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Well Posedness Under Levi Conditions for a Degenerate Second Order Cauchy Problem.

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ABSTRACT - We consider the Cauchy problem for a second order equation of hyperbolic type which degenerates both in the sense that it is weakly hyperbolic and it has non Lipschitz continuous in time coefficients. Intersections between the roots of the equation are of a finite order k, and the first time derivative of the principal part's coefficients present a blow-up phenomenon at the time $t = 0$, behaving as t^{-q}, $q \geq 1$. The mixture of these two situations gives, under an appropriate Levi condition, C^∞ or Gevrey well posedness of the Cauchy problem, depending on the dominant between the two behaviors.

1. Introduction and main results.

In this paper we study the Cauchy problem

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = 0, & (t,x) \in [0,T] \times \boldsymbol{R}^n, \\ u(0,x) = u_0(x), \\ \partial_t u(0,x) = u_1(x), \end{cases} \tag{1.1}$$

for a second order operator of the form

$$P = D_t^2 - a(t,x,D_x) + b(t,x,D_x) + c(t,x), \tag{1.2}$$

$$a(t,x,\xi) = \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j, \quad b(t,x,\xi) = \sum_{j=1}^{n} b_j(t,x)\xi_j,$$

$D = \dfrac{1}{\sqrt{-1}}\,\partial$, which is hyperbolic, i.e.

$$a(t,x,\xi) \geq 0, \quad t \in [0,T], x, \xi \in \boldsymbol{R}^n.$$

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We are going to allow (1.2) to degenerate both in the sense that its characteristic roots may coincide at some points and that its coefficients are not regular with respect to the time variable. The purpose of the paper is to obtain well posedness results for the degenerate problem (1.1). We remind that (1.1) is said to be well posed in the space X if for every $u_0, u_1 \in X$ there is a unique solution $u \in C^1([0,T]; X)$.

We know that the Cauchy problem for a weakly hyperbolic equation may be not well posed in C^∞, and that C^∞ well posedness can be achieved by asking the equation to satisfy some condition (the Levi condition) on the first order term b. For example, the Cauchy problem for

$$P = D_t^2 - t^{2\ell} D_x^2 + t^{\upsilon} D_x$$

is well posed in C^∞ if and only if $\upsilon \geq \ell - 1$; for $\upsilon < \ell - 1$, well posedness holds only in Gevrey classes of index $\sigma < (2\ell - \upsilon)/(\ell - \upsilon - 1)$, [12]. In the particular case of an effectively hyperbolic operator, C^∞ well posedness holds without assuming any Levi condition, see [16]; notice that if $a = a(t,\xi)$, (1.2) is effectively hyperbolic if $\sum_{j=0}^{2} |\partial_t^j a(t,\xi)| \neq 0$, $t \in [0,T]$, $|\xi| = 1$.

An intermediate situation between effective and non effective hyperbolicity is introduced in [9] for (1.2) with coefficients depending only on time: if there exists an integer $k \geq 2$ and a $\gamma \in [0, 1/2]$ such that

$$\sum_{j=0}^{k} |\partial_t^j a(t,\xi)| \neq 0, \quad |b(t,\xi)| \leq C a^{\gamma}(t,\xi), \ t \in [0,T], |\xi| = 1,$$

then (1.1) is C^∞ well posed provided $\gamma \geq 1/2 - 1/k$, and this choice of γ is optimal; otherwise, (1.1) is well posed in Gevrey classes of index $\sigma < (1-\gamma)/[1/2 - (\gamma + 1/k)]$. Similar results for (1.2) depending also on space variables are given in [11], [1], [2].

Strictly hyperbolic equations with non Lipschitz continuous in time coefficients of the principal part have been widely studied starting from [6]; different ways to weaken the Lipschitz regularity produce quite different effects. Here we are interested in the way considered by [7]: for (1.2) with coefficients depending only on time, and supposing the singular behavior

$$|\partial_t a(t,\xi)| \leq \frac{C}{t^q}, \quad q \geq 1, \quad t \in]0,T], |\xi| = 1,$$

the authors show that (1.1) is C^∞ well posed only if $q = 1$; otherwise, well posedness holds in Gevrey classes of index $\sigma < q/(q-1)$.The index is sharp. The same results with dependence also on space variables are in [4], [5].

