i))^{\frac{1}{\alpha}} \leq \sup_i (I_\alpha(\chi_i))^{\frac{1}{\alpha}-1} \sum_i I_\alpha(\chi_i) \leq \left(\frac{c}{n}\right)^{\frac{1}{\alpha}-1} c\,,$$

from which we get

$$\delta_n^{\frac{1}{\alpha}} \leq C n^{-(1-\alpha)}\,.$$

□

Proof of Proposition 7.2. Assuming, by contradiction that μ is α-irrigable, by (7.8) it would follow, by Remark 5.2, $(1-\alpha) \leq \left(d_r^{\frac{1}{\alpha}}\right)^{-1}$ and so $d_r^{\frac{1}{\alpha}} \leq d_\alpha$, in contradiction to our assumptions. □

8. The irrigability dimension as a resolution dimension.

In this section we show that the irrigability dimension of a measure can be seen as a resolution dimension with respect to an appropriate choice of the index p.

We shall prove the following theorem.

Theorem 8.1. *Let μ be a probability measure. Then*

a) *if $d_r^\infty(\mu) \leq 1$ then $d(\mu) = 1$;*
b) *if $d_r^\infty(\mu) > 1$ then $\exists p \geq N'$ such that $d(\mu) = d_r^p(\mu)$.*

Moreover, an exponent p for which the above inequality holds true is the unique solution of the equation

$$d_r^p(\mu) = p' \,. \tag{8.1}$$

Proof. In the case a) we have that for all $p \geq 1$,

$$d_r^p(\mu) \leq d_r^\infty(\mu) \leq 1 \leq p' \,,$$

so by applying Corollary 6.2 we have that $d(\mu) \leq p'$. The thesis follows taking the limit for $p \to +\infty$.

For the proof of the remaining part of the statement, it is sufficient to show that equation (8.1) admits a (unique) solution $p \geq N'$. Indeed, by applying Corollaries 6.2 and 7.1 one gets $d(\mu) = p'$ and therefore b) follows.

We can get a solution to (8.1) since by means of Proposition 5.3, the map $p \mapsto d_r^p(\mu)$ is a continuous map and by Proposition 5.4, being $d_r^\infty(\mu) \leq d_M(\mu) \leq N$, for $p_1 = N'$ we have $d_r^{p_1}(\mu) \leq p_1'$. On the other side, by (5.7) we get $d_r^\infty(\mu) = \lim_{p\to+\infty} d_r^p(\mu) > 1 = \lim_{p\to+\infty} p'$. So for p_2 large enough we have $d_r^{(p_2)}(\mu) \geq p_2'$. Moreover equation (8.1) admits a unique solution because the map $p \mapsto d_r^p(\mu)$ is increasing and the map $p \mapsto p'$ is strictly decreasing. □

Remark 8.1. *The uniqueness of the solution to* (8.1) *does not guarantee in any way the uniqueness of the exponent p for which $d_r^p(\mu) = d(\mu)$. In-*

deed, by Proposition 5.6 *for a measure μ which is Ahlfors regular in dimension $\beta = d(\mu)$ any exponent $p \geq 1$ gives $d_r^p(\mu) = d(\mu)$.*

As we have said in the introduction, Theorem 1.1 can be deduced from the previous result.

ALTERNATIVE PROOF OF THEOREM 1.1. If $d_r^\infty(\mu) \leq 1$, by Theorem 8.1 we have $d(\mu) = 1$ and so

$$d_c(\mu) \leq d_r^\infty(\mu) = 1 = d(\mu) \leq \max\{1,\, d_M(\mu)\}\,.$$

On the other hand, by propositions 5.4 and 5.5, we have for a suitable p

$$d_c(\mu) \leq d_r^p(\mu) = d(\mu) \leq d_M(\mu) = \max\{1,\, d_M(\mu)\}\,. \qquad \square$$

A. Appendix A - Fundamental notions, remarks and notation.

In this appendix we shall introduce some terminology which has been used in this paper, in particular we shall recall the same notation as in [4], introducing the notion of irrigable measure and referring to that paper for more details. Then we shall give some new definitions and useful tools.

Let $(\Omega,\, |\cdot|)$ be a nonatomic probability space and $\chi : \Omega \times \mathbb{R}_+ \to \mathbb{R}^N$ an irrigation pattern, as defined in Section 1. When we shall deal with subsets $\Omega' \subset \Omega$ we shall use $\chi_{|\Omega'}$ instead of $\chi_{|\Omega' \times \mathbb{R}_+}$ to denote the restriction of χ to $\Omega' \times \mathbb{R}_+$ and we shall call $\chi_{|\Omega'}$ the *subpattern* of χ defined on Ω'.

