

## A Note on Clean Abelian Groups.

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*To Luigi Salce on the occasion of his 60th birthday*

ABSTRACT - Nicholson defined a ring to be clean if every element is the sum of a unit and an idempotent. A module is clean if its endomorphism algebra is clean. We show that torsion-complete Abelian  $p$ -groups are clean and characterize the clean groups among the class of totally projective  $p$ -groups. An example is given of a clean  $p$ -group which is neither totally projective nor torsion-complete.

### Introduction.

The relationship between the general elements of a ring,  $E$ , and its units has been a central topic of interest particularly when the ring in question is the endomorphism ring (or more generally algebra) of a module. In Abelian group theory this interest was largely sparked by a question of László Fuchs in [3] and subsequently led to the notion of the unit sum number of an Abelian group; see e.g. [5], [17], [11]. In particular a great deal of attention has focused on groups and modules having the property that every endomorphism is the sum of exactly two automorphisms. Following on from this approach, the first author and his co-workers have investigated Abelian groups with the stronger property that every endomorphism is the sum of two automorphisms, one of which is an involution - see [6]. At the same time Nicholson and various co-authors introduced the notion of a *clean ring*: a ring is said to be clean if every element is the sum of a unit and an idempotent. The

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notion is extended to modules by defining a *module to be clean* if its endomorphism algebra is clean in the previous sense. The interest in this notion derived initially from the fact that clean rings are exchange rings and, if the ring has central idempotents, then it is an exchange ring if, and only if it is clean - see [12]. Notice that if 2 is a unit, then the involution property coincides with the property of being clean. There are some advantages to working with clean rings in this context: for example the field of two elements is the only vector space which does not have the involution property - in fact it doesn't even have the weaker property of having finite unit sum number - but it is clean. A rather surprising development was Ó Searcóid's result: the ring of linear transformations of a vector space (of arbitrary dimension) over a field is clean - see [14] or [13] for a generalization to vector spaces over a division ring. Since a vector space over a finite field is just an elementary Abelian group, it is natural to investigate the endomorphism rings of torsion Abelian groups. The principal result of the present work is a classification of totally projective Abelian  $p$ -groups having a clean endomorphism ring. Recall that there are several equivalent definitions of the concept of a totally projective  $p$ -group, the least technical being that the group is simply presented in the sense that it can be generated by a set of elements  $X = \{x_i\} (i \in I)$  subject only to defining relations of the form  $p^m x_i = 0$  or  $p^n x_i = x_j (i \neq j)$ , where  $m, n$  are positive integers. This class is of significance since it is the largest class of Abelian  $p$ -groups distinguishable via the so-called Ulm invariants; the class of totally projective  $p$ -groups is well known to include all countable reduced  $p$ -groups. Our final result shows that clean  $p$ -groups exist in such abundance that they are unlikely to be classifiable by any reasonable set of numerical invariants.

Our notation is largely standard and the relevant notions in Abelian group theory may be found in the texts by Fuchs [4] or Kaplansky [9]. We specifically note the following two concepts which are used repeatedly. An Abelian  $p$ -group is said to be:

(i) *reduced* if it contains no divisible subgroups i.e. no copy of the quasi-cyclic group  $\mathbb{Z}(p^\infty)$ .

(ii) *torsion-complete* if it is the torsion subgroup of the  $p$ -adic completion of a direct sum of cyclic  $p$ -groups. Recall that every  $p$ -group without elements of infinite height may be embedded in a torsion-complete group.

The notation  $\mathbb{P}$  is used to denote the set of rational primes,  $\omega$  denotes

the first infinite ordinal and  $h_G(g)$  denotes the (generalized) height of an element  $g \in G$ ; in the last case if no ambiguity results, we drop the reference to  $G$  and simply write  $h(g)$ . The cyclic group of order  $n$  will be denoted by  $\mathbb{Z}(n)$ .