The purpose of the paper is to study the two different kinds of degeneration mixed together. We consider (1.2) with the following structure:

$$\text{(1.3)} \qquad \begin{cases} P = D_t^2 - \alpha(t)Q(t,x,D_x) + b(t,x,D_x) + c(t,x), \\ \alpha \in C^\infty[0,T], \ Q(t,x,\xi) = \sum_{i,j=1}^{n} q_{ij}(t,x)\xi_i\xi_j; \end{cases}$$

the weak hyperbolicity condition is expressed by:

$$\text{(1.4)} \qquad \begin{cases} \alpha(t) \geq 0, \ t \in [0,T], \\ Q(t,x,\xi) \geq q_0|\xi|^2, \quad q_0 > 0, \ t \in [0,T], x, \xi \in \boldsymbol{R}^n. \end{cases}$$

We assume for (1.3) that:

i) there exist an integer $k \geq 2$ and a $\gamma \in [0,1/2]$ such that

$$\text{(1.5)} \qquad \begin{cases} \sum_{i=0}^{k} |\alpha^{(i)}(t)| \neq 0, \quad |\partial_x^\beta b_j(t,x)| \leq C_\beta \alpha(t)^\gamma, \\ j = 1, \dots, n, \quad t \in [0,T], x \in \boldsymbol{R}^n, \ \beta \in \boldsymbol{Z}_+^n; \end{cases}$$

ii) the coefficients of Q satisfy

$$\text{(1.6)} \qquad |\partial_t q_{ij}(t,x)| \leq \frac{C}{t^q}, \ q \geq 1, \ i,j = 1, ..., n.$$

We prove that, under these assumptions, (1.1) is:

– C^∞ well posed if $q = 1$ and $\gamma \geq 1/2 - 1/k$, see Theorem 2.1;
– well posed in Gevrey classes of index $\sigma < q/(q-1)$ if $q > 1$ and $\gamma \geq 1/2 - 1/k$, or $\sigma < \min\left\{\frac{q}{q-1}, \frac{1-\gamma}{1/2-(\gamma+1/k)}\right\}$ if $q \geq 1$ and $\gamma < 1/2 - 1/k$, see Theorem 3.1.

Both results are in line with [7] and [9].

We use an approach as similar as possible to the proofs of these results, consisting of three steps:

1) factorization of the principal part of P by means of regularized characteristic roots $\pm\tilde{\lambda}$;

2) reduction of $Pu = 0$ to an equivalent 2×2 system $LU = 0$ with $L = \partial_t - i\Lambda(t,x,D_x) + R(t,x,D_x)$, $i = \sqrt{-1}$, $\Lambda(t,x,\xi)$ a real diagonal matrix having $\pm\tilde{\lambda}$ as entries, and $R(t,x,\xi)$ such that either $\int_0^t |R(\tau,x,\xi)|d\tau \leq c_0 + \delta \log\langle\xi\rangle$, $c_0, \delta > 0$, $t \in [0,T]$, $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, in

the C^∞ case or $\int\limits_0^t |R(\tau,x,\xi)|d\tau \le \delta\langle\xi\rangle^{1/\sigma}$, $\delta>0$, $t\in[0,T]$ in the Gevrey case;

3) application of the energy estimate in Sobolev or in Gevrey-Sobolev spaces to the operator L.

2. C^∞ well posedness.

This Section is devoted to a result of C^∞ well posedness for (1.1), (1.3). We denote by $H^s = H^s(\boldsymbol{R}^n)$ the usual Sobolev space, and by $\|-\|_s$ the Sobolev norm. The usual space of symbols on $\boldsymbol{R}^{2n}$ is denoted by $S^m = S^m(\boldsymbol{R}^n \times \boldsymbol{R}^n)$. The space $B((0,T];S^m)$ consists of all functions defined in $(0,T]$ and with values in S^m that are bounded as functions of time.

We prove the following:

THEOREM 2.1. *Consider the Cauchy problem* (1.1), (1.3), *under conditions* (1.4), (1.5) *for* $\gamma \ge 1/2 - 1/k$, *and* (1.6) *with* $q=1$. *Then,* (1.1) *is* C^∞ *well posed.*

This result is consistent with these facts:

– taking $k=2$ one can choose $\gamma=0$: indeed no Levi conditions are needed for an effectively hyperbolic operator, [16];

– $\gamma=1/2$ can be reached for $k\to\infty$: under the usual C^∞ Levi condition there is no need to assume that $a=\alpha Q$ has zeros of finite order, [10], [16].