Let be $(\Omega_1,\, |\cdot|_1)$ and $(\Omega_2,\, |\cdot|_2)$ two disjoint probability spaces, let $S \in \mathbb{R}^N$ and let $\chi_1 \in \mathbf{P}_S(\Omega_1)$ and $\chi_2 \in \mathbf{P}_S(\Omega_2)$ be two irrigation pattern with the same source S. Let us consider the set $\Omega = \Omega_1 \cup \Omega_2$ endowed with the finite measure defined by setting for all $A \subset \Omega$, $|A| = |A \cap \Omega_1|_1 + |A \cap \Omega_2|_2$. Then we can consider χ_1 and χ_2 as subpatterns of a pattern $\chi \in \mathbf{P}_S(\Omega)$ defined by setting for a.e. $p \in \Omega$ and for all $t \in \mathbb{R}_+$

$$\chi(p,\, t) = \begin{cases} \chi_1(p,\, t) & \text{if } p \in \Omega_1 \\ \chi_2(p,\, t) & \text{if } p \in \Omega_2\,. \end{cases} \tag{A.1}$$

The above defined pattern will be called *bunch* of the patterns χ_1 and χ_2. It is clear that the definition of bunch of patterns can be extended to a sequence of patterns, see [3].

We recall that every set of fibers of Ω, time by time, defines an equivalence relation $\simeq_t$ on Ω by relating two points p and $q \in \Omega$ at the time

t if χ_p and χ_q coincide on $[0,\ t]$. So every set of fibers at every time t divides Ω into equivalence classes which we shall call χ-vessels. For any $p \in \Omega$, we shall denote by $[p]_t$ the χ-vessel at time t which contains p, while for any $t \geq 0$ we shall denote by $\mathcal{V}_t(\chi)$ the set of all the χ-vessels at time t. The following lemma can be trivially proved, see [4].

LEMMA A.1. *Let χ be an irrigation pattern. Then for all $0 \leq t_1 \leq t_2$ and for all $V_{t_1} \in \mathcal{V}_{t_1}(\chi)$ and $V_{t_2} \in \mathcal{V}_{t_2}(\chi)$ we have the following two alternatives:*

1) $V_{t_2} \subset V_{t_1}$
2) $V_{t_2} \cap V_{t_1} = \emptyset$.

For a set of fibers $\chi \in \mathbf{C}_S(\Omega)$, we introduce the following function $\sigma_\chi : \Omega \to \mathbb{R}_+$ which gives the absorption time of a point defined as follows

$$\forall p \in \Omega \ : \quad \sigma_\chi(p) = \inf\big\{t \in \mathbb{R}_+ \mid \chi_p(\,\cdot\,) \text{ is constant on } [t, +\infty[\big\}\ ,$$

which will be called *stopping* or *absorption function for χ*.

We shall say that a point $p \in \Omega$ is absorbed when $\sigma_\chi(p) < +\infty$. A point $p \in \Omega$ is absorbed at the time t if $\sigma_\chi(p) \leq t$. Analogously we shall say that a set $X \subset \Omega$ is an absorbed set at time t if $\sigma_\chi(p) \leq t$ for a.e. $p \in X$, in particular when the set X is a χ-vessel we shall say that X is an absorbed χ-vessel. We shall denote by $A_t(\chi)$ the set of the points of Ω which are absorbed at time t, and by $A_\chi = \bigcup_{t>0} A_t(\chi)$ the set of the absorbed points. On the contrary, the set

$$M_t(\chi) = \{p \in \Omega \mid \sigma_\chi(p) > t\} = \Omega \setminus A_t(\chi)$$

is the set of the points that, at time t, are still moving. We shall call *χ-flow* at time t any not absorbed χ-vessel, and we shall denote by $\mathcal{F}_t(\chi)$ the set of the χ-flows at time t and by $F_t(\chi)$ the union of all the χ-flows at time t.

For every pattern $\chi \in \mathbf{C}_S(\Omega)$ we introduce the *irrigation function*

$$i_\chi : A_\chi \to \mathbb{R}^N\ ,$$

defined by setting

$$\forall p \in A_\chi \ : \qquad i_\chi(p) = \chi(p, \sigma_\chi(p))$$

and giving, point by point, the absorption position of the absorbed points.

In the case in which we deal with an irrigation pattern $\chi \in \mathbf{P}_S(\Omega)$, the absorption time function σ_χ, and, for all $t \geq 0$, the vessels and the set $A_t(\chi)$ of the absorbed points at time t are both measurable (see [4]).

We remark that $i_\chi(p) = \lim_{t\to\infty} \chi(p, t)$ and so also $i_\chi : A_\chi \to \mathbb{R}^N$ is a measurable function, as a pointwise limit of a sequence of measurable functions, when $\chi \in \mathbf{P}_S(\Omega)$.

The function i_χ induces on $\mathbb{R}^N$ the image (push-forward) measure μ_χ defined by the formula

$$\mu_\chi(A) = |i_\chi^{-1}(A)| \ ,$$

for any Borel set $A \subset \mathbb{R}^N$. We shall refer to μ_χ as to the *irrigation measure* induced by the pattern χ.

For a fixed cost exponent $\alpha \in]0, 1[$, we introduce the functional cost I_α in [4], defined on the set $\mathbf{P}_S(\Omega)$ of the irrigation patterns χ, by the following formula

$$I_\alpha(\chi) = \int_{\mathbb{R}_+} c_\chi(t) dt \ ,$$

where

$$c_\chi(t) = \int_{M_t(\chi)} |[p]_t|^{\alpha-1} dp \tag{A.2}$$

is the relative density cost function.