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**Main Results.**

We begin with a result which is valid for modules over an arbitrary ring; various versions of this result have appeared in the literature but the proof given here is direct and simple. It is broadly similar to that first appearing in [7].

LEMMA 1. *If  $A$  and  $B$  are clean modules, then  $A \oplus B$  is clean.*

PROOF. The endomorphisms of  $A \oplus B$  may be regarded as matrices of the form

$$M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

where  $X \in \text{End}(A)$ ,  $W \in \text{End}(B)$ ,  $Y \in \text{Hom}(B, A)$  and  $Z \in \text{Hom}(A, B)$ . Now as  $A$  is clean we can write  $X = C + U$  where  $C$  is an idempotent and  $U$  is a unit. Note that  $ZU^{-1}Y \in \text{End}(B)$  and so we may write  $W - ZU^{-1}Y = D + V$  with  $D$  an idempotent and  $V$  a unit. Now

$$M = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} U & Y \\ Z & V + ZU^{-1}Y \end{pmatrix}.$$

Clearly the first term is an idempotent. Moreover the second term is a unit: pre-multiplying it by

$$P = \begin{pmatrix} I & 0 \\ -ZU^{-1} & I \end{pmatrix}.$$

and post-multiplying it by

$$Q = \begin{pmatrix} U^{-1} & -U^{-1}YV^{-1} \\ 0 & V^{-1} \end{pmatrix}$$

yields the identity matrix and  $P, Q$  are invertible.  $\square$

Recall that a torsion Abelian group  $G$  may always be expressed as the direct sum of its  $p$ -primary components,  $G = \bigoplus_{p \in \mathbb{P}} G_p$  - see e.g. [4, Theorem 8.4].

LEMMA 2. *A torsion Abelian group  $G$  is clean if, and only if each of its  $p$ -primary components is clean.*

PROOF. Since  $G = \bigoplus_{p \in \mathbb{P}} G_p$  and there are no homomorphisms between different primary components, the endomorphism ring  $\text{End}(G)$  is just the ring direct product over  $p \in \mathbb{P}$  of  $\text{End}(G_p)$ . However it is clear that the property of being clean is inherited by ring direct products and ring homomorphic images and so the result follows immediately.  $\square$

Our next result is also more general than we need; it may be deduced from the second author's forthcoming paper [18] by noting that divisible Abelian groups are precisely the injective  $\mathbb{Z}$ -modules but we give here a proof using standard results from Abelian group theory. It may also be deduced from a recent paper on continuous modules [1].

LEMMA 3. *A divisible Abelian group is clean.*

PROOF. A divisible Abelian group is the direct sum of copies of the additive group of rationals,  $\mathbb{Q}$ , and copies of the Prüfer quasi-cyclic groups,  $\mathbb{Z}(p^\infty)$ , for various primes  $p$ . Since a direct sum of copies of  $\mathbb{Q}$  is a vector space, it is clean by Ó Searcóid's result quoted in the introduction. It now follows from Lemmas 1 and 2 that it will suffice to show that a divisible  $p$ -group is clean.

Now if  $D$  is a divisible  $p$ -group then the Jacobson radical of  $\text{End}(D)$  is just  $p\text{End}(D)$  and  $\text{End}(D)/J(\text{End}(D))$  is isomorphic to the ring of linear transformations of the  $\mathbb{Z}/p\mathbb{Z}$ -vector space  $D[p]$  - see e.g. [10, Lemma 3.5, Theorem 3.9]. This latter is, of course, clean. Moreover, since  $D$  is torsion, its endomorphism ring is complete in the  $p$ -adic topology - see e.g. [4, Theorem 46.1] - which here coincides with the  $J$ -adic topology. Thus both

units and idempotents lift from the quotient  $\text{End}(D)/J(\text{End}(D))$  to  $\text{End}(D)$  which is thus clean.  $\square$