In the proof of Theorem 2.1 we are going to make use of the following result [2]:

THEOREM 2.2. *Consider the operator*

$$L = \partial_t - i\begin{pmatrix} \tilde\lambda_1(t,x,D_x) & 0 \\ 0 & \tilde\lambda_2(t,x,D_x)\end{pmatrix} + R(t,x,D_x) \tag{2.1}$$

$t\in[0,T]$, $x\in\boldsymbol{R}^n$, *where* $\tilde\lambda_j(t,x,\xi)\in\boldsymbol{R}$, $\tilde\lambda_j\in L^1([0,T];S^1)$ *for* $j=1,2$, *and* R *is a* 2×2 *matrix satisfying*

$$\begin{cases} R\in L^1([0,T];S^1), \\ |R(t,x,\xi)|\le\varphi(t,\xi), \quad \varphi\in L^1([0,T];S^1), \\ \int\limits_0^T |\partial_\xi^\beta\varphi(t,\xi)|dt \le \delta_\beta\langle\xi\rangle^{-|\beta|}\log(1+\langle\xi\rangle), \qquad \beta\ge 0. \end{cases} \tag{2.2}$$

Then, there exists $\delta > 0$ *such that the energy estimate*

$$\|U(t)\|_{\mu}^{2} \leq C_{\mu}\left(\|U(0)\|_{\mu+\delta}^{2} + \int_{0}^{t} \|LU(\tau)\|_{\mu+\delta}^{2} d\tau\right)$$

holds for all $U \in C^1([0,T]; H^{\mu+\delta}) \cap C([0,T]; H^{\mu+\delta+1})$.

Theorem 2.2 implies well posedness, with the loss of δ derivatives, in $H^{+\infty} = \cap_s H^s$ and $H^{-\infty} = \cup_s H^s$ of the Cauchy problem for L.

PROOF OF THEOREM 2.1. The characteristic roots of P are

$$\pm\lambda(t,x,\xi) = \pm\sqrt{\alpha(t)Q(t,x,\xi)}$$

with $Q \in C([0,T]; S^2)$, $\partial_t Q \in B((0,T]; S^2)$. Given a function $\rho \in C_0^\infty(\boldsymbol{R})$, $0 \leq \rho \leq 1$, $\int \rho(\tau)d\tau = 1$, and extended the symbol Q on $\boldsymbol{R}_t$ by setting $Q(\tau,x,\xi) = Q(T,x,\xi)$ for $\tau \geq T$, $Q(\tau,x,\xi) = Q(0,x,\xi)$ for $\tau \leq 0$, we approximate $\pm\lambda$ by defining new symbols $\pm\tilde{\lambda}(t,x,\xi)$ as follows:

$$\tilde{\lambda}(t,x,\xi) = \sqrt{\alpha(t) + \langle\xi\rangle^{-2}} \int \sqrt{Q(\tau,x,\xi)}\rho(\langle\xi\rangle(t-\tau))\langle\xi\rangle d\tau. \tag{2.3}$$

Then we use $\pm\tilde{\lambda}$ to factorize (1.3). In the factorization we need to deal with the symbols

$$\partial_t\tilde{\lambda}(t,x,\xi) = \frac{i\alpha'(t)}{2\sqrt{\alpha(t)+\langle\xi\rangle^{-2}}} I_1(t,x,\xi) + \sqrt{\alpha(t)+\langle\xi\rangle^{-2}} I_2(t,x,\xi) \tag{2.4}$$

and

$$\begin{aligned}\tilde{\lambda}(t,x,\xi) - \lambda(t,x,\xi) = {} & \sqrt{\alpha(t)+\langle\xi\rangle^{-2}} I_3(t,x,\xi) \\ & + \left(\sqrt{\alpha(t)+\langle\xi\rangle^{-2}} - \sqrt{\alpha(t)}\right)\sqrt{Q(t,x,\xi)}\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
I_1(t,x,\xi) &= \int \sqrt{Q(\tau,x,\xi)}\rho(\langle\xi\rangle(t-\tau))\langle\xi\rangle d\tau, \\
I_2(t,x,\xi) &= \int \sqrt{Q(\tau,x,\xi)}\rho'(\langle\xi\rangle(t-\tau))\langle\xi\rangle^2 d\tau, \\
I_3(t,x,\xi) &= \int \left(\sqrt{Q(\tau,x,\xi)} - \sqrt{Q(t,x,\xi)}\right)\rho(\langle\xi\rangle(t-\tau))\langle\xi\rangle d\tau.
\end{aligned}$$