Remark A.1. *Let $(\chi_n)_{n\in\mathbb{N}}$ be a sequence of patterns $\chi_n : \Omega_n \times \mathbb{R}_+ \to \mathbb{R}^N$ all with the same source $S \in \mathbb{R}^N$, let χ be the bunch of the sequence $(\chi_n)_{n\in\mathbb{N}}$. Then it is easy to show that for any $\alpha \in]0, 1[$ we have*

$$I_\alpha(\chi) \leq \sum_n I_\alpha(\chi_n) \tag{A.3}$$

and

$$\mu_\chi = \sum_n \mu_{\chi_n} \ . \tag{A.4}$$

We introduce some more definitions.

Definition A.1. *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω, we will say that*

$$F_\chi = \{x \in \mathbb{R}^N \mid \exists t > 0, \ \exists A \in \mathcal{F}_t(\chi) \text{ s.t. } x = \chi(p, t), \ p \in A\}$$

is the flow zone of χ.

The following lemma has been proved in [3].

Lemma A.2. *For any pattern χ of finite cost, F_χ is a Borel set and $d(F_\chi) = 1$.*

Definition A.2. *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω, then the set*

$$D_\chi = \{p \in \Omega \mid p \in F_{\sigma_\chi}(p)\} \tag{A.5}$$

will be called dispersion *of the pattern χ. Moreover we shall say that*

- χ has a complete dispersion *(or equivalently χ is totally dispersed) if $|\Omega \setminus D_\chi| = 0$*
- χ *is* a pattern with dispersion *if $|D_\chi| > 0$*
- χ *is* a pattern without dispersion *if $|D_\chi| = 0$.*

Remark A.2. *Let χ be an irrigation pattern. Then the irrigation function sends the dispersion D_χ in the flow zone F_χ, i.e.*

$$i_\chi(D_\chi) \subset F_\chi \,. \tag{A.6}$$

As a consequence, by the definition of irrigation measure induced by χ, we have

$$|D_\chi| \leq \mu_\chi(F_\chi) \,. \tag{A.7}$$

Therefore to get a pattern without dispersion it is sufficient to check that $\mu_\chi(F_\chi) = 0$.

Hence, when a pattern χ has a complete dispersion, every point is absorbed just because it stops its motion while it still belongs to a flow.

Remark A.3. *Let $\chi \in P_S(\Omega)$ be an irrigation pattern of Ω, then the subpattern of χ restricted to $\Omega \setminus D_\chi$ is a pattern without dispersion.*

Definition A.3. *Let χ be an irrigation pattern, p, $q \in \Omega$ and $t \geq 0$. We shall introduce the separation time $s_\chi(p,q)$ of the two points p and q defined as*

$$s_\chi(p,q) = \inf\{t \geq 0 \,|\, \chi(p,t) \neq \chi(q,t)\} \tag{A.8}$$

Definition A.4. *Let χ be an irrigation pattern, $p \in \Omega$ and $t \geq 0$. We shall say that two points p, $q \in \Omega$ are strictly equivalent at time t, and we shall write $p \simeq_t^s q$, if there exists $\varepsilon > 0$ such that $p \simeq_{t+\varepsilon} q$. We shall call $[p]_t^s$ strict equivalence class defined by p at time t or equivalently strict vessel of* *the point p at time t, the following set*

$$[p]_t^s = \{q \in \Omega \,|\, p \simeq_t^s q\}$$

and we shall denote by $\mathcal{V}_t^s(\chi)$ *the set of the strict vessels at time t according to the pattern* χ.

REMARK A.4. *Let* χ *be an irrigation pattern,* $p \in \Omega$ *and* $t \geq 0$. *Then the strict equivalence class defined by p at time t coincides with the union of the equivalence classes* $[p]_{t'}$ *defined by p at times* $t' > t$, *i.e.*

$$[p]_t^s = \bigcup_{t'>t} [p]_{t'} = \bigcup_{t'>t} [p]_{t'}^s \,. \tag{A.9}$$

REMARK A.5. *For a.e.* $p,\, q \in \Omega$ *and for all* $t \geq 0$*:*

- $p \simeq_t q$ *for all* $t \leq s_\chi(p,q)$
- $p \simeq_t^s q$ *for all* $t < s_\chi(p,q)$.

DEFINITION A.5. *Let* $\chi \in P_S(\Omega)$ *be an irrigation pattern of* Ω. *For any pair* $(p,\, t) \in \Omega \times \mathbb{R}_+$ *the function* $\chi_{(p,\,t)} : [p]_t \times \mathbb{R}_+ \to \mathbb{R}^N$, *defined, for all* $(q,\, s) \in [p]_t \times \mathbb{R}_+$, *by* $\chi_{(p,\,t)}(q,\, s) = \chi(q, s + t)$ *is the branch of* χ *starting from* $\chi(p,\, t)$.

REMARK A.6. *Let* $\chi \in P_S(\Omega)$ *be an irrigation pattern of* Ω. *Then for any* $(p,\, t) \in \Omega \times \mathbb{R}_+$ *the branch of* χ *starting from* $\chi(p,\, t)$ *does not depends on p but only on the* χ*-vessel* $[p]_t$. *Moreover to get nontrivial (constant) branches one must require the vessel* $[p]_t$ *to be a flow.*

DEFINITION A.6. *Let* $\chi \in P_S(\Omega)$ *be an irrigation pattern of* Ω. *For any pair* $(p,\, t) \in \Omega \times \mathbb{R}_+$ *the function* $\chi'_{(p,\,t)} : [p]_t \times \mathbb{R}_+ \to \mathbb{R}^N$, *defined, for all* $(q,\, s) \in [p]_t^s \times \mathbb{R}_+$, *by* $\chi'_{(p,\,t)}(q,\, s) = \chi(q, s + t)$, *where* $[p]_t^s$ *is the strict* χ*-vessel of p at time t, is the single branch of* χ *starting from* $\chi(p,\, t)$.