**COROLLARY 4.** *If  $G$  is an arbitrary  $p$ -group of the form  $G = D \oplus G_R$ , where  $D$  is divisible and  $G_R$  is reduced, then  $G$  is clean if, and only if  $G_R$  is clean.*

**PROOF.** If  $G_R$  is clean then the result follows from Lemma 3 and Lemma 1. Conversely suppose that  $G$  is clean. Then, since there are no non-zero homomorphisms from a divisible group into a reduced group, an endomorphism of  $G$  may be represented as a matrix of the form

$$M = \begin{pmatrix} X & Y \\ 0 & W \end{pmatrix}$$

where  $X \in \text{End}(D)$ ,  $W \in \text{End}(G_R)$ ,  $Y \in \text{Hom}(G_R, D)$ . But then the mapping sending  $M$  to  $W$  is a *ring* epimorphism from  $\text{End}(G)$  onto  $\text{End}(G_R)$  and thus  $E(G_R)$  is also clean.  $\square$

**REMARK.** It is clear that the proof of Corollary 4 extends to a much wider context. For example, it follows *mutatis mutandis* that if  $G = A \oplus B$  and  $\text{Hom}(A, B) = 0$ , then  $G$  is clean if, and only if both  $A$  and  $B$  are clean. In light of Corollary 4 it makes sense to focus our attention on reduced  $p$ -groups.

The determination of the Jacobson radical of the endomorphism of an Abelian  $p$ -group is not easy even when the group is a direct sum of cyclic groups. It is, however, possible to give an upper bound for the Jacobson radical. Denote by  $H(G)$  the set of endomorphisms of  $G$  which are strictly height-increasing on the socle  $G[p]$  of  $G$ :

$$H(G) = \{\phi \in \text{End}(G) \mid h(x) < \infty \text{ implies } h(\phi(x)) > h(x) \text{ for all } x \in G[p]\}.$$

The set  $H(G)$  is actually a two-sided ideal in  $\text{End}(G)$ ; it was introduced by Pierce in his seminal paper [15] and is often referred to as the Pierce radical of  $G$ . It was shown in [15, Lemma 14.4] that  $J(\text{End}(G)) \subseteq H(G)$ .

**LEMMA 5.** *Let  $G$  be a reduced Abelian  $p$ -group, then  $G$  is clean only if  $J(\text{End}(G)) = H(G)$ .*

**PROOF.** As noted above one inclusion always holds. Suppose that  $G$  is clean and that  $\phi \in H(G)$ . Then we may write  $\phi = \pi + u$  where  $\pi$  is an idempotent and  $u$  is a unit. Since  $H(G)$  is an ideal it will suffice to show that

$1 - \phi$  is a unit. We claim that for all  $g \in G$ ,  $\pi(g) \neq 0$ . Suppose for a contradiction that  $\pi(g) = 0$  for some  $g \in G$ . Since  $g$  is a torsion element there is  $0 \neq x \in G[p]$  such that  $\pi(x) = 0$ . But then  $h(\phi(x)) = h(\pi(x) + u(x)) = h(u(x)) = h(x) - \text{contradiction}$ . Since  $\pi$  is an idempotent which is never zero, we conclude that  $\pi = 1$ . However this then means that  $1 - \phi = -u$ , a unit as required.  $\square$

LEMMA 6. *If the reduced Abelian  $p$ -group  $G$  is clean, then  $G$  does not have a summand which is a direct sum of cyclic groups whose orders are unbounded.*

PROOF. Suppose that  $G$  has a summand of the form  $B = \bigoplus_{i < \omega} \mathbb{Z}(p^{n_i})$  with  $n_i < n_j$  for all  $i < j$ . Let  $G = B \oplus K$  and choose generators  $e_i$  for each cyclic summand  $\mathbb{Z}(p^{n_i})$  of  $B$ . Next, define a mapping  $\phi : G \rightarrow G$  by

$$\phi(x) = \begin{cases} e_i + p^{n_j - n_i} e_{i+1} & \text{if } x = e_i; \\ x & \text{if } x \in K \end{cases}$$