The change of variable $\langle\xi\rangle(t-\tau) = s$ clearly gives $I_1 \in L^1([0,T]; S^1)$. Using

$\int \rho'(s)ds = 0$ we can rewrite

$$I_2(t,x,\xi) = \int \Big(\sqrt{Q(\tau,x,\xi)} - \sqrt{Q(t,x,\xi)}\Big)\rho'(\langle\xi\rangle(t-\tau))\langle\xi\rangle^2 d\tau;$$

by the hypothesis on Q we get

$$I_2 \in C([0,T];S^2),\ \ tI_2 \in B((0,T];S^1),$$

$$I_3 \in C([0,T];S^1),\ \ tI_3 \in B((0,T];S^0).$$

The same arguments give

$$[\lambda(t,x,D_x)]^2 - [\tilde\lambda(t,x,D_x)]^2 = \Big(\sqrt{\alpha(t) + \langle D_x\rangle^{-2}} + \alpha(t)\Big) I_4(t,x,D_x), \tag{2.6}$$

with

$$I_4(t,x,\xi) \in C([0,T];S^2),\ \ tI_4 \in B((0,T];S^1).$$

From (2.3), (2.4), (2.6) we come to the factorization

$$P = (D_t - \tilde\lambda(t,x,D_x))(D_t + \tilde\lambda(t,x,D_x)) + \frac{\alpha'(t)}{\sqrt{\alpha(t) + \langle D_x\rangle^{-2}}} S_1(t,x,D_x) \tag{2.7}$$

$$+(\sqrt{\alpha(t) + \langle D_x\rangle^{-2}} + \alpha(t)) S_2(t,x,D_x) + b(t,x,D_x) + c(t,x,D_x),$$

with

$$S_1(t,x,\xi) \in L^1([0,T];S^1),$$

$$S_2(t,x,\xi) \in C([0,T];S^2),\ \ tS_2(t,x,\xi) \in B((0,T];S^1).$$

To perform now the reduction to a first order system, we consider the operator $\omega(t,D_x)$ with symbol

$$\omega(t,\xi) = \sqrt{\alpha(t) + \langle\xi\rangle^{-2}}\langle\xi\rangle,$$

and we define the vector $V = (v_0, v_1)$ as follows:

$$\begin{cases} v_0 = \omega(t,D_x)u, \\ v_1 = (D_t + \tilde\lambda(t,x,D_x))u. \end{cases} \tag{2.8}$$

By (2.8), and after a straightforward diagonalization of matrix $M(t,x,D_x)$, we obtain that the scalar problem (1.1) is equivalent to

$$\begin{cases} LU = 0, \\ U(0,x) = U_0, \end{cases} \tag{2.9}$$

where $U = M(t,x,D_x)V$ and

$$L = \partial_t - i\begin{pmatrix} -\tilde{\lambda}(t,x,D_x) & 0 \\ 0 & \tilde{\lambda}(t,x,D_x) \end{pmatrix} + \frac{\alpha'(t)}{\alpha(t)+\langle D_x\rangle^{-2}}A(t,x,D_x)$$

$$+\frac{b(t,x,D_x)\langle D_x\rangle^{-1}}{\sqrt{\alpha(t)+\langle D_x\rangle^{-2}}}B(t,x,D_x) + C(t,x,D_x) + E(t,x,D_x), \tag{2.10}$$

with 2×2 matrices A, B, C, E such that

$$A, B, E \in L^1([0,T];S^0),\ \ C \in C([0,T];S^1),\ \ tC \in B((0,T];S^0).$$

Operator (2.10) has the structure (2.1) for a remainder

$$R(t,x,\xi) = \frac{\alpha'(t)}{\alpha(t)+\langle\xi\rangle^{-2}}A(t,x,\xi) + C(t,x,\xi)$$

$$+\frac{b(t,x,\xi)\langle\xi\rangle^{-1}}{\sqrt{\alpha(t)+\langle\xi\rangle^{-2}}}B(t,x,\xi) + E(t,x,\xi).$$

We only have to check that R fulfills condition (2.2); then an application of Theorem 2.2 will give C^∞ well posedness of (1.1) by usual arguments in the energy method.