Where $[p]_t \neq [p]_t^s$ we shall have branches which are not single ones and, in order to point out that the point $\chi(p,\, t)$ give rise to more than one single branch, we shall call $[p]_t$ *multiple branch*.

We introduce the notion of a simple pattern which will allow us to extend Definitions A.5 and A.6 to any point $x \in F_\chi$.

DEFINITION A.7. *Let* $\chi \in P_S(\Omega)$ *be an irrigation pattern of* Ω, *we will say that* χ *is a simple pattern if:*

- for a.e. point $p \in \Omega$ the χ-fiber of the point p, i.e. the function $\chi_p : \mathbb{R}_+ \to \mathbb{R}^N$ is a simple curve up to the stopping time $\sigma_\chi(p)$, i.e. once restricted to the interval $[0, \sigma_\chi(p)]$
- for a.e. pair of points p and q of Ω: $\chi_p(s) \neq \chi_q(t)$ for all $s,\ t > s_\chi(p, q)$.

It is easy to show that any subpattern of a simple pattern is simple too. Moreover, when we deal with an irrigable measure μ, it is always possible to irrigate μ by means of a simple pattern, see [3, Lemma 6.15]. Lemma A.2 and remarks A.2 and A.3 allow us to assume that the restriction of μ out of a 1-dimensional set can be irrigated by a simple pattern without dispersion.

REMARK A.7. *It is worth to remark, see* [3, Lemma 6.15], *that one can say that a pattern* χ *is simple if*

$$\forall x \in F_\chi, \exists | t \geq 0,\ \exists | V = [p]_t \in \mathcal{F}_\chi(t) \text{ s.t. } x = \chi(p,\ t)\,.$$

The above remark allows us to give, for simple patterns χ, the definition of branch of χ which starts from a point $x \in F_\chi$, according to definitions A.5 and A.6, applied to any pair $(p,\ t)$ such that $x = \chi(p, t)$.

DEFINITION A.8. *Let* χ *be a simple irrigation pattern with source point S. For any branch* χ' *from* $x = \chi(p,\ t)$ *of* χ*, we shall call* pattern χ "stumped" of the branch χ', *the restriction of* χ *to* $\Omega \setminus [p]_t$ *and we shall denote it by* $\chi \setminus \chi'$.

The following lemma can be trivially proved.

LEMMA A.3. *Let* χ *be an irrigation pattern and let* χ' *be a branch of* χ*. Then*

$$\mu_\chi = \mu_{\chi'} + \mu_{\chi \setminus \chi'} \tag{A.10}$$

moreover

$$I_\alpha(\chi) \geq I_\alpha(\chi') + I_\alpha(\chi \setminus \chi'). \tag{A.11}$$

DEFINITION A.9. *Given a pattern* χ *and a vessel* $V = [p]_t$ *at a time t, we shall call* cost of the vessel V and we shall denote it by $I_\alpha(V,\ t)$ or, when there is no doubt about the time at which one refers, by $I_\alpha(V)$ the cost $I_\alpha(\chi')$, where χ' is the branch of χ which starts from $\chi(p,\ t)$.

REMARK A.8. *Let χ be an irrigation pattern, then for all $t \geq 0$*

$$\sum_{V \in \mathcal{V}_t(\chi)} I_\alpha(V) = \int_t^{+\infty} c_\chi(s) ds. \tag{A.12}$$

The analogous property also holds true for the strict vessels.

We recall the definition of χ-vessel evolution introduced in [4]

DEFINITION A.10. *Let $I \subset \mathbb{R}_+$. We shall say that the one-parameter family of sets $V_t = (V_t)_{t \in I}$ is a χ-vessel evolution if:*

- *it is decreasing under inclusion*
- *$V_t \in \mathcal{V}_t(\chi)$ for every $t \in I$.*

REMARK A.9. *Let V_t be a χ-vessel evolution, then the family $(I_\alpha(V_t))_{t \in \mathbb{R}_+}$ which, time by time, gives the cost of the vessel V_t, is decreasing but it is, in general, not continuous because of the possible "multiple branching" of the pattern χ at some time.*

The function $\varphi_\chi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is defined by setting for a.e. $p \in \Omega$ and for all $t \in \mathbb{R}_+$

$$\varphi_\chi(p, t) = |[p]_t|^{\alpha-1} \mathbb{1}_{M_t(\chi)}(p) \,, \tag{A.13}$$

so that $c_\chi(t) = \int_\Omega \varphi_\chi(p, t) dp$.