Note first, that  $\phi$  is not onto and hence not an automorphism: a straightforward calculation shows that  $e_1$  is not in the image of  $\phi$ . Suppose, for a contradiction, that  $\phi$  can be expressed as  $\phi = \pi + \alpha$ , where  $\pi$  is an idempotent and  $\alpha$  is an automorphism of  $G$ . Then pre-multiplication by  $\pi$  yields  $\pi(\phi - 1) = \pi\alpha$  and so  $\pi(\phi - 1)(G) = \pi\alpha(G) = \pi(G)$ . Hence  $\pi(G) = \pi(\phi - 1)(G) \subseteq pB$ , and so, since  $\pi$  is an idempotent,  $\pi(G) \subseteq p^\omega B = 0$ . Thus  $\pi = 0$  and so  $\phi$  is an automorphism – contradiction.  $\square$

COROLLARY 7. *If  $G$  is the direct sum of infinitely many unbounded  $p$ -groups  $G_i$ , then  $G$  is not clean.*

PROOF. Since an unbounded  $p$ -group has a cyclic direct summand of order exceeding any bound, we can construct a summand of  $G$  by choosing cyclic summands of strictly increasing order from the various  $G_i$ . The result now follows from Lemma 6.  $\square$

In light of Lemma 6 it is reasonable to give some examples of  $p$ -groups which are clean.

LEMMA 8. *Torsion-complete  $p$ -groups are clean. In particular, bounded  $p$ -groups (and a fortiori finite  $p$ -groups) are also clean.*

PROOF. Let  $B = \bigoplus_{n < \omega} B_n$  be a direct sum of cyclic groups where each  $B_n$  is a direct sum of cyclic groups of order  $p^{n+1}$ . Let  $\bar{B}$  denote the torsion completion of  $B$ . The proof is based on an observation of Pierce, [15, Theorem 14.3], that the quotient  $\text{End}(\bar{B})/J(\text{End}(\bar{B}))$  is isomorphic to a product of full matrix rings over the field  $\mathbb{Z}/p\mathbb{Z}$ :  $\text{End}(\bar{B})/J(\text{End}(\bar{B})) \cong \prod_{\text{ring } n < \omega} \text{End}_{\mathbb{Z}/p\mathbb{Z}}(B_n[p])$ . Since each  $B_n[p]$  is a vector space, it is clean and it is easily seen that ring direct products of clean rings are, again, clean. But then, exactly as in the proof of Lemma 3, both idempotents and units lift from  $\text{End}(\bar{B})/J(\text{End}(\bar{B}))$  to  $\text{End}(\bar{B})$  and so the latter is also clean. Since a bounded group, and hence a finite group, is torsion-complete, these groups are clean as well.

**THEOREM 9.** *A countable  $p$ -group  $G$  is clean if, and only if  $G = D \oplus B$ , where  $D$  is divisible and  $B$  is bounded.*

PROOF. If  $G = D \oplus B$  then the result follows from Corollary 4 and Lemma 6. (Note that there is no need for countability here.) Conversely, suppose that  $G$  is clean and countable; it clearly suffices to show that the reduced part of  $G$  is bounded and so there is no loss in assuming now that  $G$  is reduced. Suppose, for a contradiction, that  $G$  is unbounded. Since  $G$  is also countable we may write  $G = \bigoplus_{i \in I} G_i$  where the index set  $I$  is countably infinite and each  $G_i$  is unbounded – this non-trivial fact follows from Zippen’s Theorem; see [4, Proposition 77.5 and Exercise 77.8(a)]. It follows immediately that  $G$  has a direct summand which is an unbounded direct sum of cyclic groups, contrary to Lemma 6.  $\square$

REMARK. It is essential in the proof of Theorem 9 that we can deduce that the groups  $G_i$  constructed are unbounded. A direct sum of infinitely many reduced  $p$ -groups may be clean without the group being bounded: take  $G = \bigoplus_{i < \omega} A_i \oplus \bar{B}$  where each  $A_i$  is cyclic of order  $p$  and  $\bar{B}$  is an unbounded torsion-complete group. Then  $G$  is clean by Lemmas 1 and 8 but clearly  $G$  is not bounded. This example is, of course, not in conflict with Theorem 9, since the group  $G$  is necessarily uncountable.