For any $N \geq 2$,

$$\frac{\alpha'(t)}{\alpha(t)+\langle\xi\rangle^{-2}}A(t,x,\xi) = \frac{\alpha'(t)}{(\alpha(t)+\langle\xi\rangle^{-2})^{1-1/N}}\cdot\frac{A(t,x,\xi)}{(\alpha(t)+\langle\xi\rangle^{-2})^{1/N}};$$

we know from Lemma 1 in [10] that if $f \in C^N[0,T]$ is real valued and non negative, then $f^{1/N}$ is absolutely continuous on $[0,T]$; we apply the Lemma to $f(t) = \alpha(t) + \langle\xi\rangle^{-2}$ and we obtain

$$\alpha'(t)A(t,x,\xi)/\big(\alpha(t)+\langle\xi\rangle^{-2}\big) \in L^1([0,T];S^{\frac{2}{N}}), N \geq 2.$$

Using the Levi condition, we split

$$\frac{b(t,x,\xi)\langle\xi\rangle^{-1}B(t,x,\xi)}{\big(\alpha(t)+\langle\xi\rangle^{-2}\big)^{1/2}} = \frac{b(t,x,\xi)\langle\xi\rangle^{-1}}{(\alpha(t)+\langle\xi\rangle^{-2})^{\gamma}}\cdot\frac{B(t,x,\xi)}{(\alpha(t)+\langle\xi\rangle^{-2})^{1/2-\gamma}},$$

and get

$$b(t,x,\xi)\langle\xi\rangle^{-1}B(t,x,\xi)\Big(\alpha(t)+\langle\xi\rangle^{-2}\Big)^{-1/2} \in L^1([0,T];S^{1-2\gamma}).$$

The first condition in (2.2) holds true.

Now we have to construct φ as in (2.2). To this aim we take a smooth function ψ, $0 \le \psi \le 1$, $\psi(y) = 1$ for $|y| \le 1$, $\psi(y) = 0$ for $|y| \ge 2$, and we introduce the function

$$\tilde{\varphi}(t,\xi) = \psi(t\langle\xi\rangle)\delta\langle\xi\rangle + \frac{\delta}{t}(1 - \psi(t\langle\xi\rangle)), \ \delta > 0.$$

We can choose δ so large that $|C(t,x,\xi)| \le \tilde{\varphi}(t,\xi)$, for all (t,x,ξ). We define

$$\varphi(t,\xi) = K\left(\frac{\alpha'(t)}{\alpha(t) + \langle\xi\rangle^{-2}} + \frac{1}{(\alpha(t) + \langle\xi\rangle^{-2})^{1/2-\gamma}} + \tilde{\varphi}(t,\xi)\right),$$

$K > 0$ and large enough to have $|R(t,x,\xi)| \le \varphi(t,\xi)$ for all (t,x,ξ).

We get, from Lemma 1 and Lemma 2 of [9], that

$$\begin{cases} \displaystyle\int_0^T \frac{|\alpha'(t)|}{\alpha(t) + \langle\xi\rangle^{-2}}\,dt \le c_0 + \delta\log\langle\xi\rangle, \\ \displaystyle\int_0^T \frac{1}{(\alpha(t) + \langle\xi\rangle^{-2})^{1/2-\gamma}}\,dt \le c_0 + \delta\log\langle\xi\rangle, \end{cases}$$

for some $c_0 > 0$, $\delta > 0$, and thanks to $\gamma \ge 1/2 - 1/k$. As to $\tilde{\varphi}$, we have

$$\int_0^T \tilde{\varphi}(t,\xi)dt \le \delta \int_0^{2\langle\xi\rangle^{-1}} \langle\xi\rangle dt + \delta \int_{2\langle\xi\rangle^{-1}}^T (1/t)dt \le c_0 + \delta\log\langle\xi\rangle.$$

Condition (2.2) is fulfilled. Theorem (2.1) is proved.

3. Gevrey well posedness.

In this Section we prove a result of Gevrey well posedness for (1.1), (1.3). For $\sigma \ge 1$, we denote by $G^\sigma = G^\sigma(\boldsymbol{R}^n)$ the Gevrey class of index σ, consisting of all functions f such that

$$|\partial_x^\beta f(x)| \le CA^{|\beta|}\beta!^\sigma, \ C, A > 0, \ \forall\beta \in \boldsymbol{Z}_+^n.$$

In the proof we use Gevrey-Sobolev spaces $H^{s,\varepsilon,\sigma} = H^{s,\varepsilon,\sigma}(\boldsymbol{R}^n)$, defined for $\varepsilon \ge 0$ and $\sigma \ge 1$ by

$$H^{s,\varepsilon,\sigma}(\boldsymbol{R}^n) = \{u(x); e^{\varepsilon\langle D_x\rangle^{1/\sigma}}u \in H^s(\boldsymbol{R}^n)\};$$

there the norm is $\|u\|_{s,\varepsilon,\sigma} = \|e^{\varepsilon\langle D_x\rangle^{1/\sigma}} u\|_s$. Notice that $H^{s,\varepsilon,\sigma} \subset G^\sigma$, $\varepsilon > 0$.