DEFINITION A.11. *Let χ be an irrigation pattern. Then χ induces on $\Omega \times \mathbb{R}_+$ a positive measure ν defined in the following way:*

$$\nu(A) = \int_A |[p]_t|^{\alpha-1} \mathbb{1}_{M_t}(p) dp\, dt = \int_A \varphi_\chi(p, t) dp\, dt \,.$$

REMARK A.10. *Let χ be an irrigation pattern. Then*

$$\int_{\mathbb{R}_+} c_\chi(t) dt = \nu(\Omega \times \mathbb{R}_+) = I_\alpha(\chi) \,.$$

REMARK A.11. *Let χ be an irrigation pattern, then for all $t \geq 0$ and for all vessel $V_t \in \mathcal{V}_t(\chi)$ at time t*

$$I_\alpha(V_t) = \nu(V_t \times [t, +\infty[) \,.$$

B. Appendix B.

This second appendix is devoted to the proof of some tools and propositions stated and used in Section 5.

LEMMA B.1. *Let μ be a probability measure. Then for all $d' > d_r^1(\mu)$ and for all $\varepsilon > 0$ $\exists A_\varepsilon \subset \mathbb{R}^N$ such that*

$$\mu(\mathbb{R}^N \setminus A_\varepsilon) < \varepsilon \text{ and } d_M(A_\varepsilon) \le d' \,. \tag{B.1}$$

PROOF. Let us call $d = d_r^1(\mu)$ and let us fix $d' > d$ and $\varepsilon > 0$. For any $k \in \mathbb{N}$ let us set $\varepsilon_k = 2^{-k}\varepsilon$. Fix $k \in \mathbb{N}$, since $\lim_{n\to+\infty} d_1(\mu, D_n) = 0$ there exists $n_k \in \mathbb{N}$ and $\mu_{n_k} \in D_{n_k}$ such that

$$d_1(\mu, \mu_{n_k}) < \varepsilon_k \,. \tag{B.2}$$

Let $\{x_1, x_2, \dots, x_{n_k}\} = \operatorname{supp}(\mu_{n_k})$ and $U_k = \bigcup_{i=1}^{n_k} B(x_i, \varepsilon_k^{\frac{d}{d'}})$. By (B.2)

$$\mu(\mathbb{R}^N \setminus U_k) < \varepsilon_k^{1-\frac{d}{d'}} \,. \tag{B.3}$$

Let us call $A_\varepsilon = \bigcap_{k\in\mathbb{N}} U_k$. Then, by (B.3), being $d < d'$, we have

$$\mu(\mathbb{R}^N \setminus A_\varepsilon) \le \sum_{k\in\mathbb{N}} \mu(\mathbb{R}^N \setminus U_k) < \sum_{k\in\mathbb{N}} \varepsilon_k^{1-\frac{d}{d'}} = \varepsilon^{1-\frac{d}{d'}} \sum_{k\in\mathbb{N}} (2^{-k})^{1-\frac{d}{d'}} = \frac{\varepsilon^{1-\frac{d}{d'}}}{1-\left(\frac{1}{2}\right)^{1-\frac{d}{d'}}} \,.$$

Replacing ε by $\left(\left(1-\left(\frac{1}{2}\right)^{\frac{d'-d}{d'}}\right)\varepsilon\right)^{\frac{d'}{d'-d}}$ we get the first inequality in (B.1).

Let us remark that, for all $k \in \mathbb{N}$ large enough, we can fix n_k in order to have

$$n_k < \varepsilon_k^{-d'} \,. \tag{B.4}$$

Indeed, being $d < d'$, by (5.6), we have, for large enough n, $\log_n \delta_n^1 < -\dfrac{1}{d'}$ i.e. $\delta_n^1 < n^{-\frac{1}{d'}}$, namely

$$n < (\delta_n^1)^{-d'} \,. \tag{B.5}$$

Then set $n_k = \min\{n \mid \delta_n^1 \le \varepsilon_k\}$ we have $\delta_{n_k-1}^1 > \varepsilon_k$ and so, by (B.5), $n_k \le \varepsilon_k^{-d'} + 1$, from which (B.4) follows by the arbitrariness of d'.

Now let us consider $k = \min\{h \in \mathbb{N} \mid \varepsilon_h^{\frac{d}{d'}} \le \delta\}$, where $\delta > 0$ is a fixed real positive number small enough to have $k \ge 1$ ($\delta < \varepsilon^{\frac{d}{d'}}$ is enough), then

$$\delta < \varepsilon_{k-1}^{\frac{d}{d'}} = 2^{\frac{d}{d'}} \varepsilon_k^{\frac{d}{d'}} \,. \tag{B.6}$$

Being $A_\varepsilon \subset U_k$, A_ε can be covered by using n_k balls with a radius $\varepsilon_k^{\frac{d}{d'}} \le \delta$. Combining (B.4) with (B.6) we have

$$n_k \le 2^{d'} \delta^{-\frac{(d')^2}{d}} .$$

This last inequality gives, by Lemma 1.2, $d_M(A_\varepsilon) \le \dfrac{(d')^2}{d}$. By the arbitrariness of d' the thesis follows. □

Proof of Proposition 5.5. We shall prove that, for any $d' > d_r^1(\mu)$ we have $d_c(\mu) \le d'$. Let $d' > d_r^1(\mu)$, then by applying Lemma B.1 to $\varepsilon = \dfrac{1}{n}$ we get the existence of a sequence of sets $(A_n)_{n\in\mathbb{N}}$ such that

$$\mu(\mathbb{R}^N \setminus \bigcup_{n=1}^{+\infty} A_n) = 0 \tag{B.7}$$

and

$$d(\bigcup_{n=1}^{+\infty} A_n) \le \sup_{n\in\mathbb{N}} (d(A_n)) \le \sup_{n\in\mathbb{N}} (d_M(A_n)) \le d' . \tag{B.8}$$