It is possible to improve considerably on Theorem 9. As noted in the introduction, if  $G$  is a clean  $p$ -group and  $p \neq 2$ , then every endomorphism of  $G$  is the sum of an involution and a unit; in particular  $G$  has unit sum number 2. Thus to find clean  $p$ -groups it is enough to restrict attention to

groups with unit sum number 2. A large and important class of such reduced  $p$ -groups is the class of totally projective groups – see [4, §82] for details of these groups and see [8] for their unit sum property.

**THEOREM 10.** *A totally projective  $p$ -group  $G$  is clean if, and only if,  $G$  is bounded.*

**PROOF.** Only the necessity needs to be established by virtue of Lemma 8. In light of Lemma 6 it suffices then to show that an unbounded totally projective  $p$ -group must have a direct summand which is an unbounded direct sum of cyclic groups. We establish this by induction on the length,  $l(G)$ , of  $G$ . Since  $G$  is unbounded, it has infinite length and if  $l(G) \leq \omega 2 = \omega + \omega$ , then it follows from e. g. [16, Corollario 26.3], that  $G$  is a direct sum of countable  $p$ -groups. It follows immediately as in the proof of Theorem 9, that  $G$  has the required summand. Suppose then that  $l(G) > \omega 2$ . Since  $G$  is totally projective, it certainly is a  $C_{\omega 2}$ -group – see [16, §30] – and the cofinality of  $\omega 2$  is clearly  $\omega$ . It follows then from [16, Teorema 30.4], that  $G$  has an  $\omega 2$ -basic subgroup,  $B$  say. By definition  $B$  is a totally projective group of length  $l(B) \leq \omega 2$ , and hence, as noted above, is a direct sum of countable groups. Moreover  $B$  must be unbounded: since it is  $\omega 2$ -basic, it is  $\omega$ -dense and so  $G = B + p^\omega G$ . If  $B$  were bounded this would contradict  $l(G) > \omega 2$ . Then, as above, we can decompose  $B = C \oplus D$ , where  $C$  is an unbounded direct sum of cyclic groups. But now  $B$  satisfies the hypotheses of [16, Lemma 30.2] with  $\mu = \omega$ , and so  $G$  decomposes as  $G = C \oplus (D + p^\omega G)$ .  $\square$

The classes of totally projective and torsion-complete  $p$ -groups are essentially the only well-behaving classes of  $p$ -groups and we have just established that clean totally projective groups are bounded and hence torsion-complete. It is therefore rather natural to look for groups which are clean but not torsion-complete; existence of such groups would tend to indicate that it is unlikely that any reasonable classification of clean  $p$ -groups exists. Alas, we can indeed exhibit such groups! Pierce [15] has exhibited a  $p$ -group  $G$  which has as basic subgroup the group  $B = \bigoplus_{i < \omega} \mathbb{Z}(p^{i+1})$ , and furthermore  $\text{End}(G) = J_p \oplus \text{End}_s(G)$ , where  $\text{End}_s(G)$  denotes the (two-sided) ideal of small endomorphisms. Recall that an endomorphism  $\phi : G \rightarrow G$  is said to be *small* if given any positive integer  $e$ , there exists a positive integer  $n$  such that  $\phi(p^n G)[p^e] = 0$ ; see Pierce's original paper [15] or [4, §46] for further details. Pierce's original example is a group of cardinality  $2^{\aleph_0}$ , but it is possible, using recent techniques based on