A class of bounded pseudodifferential operators in Sobolev-Gevrey spaces is the class $S^{m,\sigma} = S^{m,\sigma}(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ of Gevrey symbols, defined for $\sigma \geq 1$ as the space of all functions $a(x,\xi)$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_\alpha A^{|\beta|} \beta!^\sigma \langle\xi\rangle^{m-|\alpha|}, \ \alpha,\beta \in \boldsymbol{Z}_+^n, \ A, c_\alpha > 0,$$

which is the limit space

$$S^{m,\sigma} := \lim_{\substack{\longleftarrow \\ \ell\to+\infty}} S_\ell^{m,\sigma}, \quad S_\ell^{m,\sigma} := \lim_{\substack{\longrightarrow \\ A\to+\infty}} S_{\ell,A}^{m,\sigma}$$

of the Banach spaces $S_{A,\ell}^{m,\sigma}$ of all symbols such that

$$|a|_{m,\sigma,A,\ell} := \sup_{|\alpha|\leq\ell,\beta\in\mathbf{Z}_+^n} \ \sup_{x,\xi} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| A^{-|\beta|} \beta!^{-\sigma} \langle\xi\rangle^{-m+|\alpha|} < +\infty.$$

In what follows, $B((0,T]; S^{m,\sigma})$ will denote the space of all functions defined in $(0,T]$ and with values in $S^{m,\sigma}$ that are bounded as functions of time.

We are going to prove the following:

THEOREM 3.1. *Consider the Cauchy problem* (1.1), (1.3) *under conditions* (1.4), (1.5) *with* $C_\beta = CA^{|\beta|}\beta!^\sigma$, (1.6).

If $\gamma \geq 1/2 - 1/k$ *and* $q > 1$, *then problem* (1.1) *is* G^σ *well posed provided*

$$1 \leq \sigma < \frac{q}{q-1}. \tag{3.1}$$

If $\gamma < 1/2 - 1/k$ and $q \geq 1$, then problem (1.1) is G^σ well posed for

$$1 \leq \sigma < \sigma_0 = \min\left\{\frac{q}{q-1}, \frac{1-\gamma}{1/2-(\gamma+1/k)}\right\}. \tag{3.2}$$

REMARK 3.2. In the proof of Theorem 3.1, we are precisely going to show that:

- $1 \leq q < \dfrac{2k}{k+2}$, $0 \leq \gamma < 1 - q\left(\dfrac{1}{2}+\dfrac{1}{k}\right) \Rightarrow \sigma_0 = \dfrac{1-\gamma}{1/2-(\gamma+1/k)}$;
- $1 \leq q < \dfrac{2k}{k+2}$, $1 - q\left(\dfrac{1}{2}+\dfrac{1}{k}\right) \leq \gamma \leq \dfrac{1}{2} \Rightarrow \sigma_0 = \dfrac{q}{q-1}$;
- $q \geq \dfrac{2k}{k+2}$, $0 \leq \gamma \leq \dfrac{1}{2} \Rightarrow \sigma_0 = \dfrac{q}{q-1}$.

It becomes easy to compare these results with [7], [9] if only we notice that

$$1-q(1/2+1/k)\le 1/2-1/k,\quad \forall q\ge 1,\ k\ge 2,$$
$$1-q(1/2+1/k)\le 0\ \text{ for } q\ge 2k/(k+2).$$

In the proof of Theorem 3.1 we apply the following result [2]:

THEOREM 3.3. *Consider the operator L in* (2.2), *suppose $\tilde\lambda_j\in L^1([0,T];S^{1,\sigma})$ for $j=1,2$ and*

$$\tag{3.3}\begin{cases}|R(t,x,\xi)|\le\varphi(t,\xi),\quad \varphi\in L^1([0,T];S^1),\\ \displaystyle\int_0^t\varphi(\tau,\xi)d\tau\in C([0,T];S^{1/\sigma}).\end{cases}$$