By (B.7) and (B.8) we get that $d_c(\mu) \le d'$ and therefore the thesis. □

We shall now prove Proposition 5.4 which states that the resolution dimension of index $p = +\infty$ coincides with the Minkowski dimension d_M introduced in Definition 1.7. This circumstance explains why we have used the notation $d_r^\infty(\mu)$ to refer to $\sup_{p\ge 1} d_r^p(\mu)$ which is, according to Proposition 5.3, a weaker option see Remark 5.4. However in the following part of this appendix, to the aim of proving the equivalence with the Minkowski dimension, we shall need to use the notation d_r^∞ according to Definition 5.5, taking $p = +\infty$ in (5.6).

By the definition of Kantorowich-Wasserstein distance we can easily deduce the following remark.

Remark B.1. *Let μ be a probability measure, then for all $n \in \mathbb{N}$ δ_n^∞ is the infimum of the numbers $\delta > 0$ such that there exists a δ-net of* supp (μ) *of cardinality n.*

Proof of Proposition 5.4. Let $\beta > d_M(\mu)$. Given $n \in \mathbb{N}$ large enough, let $\delta = n^{-\frac{1}{\beta}}$. By Lemma 1.1 we can cover supp (μ) by using $\delta^{-\beta} = n$ balls of radius δ and so, by Remark B.1, $\delta_n^\infty \le \delta$. Therefore $\limsup_{n\to+\infty} \log_n \delta_n^\infty \le -\dfrac{1}{\beta}$, namely $d_r^\infty \le \beta$. By the arbitrariness of β, $d_r^\infty \le d_M$ follows. Conversely, let $\beta > d_r^\infty$, namely $-\dfrac{1}{\beta} > \limsup_{n\to+\infty} \log_n \delta_n^\infty$. Then $\delta_n^\infty < n^{-\frac{1}{\beta}}$, namely $n < (\delta_n^\infty)^{-\beta}$ for n large enough and, by Remark B.1, supp (μ) has a δ_n^∞-net of cardinality n. Since $n < (\delta_n^\infty)^{-\beta}$, we can deduce from Lemma 1.2 that $d_M \le \beta$ and, by the arbitrariness of β, that $d_M \le d_r^\infty$. □

We know by Proposition 5.3 that $\sup_{p\ge 1} d_r^p(\mu) \le d_r^\infty(\mu) = d_M(\mu)$. We shall show by the following example that, in general, the two values are different, so the weaker definition of d_r^∞ gives a different dimension.

EXAMPLE B.1. *There exist probability measures μ such that $d_M(\mu) = N$ and $\sup_{p\ge 1} d_r^p(\mu) = 0$.*

PROOF. Let $\mu = \sum_{n=1}^{\infty} m_n \delta_{x_n}$ where, for all $n \in \mathbb{N}$, $x_n \in \mathbb{Q}^N \cap B_{\frac{1}{2}}(S)$ and $m_n = ce^{-n} > 0$, where the constant $c > 0$ is a normalization constant which allows $\sum_{n\ge 1} m_n = 1$. By construction $d_M(\mu) = d_M(\operatorname{supp}(\mu)) = d_M(B_{\frac{1}{2}}(S)) = N$. For any $p \ge 1$ we shall bound δ_n^p by considering as an element of $\mathcal{D}_n$ the sum of the first n masses of μ.

So

$$\delta_n^p \le d_p\Big(\mu, \sum_{k=1}^{n} m_k \delta_{x_k}\Big) \le \left(\sum_{k=n+1}^{\infty} m_k\right)^{\frac{1}{p}} \le \left(\int_n^\infty ce^{-x}dx\right)^{\frac{1}{p}} = (ce^{-n})^{\frac{1}{p}} .$$

Therefore

$$\log_n(\delta_n^p) \le \frac{1}{p}\log_n c - \frac{n}{p}\log_n e = \frac{1}{p}\log_n c - \frac{n}{p\log n} \to -\infty ,$$

which, taking into account (5.6) easily leads to $d_r^p(\mu) = 0$ for all $p \ge 1$. □

In the following example we shall show a probability measure (which cannot be Ahlfors regular, see Proposition 5.6) for which there is a real the dependence of $d_r^p(\mu)$ on the index $p \ge 1$. In particular, we shall show that $\dfrac{q}{p}$ is the best possible constant in (5.8) while any Ahlfors regular μ shows the optimality of (5.7).