the so-called Shelah Black Box Principle, a powerful combinatorial principle, to exhibit such groups of arbitrary large cardinality – see for example [2]. (It is, of course, necessary to choose the basic subgroup as a “large” direct sum of standard basic groups of the type  $B$  above.) Moreover it is possible to exhibit “essentially-rigid systems” of such groups i.e. a family of groups  $G_i$  ( $i \in I$ ), in which the only homomorphisms from  $G_i \rightarrow G_j$  ( $i \neq j$ ) are small. In particular it is possible to exhibit a family of  $2^{2^{\aleph_0}}$  non-isomorphic groups  $G_i$  all having a standard basic subgroup  $B = \bigoplus_{n < \omega} \mathbb{Z}(p^{n+1})$ . It is well known and easy to show that any group  $G$  with endomorphism ring  $\text{End}(G) = J_p \oplus \text{End}_s(G)$  is *essentially indecomposable* in the sense that in any direct decomposition  $G = M \oplus N$ , one of  $M, N$  is bounded.

We begin by showing that such essentially indecomposable groups obey the condition of Lemma 5:  $J(\text{End}(G)) = H(G)$ .

LEMMA 11. *If  $\text{End}(G) = J_p \oplus \text{End}_s(G)$ , then  $J(\text{End}(G)) = H(G)$  and each element of  $H(G)$  is locally nilpotent.*

PROOF. As we have already noted the Jacobson radical is always a subset of the Pierce radical, so to establish equality it suffices to show the reverse. Suppose that  $\psi = r + \phi$  belongs to the Pierce radical, where  $r \in J_p$  and  $\phi$  is small. Then there is a positive integer  $N$  such that  $\phi(p^N G)[p] = 0$ . Since, by assumption,  $\psi$  is strictly height increasing on the socle, the  $p$ -adic integer  $r$  cannot be a unit, so  $r = p\pi$ . But then  $\psi$  and  $\phi$  agree on  $G[p]$ . Thus  $\phi$  is both small and strictly height increasing on the socle. Hence for any  $x \in G[p]$ ,  $\phi^{N+1}(x) = \psi^{N+1}(x) = 0$  i.e.  $\psi$  is locally nilpotent on the socle,  $G[p]$ . We claim that this forces  $\psi$  to be locally nilpotent on  $G$ . Suppose by induction that the result is true for  $G[p^k]$  and let  $x \in G[p^{k+1}]$ . Now  $px \in G[p^k]$  and so  $\psi^{N_k}(px) = 0$  for some  $N_k$ . But then  $\psi^{N_k}(x) \in G[p]$  and so  $\psi^{N+1}(\psi^{N_k}(x)) = 0$  i.e.  $\psi^{N_{k+1}}(x) = 0$ , where  $N_{k+1} = N_k + N + 1$  and so  $\psi$  is locally nilpotent on  $G[p^{k+1}]$  and hence by induction, on  $G$ . Thus our claim is established and with it the second statement of the lemma. But it follows immediately that  $1 - \psi$  is an automorphism of  $G$ , or equivalently that  $\psi \in J(\text{End}(G))$ .  $\square$

The structure of the quotient  $\text{End}(G)/H(G)$  is known from [15, Theorem 14.3]: it is a subring with 1, lying between a countable direct sum and direct product of matrix rings over the Galois field of  $p$  elements. When the basic subgroup is standard, or equivalently when each Ulm invariant of the group is equal to 1, this takes on a particularly simple form:  $\text{End}(G)/H(G) \cong R$ , where  $1 \in R$  and  $S \leq R \leq P$  with  $S = \bigoplus_{\aleph_0} \mathbb{Z}(p)$  and  $P$  the corresponding direct product.