Then, there exist $\lambda_0>0$ and $b(t)\in L^1[0,T]$, $b(t)\ge 0$, such that for

$$w(t,\xi)=\exp\left(\int_0^t\left(\varphi(\tau,\xi)+b(\tau)\langle\xi\rangle^{1/\sigma}\right)d\tau\right)$$

the energy estimate

$$\|w(t,D_x)U(t)\|^2_{\mu,\lambda,\sigma}\le C_\mu\left(\|U(0)\|^2_{\mu,\lambda,\sigma}+\int_0^t\|w(\tau,D_x)LU(\tau)\|^2_{\mu,\lambda,\sigma}d\tau\right),$$
$$0\le t\le T^*,\ \ w(T^*,\xi)\le e^{\lambda\langle\xi\rangle^{1/\sigma}},$$

holds for all $U\in C^1([0,T];H^{\mu,\lambda,\sigma})\cap C([0,T];H^{\mu+1,\lambda,\sigma})$, $0<\lambda\le\lambda_0$.

PROOF OF THEOREM 3.1. We define

$$\tilde\lambda(t,x,\xi)=\sqrt{\alpha(t)+\langle\xi\rangle^{-\frac{1}{1-\gamma}}}\int\sqrt{Q(\tau,x,\xi)}\rho(\langle\xi\rangle(t-\tau))\langle\xi\rangle d\tau;$$

then we use the same arguments of the proof of Theorem 2.1 in the frame of the calculus of Gevrey symbols $S^{m,\sigma}$ and we come to the following factorization for P:

$$\begin{aligned}P=\ &(D_t-\tilde\lambda(t,x,D_x))(D_t+\tilde\lambda(t,x,D_x))+\frac{\alpha'(t)}{\sqrt{\alpha(t)+\langle D_x\rangle^{-\frac{1}{1-\gamma}}}}S_1(t,x,D_x)\\ &+\left(\sqrt{\alpha(t)+\langle D_x\rangle^{-\frac{1}{1-\gamma}}}+\alpha(t)\right)S_2(t,x,D_x)+S_3(t,x,D_x)\\ &+b(t,x,D_x)+c(t,x,D_x),\end{aligned}$$

where

$$S_1(t,x,\xi) \in L^1([0,T]; S^{1,\sigma}), S_3 \in L^1([0,T]; S^{2-\frac{1}{1-\gamma},\sigma}),$$

$$S_2(t,x,\xi) \in C([0,T]; S^{2,\sigma}),\ t^q S_2(t,x,\xi) \in B((0,T]; S^{1,\sigma}).$$

Defining this time

$$\omega(t,\xi) = \sqrt{\alpha(t) + \langle\xi\rangle^{-\frac{1}{1-\gamma}}}\langle\xi\rangle,$$

we retrace the reduction to a first order system of Theorem 2.1, coming to the equivalent Cauchy problem (2.9) for

$$\begin{aligned} L = \quad & \partial_t - i\begin{pmatrix} -\tilde{\lambda}(t,x,D_x) & 0 \\ 0 & \tilde{\lambda}(t,x,D_x) \end{pmatrix} \\ & + \frac{\alpha'(t)}{\alpha(t) + \langle D_x\rangle^{-\frac{1}{1-\gamma}}} A(t,x,D_x) + \frac{b(t,x,D_x)\langle D_x\rangle^{-1}}{\sqrt{\alpha(t) + \langle D_x\rangle^{-\frac{1}{1-\gamma}}}} B(t,x,D_x) \\ & + C(t,x,D_x) + \frac{D(t,x,D_x)}{\sqrt{\alpha(t) + \langle D_x\rangle^{-\frac{1}{1-\gamma}}}} + E(t,x,D_x), \end{aligned} \tag{3.4}$$

where

$$A, B, E \in L^1([0,T]; S^{0,\sigma}),\ D \in L^1([0,T]; S^{1-\frac{1}{1-\gamma},\sigma}),$$

$$C \in C([0,T]; S^{1,\sigma}),\ t^q C \in B((0,T]; S^{0,\sigma}).$$

System (3.4) has the structure (2.1) with

$$\begin{aligned} R(t,x,\xi) = \quad & \frac{\alpha'(t)}{\alpha(t) + \langle\xi\rangle^{-\frac{1}{1-\gamma}}} A(t,x,\xi) + \frac{b(t,x,\xi)\langle\xi\rangle^{-1}}{\sqrt{\alpha(t) + \langle\xi\rangle^{-\frac{1}{1-\gamma}}}} B(t,x,\xi) \\ & + C(t,x,\xi) + \frac{D(t,x,\xi)}{\sqrt{\alpha(t) + \langle\xi\rangle^{-\frac{1}{1-\gamma}}}} + E(t,x,\xi). \end{aligned}$$

We are going to show that there exist $\delta > 0$ and $0 < h < 1/\sigma$, σ as in (3.1) or (3.2), such that

$$\int_0^t R(s,x,\xi)ds \le \delta\langle\xi\rangle^h,\ t \in [0,T]. \tag{3.5}$$

Then, an application of Theorem 3.3 with $\varphi(\xi) = \delta\langle\xi\rangle^h$ will give Gevrey well posedness of (1.1) by usual arguments.