EXAMPLE B.2. *For any $p < q$, the constant $\dfrac{q}{p}$ in* (5.8) *of* Proposition 5.3 *cannot be improved.*

PROOF. Let $N = 1, S = 0$ and let us consider a $\mu = \sum_{i \in \mathbb{N}} m_i \delta_{x_i}$ where, for all $i \in \mathbb{N}$, δ_i denotes the Dirac mass concentrated in a point $x_i \in \mathbb{R}$. Let us fix two exponents $\gamma, \beta > 1$ and take $\forall i \in \mathbb{N}$,

$$r_i = x_{i+1} - x_i = \operatorname{dist}(x_i, \{x_j \,|\, j \neq i\}) = i^{-\gamma} \tag{B.9}$$

and

$$m_i = i^{-\beta}\,. \tag{B.10}$$

By using the triangular inequality, we get that

$$\forall x \in \mathbb{R} \;\exists \text{ at most one } i \in \mathbb{N} \text{ s.t. } |x - x_i| \leq \frac{r_i}{2}\,. \tag{B.11}$$

For a given $p \geq 1$ and for a fixed $n \in \mathbb{N}$ let us evaluate δ_n^p. Let us set $\mu_n = \sum_{i=1}^{n} m_i' \delta_{x_i}$ where for Pall $i < n$, $m_i' = m_i$, while $m_n' = \sum_{j \geq n} m_j$. Then we can bound δ_n^p by $C\, n^{1-\gamma-\frac{\beta-1}{p}}$. Indeed, by (B.10) and (B.9)

$$\delta_n^p \leq d_p(\mu,\, \mu_n) \leq \left[\left(\sum_{i=n+1}^{\infty} m_i\right)\left(\sum_{i=n+1}^{\infty} r_i\right)^p\right]^{\frac{1}{p}} \tag{B.12}$$

$$= \left(\sum_{i=n+1}^{\infty} i^{-\beta}\right)^{\frac{1}{p}} \sum_{i=n+1}^{\infty} i^{-\gamma} \leq C\, n^{1-\gamma-\frac{\beta-1}{p}}\,.$$

Fix now an arbitrary discretization $\mu_n \in D_n$ of μ. By (B.11) for any point x of $\operatorname{supp}(\mu_n)$ we find at most a point x_i such that $|x - x_i| \leq \frac{r_i}{2}$. So we can find at most n points x_i which are at a distance less or equal to $\frac{r_i}{2}$ from $\operatorname{supp}(\mu_n)$. We can assume, being $(m_i)_{i \in \mathbb{N}}$ and $(r_i)_{i \in \mathbb{N}}$ decreasing sequences, that such points x_i are the first n ones. Therefore we have

$$\delta_n^p = d_p(\mu, \mu_n) \geq \left(\sum_{i=n+1}^{\infty} m_i \left(\frac{r_i}{2}\right)^p\right)^{\frac{1}{p}} \geq c \left(\sum_{i=n+1}^{\infty} i^{-\beta} i^{-\gamma\, p}\right)^{\frac{1}{p}} \geq c\, n^{-\gamma-\frac{\beta-1}{p}}\,. \tag{B.13}$$

From (B.13) and (B.12) we have the existence of two positive constants, c and C such that for all $p \geq 1$ and $n \in \mathbb{N}$:

$$c\, n^{-\gamma-\frac{\beta-1}{p}} \leq \delta_n^p \leq C\, n^{1-\gamma-\frac{\beta-1}{p}}\,. \tag{B.14}$$

Taking the $\log_n$ and then taking the lim sup of the three members of (B.14) as in (5.6), we have for all $p \geq 1$

$$\frac{p}{p\gamma+\beta-1} \leq d_r^p(\mu) \leq \frac{p}{p(\gamma-1)+\beta-1} \tag{B.15}$$

and therefore

$$\frac{q}{p}\,\frac{p(\gamma-1)+\beta-1}{q\gamma+\beta-1} \leq \frac{d_r^q(\mu)}{d_r^p(\mu)} \leq \frac{q}{p}\,\frac{p\gamma+\beta-1}{q(\gamma-1)+\beta-1}\,. \tag{B.16}$$

Taking into account that, for a fixed value of $\gamma > 1$, both the bounds in (B.16) go to $\frac{q}{p}$ as $\beta \to +\infty$ we conclude the proof. □

C. Appendix C - Index of the main notation.

The order first follows the exposition in Appendix A and then the exposition of the paper:

- $(\Omega, |\cdot|)$ a nonatomic probability space
- $\chi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^N$ = set of fibers
- $\chi(p,t) \in \mathbb{R}^N$ position of the point $p \in \Omega$ at the time t
- $\chi_p = t \mapsto \chi(p,t)$ = fiber of p
- $\mathbf{C}_S(\Omega)$ = set of sets of fibers of Ω
- $\mathbf{P}_S(\Omega)$ = set of all irrigation patterns, i.e. the set of all the measurable sets of fibers of Ω
- $[p]_t$ = equivalence class of p under the equivalence $p \simeq_t q$ if $\chi_p(s) = \chi_q(s)$ for all $s \in [0,t]$
- *χ-vessels* = class of equivalence at time t under $\simeq_t$
- $\mathcal{V}_t(\chi) = \Omega/\simeq_t$ = set of χ-vessels at time t
- $\sigma_\chi(p) = \inf\{t \in \mathbb{R}_+ \mid \chi_p(s)$ is constant on $[t,+\infty[\}$: absorption (stopping) time of p, p is absorbed at time t if $\sigma_\chi(p) \leq t$
- $X \subset \Omega$ is an absorbed set at time t if $\sigma_\chi(p) \leq t$ for a.e. $p \in X$
- χ-flow = non absorbed χ-vessel (has positive measure in Ω)
- $\mathcal{F}_t(\chi)$ = set of χ-flows at time t
- $A_t(\chi)$ = set of the points of Ω which are absorbed at time t
- $A_\chi = \bigcup_{t>0} A_t(\chi)$ = set of the absorbed points
- $M_t(\chi) = \Omega \setminus A_t(\chi)$ = set of the points of Ω that at time t are still moving
- $F_t(\chi) = \bigcup_{A \in \mathcal{F}_t(\chi)} A$ = union of the χ-flows at time t