LEMMA 12. *If  $R$  is a subring with 1 such that  $\bigoplus_{\mathbb{N}_0} \mathbb{Z}(p) \leq R \leq \prod_{\mathbb{N}_0} \mathbb{Z}(p)$ , then  $R$  is clean.*

PROOF. Note firstly that if  $x = (x_1, x_2, \dots) \in R$  has the property that each  $x_i$  is nonzero, then  $x$  is a unit in  $R$ : it follows immediately from the Fermat theorem that the inverse is  $x^{p-2}$ , an element of  $R$ . If  $\Phi(t)$  denotes the  $p$ th cyclotomic polynomial, then  $\Phi(x) \in R$  has the property that each co-ordinate is either 0 or 1, while each co-ordinate of  $x - \Phi(x)$  is nonzero. Hence the decomposition  $x = (x - \Phi(x)) + (\Phi(x))$  expresses  $x$  as the sum of a unit and an idempotent of  $R$ . □

THEOREM 13. *If  $G$  is a separable  $p$ -group with  $\text{End}(G) = J_p \oplus \text{End}_s(G)$  and each Ulm invariant  $f_G(n) = 1$ , then  $G$  is clean.*

PROOF. It follows from Lemmas 11 and 12 that  $\text{End}(G)/J(\text{End}(G))$  is clean. Put  $J = J(\text{End}(G))$ . Then if  $\phi \in \text{End}(G)$ , we have that  $\phi = \pi + u$ , with  $\pi$  an idempotent mod  $J$  and  $u$  a unit mod  $J$ . The usual method of lifting idempotents modulo a nil ideal can be adapted to our situation. We shall find an element  $\alpha \in J$  which commutes with  $\pi$  so that  $\pi' = \pi + \alpha(1 - 2\pi)$  is an idempotent in  $\text{End}(G)$ . Then, since  $\alpha \in J$ , we see that  $\pi - \pi' \in J$  will follow. The requirement that  $\pi'$  be an idempotent is equivalent (still assuming  $\alpha$  commutes with  $\pi$ ) to  $\alpha$  satisfying the quadratic  $(1 + 4\rho)x^2 - (1 + 4\rho)x + \rho$  where  $\rho = \pi^2 - \pi \in J$ . Solving this quadratic yields the solution  $\frac{1}{2}(1 - (1 + 4\rho)^{-1/2})$ . Using the binomial expansion to compute this, we are led to seek  $\alpha$  in the form of the formal sum

$$(\star) \quad \alpha = \sum_{i=1}^{\infty} a_i \rho^i \text{ where } a_i = (-1)^{i+1} \frac{1}{2} \binom{2i}{i} \in \mathbb{Z}, \quad 1 \leq i < \infty.$$

Note that the coefficients  $a_i$  are indeed integers. Now we know from Lemma 11 that each element of  $J$  is locally nilpotent and so if  $g \in G$ , then  $\rho^n(g) = 0$  for some integer  $n$ . Hence we can define an endomorphism  $\alpha$  of  $G$  by the power series  $(\star)$ , this is then a well-defined element of  $\text{End}(G)$ . Also, since  $\rho$  commutes with  $\pi$ ,  $\alpha$  commutes with  $\pi$  as well. Moreover, since the Jacobson radical coincides with the Pierce radical in this situation, to show that  $\alpha \in J$ , it will suffice to show that  $\alpha$  is strictly height increasing on the socle  $G[p]$ . This however is immediate since the power series defining  $\alpha$  has no constant term and  $\rho$  is strictly height increasing.

We now have that  $\phi = \pi + u = \pi' + \pi - \pi' + u = \pi' + u + \rho$  with  $\pi'$  an idempotent in  $\text{End}(G)$ ,  $\rho \in J$  and  $u$  a unit mod  $J$ . But then it follows from

standard properties of the Jacobson radical that  $u' = u + \rho$  is also a unit in  $\text{End}(G)$ . Thus  $\phi = \pi' + u'$  with  $\pi'$  an idempotent and  $u'$  a unit in  $\text{End}(G)$ .  $\square$

Note that the  $p$ -group described in the theorem above is not torsion-complete.

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