- $i_\chi : A_\chi \to \mathbb{R}^N$ = irrigation function defined by $i_\chi(p) = \chi(p, \sigma_\chi(p))$
- μ_χ = irrigation measure induced by the pattern χ by setting $\mu_\chi(A) = |i_\chi^{-1}(A)|$ for any Borel set $A \subset \mathbb{R}^N$
- $c_\chi(t) = \int_{M_t(\chi)} |[p]_t|^{\alpha-1} dp$ = density cost function see (A.2)
- $I_\alpha(\chi) = \int_{\mathbb{R}_+} c_\chi(t)dt$ = cost of the pattern χ
- $F_\chi = \{x \in \mathbb{R}^N \mid \exists t > 0, \exists A \in \mathcal{F}_t(\chi) \text{ s.t. } x = \chi(p, t), p \in A\}$ = flow zone of χ, see Definition A.1
- $D_\chi = \{p \in \Omega \mid p \in \mathcal{F}_{\sigma_\chi(p)}(\chi)\}$ = dispersion of the pattern χ, see Definition A.2
- $s_\chi(p, q) = \inf\{t \geq 0 \mid \chi(p, t) \neq \chi(q, t)\}$ = separation time of the two points p and q, see Definition A.3
- $[p]_t^s$ = equivalence class of p under the equivalence $p \simeq_t^s q$ if there exists $\varepsilon > 0$ s.t. $p \simeq_{t+\varepsilon} q$, see Definition A.4
- *strict χ-vessels* = class of equivalence at time t under $\simeq_t^s$, see Definition A.4
- $\mathcal{V}_t^s(\chi) = \Omega/\simeq_t^s$ = set of the strict χ-vessels at time t, see Definition A.4
- $\chi' : [p]_t \times \mathbb{R}_+ \to \mathbb{R}^N$ = branch of χ starting from $\chi(p, t)$ defined by setting $\chi'(q, \cdot) = \chi(q, \cdot + t)$, see Definition A.5
- $\chi \setminus \chi'$ = pattern χ stumped of the branch χ', see Definition A.8
- $I_\alpha(V, t) = I_\alpha(V)$ = cost of the vessel V at time t, see Definition A.9
- $\varphi_\chi(p, t) = |[p]_t|^{\alpha-1} 1_{M_t(\chi)}(p)$, see (A.13)
- $\nu(A) = \int_A \varphi_\chi(p, t)dp\,dt$, see Definition A.11
- $d_\alpha = \dfrac{1}{1-\alpha} = \left(\dfrac{1}{\alpha}\right)'$ = critical dimension of the exponent α, see Definition 1.1
- $d(\mu) = \inf\{d_\alpha \mid \mu \text{ is irrigable with respect to } \alpha\}$ = irrigability dimension of μ, see Definition 1.3
- $d(B)$ = Hausdorff dimension of the set B
- $d_c(\mu) = \inf\{d(B) \mid \mu \text{ is concentrated on } B\}$, see Definition 1.4
- $d_s(\mu)$ = Hausdorff dimension of the $\operatorname{supp}(\mu)$, see Definition 1.5
- $d_M(X)$ = Minkowski dimension of the set X, see Definition 1.6
- $N_\delta(X) = \{y \in \mathbb{R}^N \mid d(y, X) < \delta\}$, see Definition 1.6
- $d_M(\mu) = \inf\{d_M(X) \mid \mu \text{ is concentrated on } X\}$ = Minkowski dimension of μ or equivalently strong resolution dimension of index $+\infty$, see Definition 1.7, Definition 5.5 and Proposition 5.4
- *resolution* of $\mu = \operatorname{card}(\operatorname{supp}(\mu)) < \infty$, see Definition 5.1
- D_n = set of all the convex combinations of n Dirac masses, see

Definition 5.2

- $d_p(\mu, \nu) = \left(\min_\sigma \int_{\Omega\times\Omega} |x-y|^p d\sigma\right)^{\frac{1}{p}}$ = Kantorovitch-Wasserstein distance of index p between μ and ν, see Definition 5.3
- $\delta_n^p = d_p(\mu, D_n)$, see (5.3)
- $d_r^p(\mu) = \left(-\limsup_{n\to+\infty} \log_n(\delta_n^p)\right)^{-1}$ = resolution dimension of μ of index p, see Definition 5.5
- $d_r^\infty(\mu) = \sup_{p\geq 1} d_r^p(\mu)$ = weak resolution dimension of index $+\infty$, see Remark 5.4
- $W_\chi(t) = \max\{I_\alpha(V) \,|\, V \in \mathcal{V}_t(\chi)\}$, see (7.3)
- $S_\chi(t) = \max\{I_\alpha(V) \,|\, V \in \mathcal{V}_t^s(\chi)\}$, see (7.4).

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