By Lemma 1 in [9] we have again

$$\int_0^T \frac{|\alpha'(t)|}{\alpha(t)+\langle\xi\rangle^{-\frac{1}{1-\gamma}}}dt \le c_0+\delta\log\langle\xi\rangle,\ \ c_0,\delta>0.$$

By the Levi condition in (1.5) we split

$$\frac{b(t,x,\xi)\langle\xi\rangle^{-1}B(t,x,\xi)}{\big(\alpha(t)+\langle\xi\rangle^{-\frac{1}{1-\gamma}}\big)^{1/2}} = \frac{b(t,x,\xi)\langle\xi\rangle^{-1}}{\big(\alpha(t)+\langle\xi\rangle^{-\frac{1}{1-\gamma}}\big)^{\gamma}}\cdot\frac{B(t,x,\xi)}{\big(\alpha(t)+\langle\xi\rangle^{-\frac{1}{1-\gamma}}\big)^{1/2-\gamma}},$$

and Lemma 2 in [9] gives

$$\int_0^T \frac{1}{(\alpha(t)+\langle\xi\rangle^{-\frac{1}{1-\gamma}})^{1/2-\gamma}}dt \le \begin{cases} c_1+\delta\log\langle\xi\rangle, \text{ if } \gamma\ge 1/2-1/k \\ \delta\langle\xi\rangle^{(1/2-\gamma-1/k)/(1-\gamma)}, \text{ if } \gamma<1/2-1/k\end{cases}$$

for positive constants δ, c_1.

As to the new term $D(t,x,\xi)(\alpha(t)+\langle\xi\rangle^{-\frac{1}{1-\gamma}})^{-1/2}$, by Lemma 2 we obtain

$$\langle\xi\rangle^{1-\frac{1}{1-\gamma}}\int_0^T \frac{1}{\sqrt{\alpha(t)+\langle\xi\rangle^{-\frac{1}{1-\gamma}}}}dt \le \delta\langle\xi\rangle^{(1/2-\gamma-1/k)/(1-\gamma)},\ \ \delta>0.$$

Finally, if $q=1$ we have

$$|C(t,x,\xi)|\le\tilde\varphi(t,\xi),\qquad \int_0^T\tilde\varphi(t,\xi)dt\le c_2+\delta\,\log\langle\xi\rangle,\ \ c_2,\delta>0,$$

while if $q>1$ we need to mix the two estimates $C\in C([0,T];S^{1,\sigma})$, $t^qC\in B((0,T];S^{0,\sigma})$ to find an intermediate one. Given any $\varepsilon\in(0,1)$, we introduce a separation in the phase space:

- for $t^{1-\varepsilon}\langle\xi\rangle^{(1-\varepsilon)/q}\le 1$ we use $C\in C([0,T];S^{1,\sigma})$,
- for $t^{1-\varepsilon}\langle\xi\rangle^{(1-\varepsilon)/q}\ge 1$ we use $t^qC\in B((0,T];S^{0,\sigma})$,

and we obtain

$$|\partial_\xi^\gamma\partial_x^\beta C(t,x,\xi)|\le\frac{1}{t^{1-\varepsilon}}\langle\xi\rangle^{1-(1-\varepsilon)/q-|\gamma|},\ \ \gamma,\beta\in\boldsymbol{Z}_+^n,$$

that is the global estimate

$$t^{1-\varepsilon}C\in B((0,T];S^{1-(1-\varepsilon)/q,\sigma}),$$

wich under the integral sign becomes

$$\int_0^T |C(t,x,\xi)|dt \le \delta\langle\xi\rangle^{1-(1-\varepsilon)/q}, \delta > 0.$$

Thus:

– if $q > 1$ and $\gamma \ge 1/2 - 1/k$, then (3.5) holds true with $h = 1 - \dfrac{1-\varepsilon}{q}$; it is possible to choose an $\varepsilon \in (0,1)$ such that $h < 1/\sigma$, thanks to (3.1);

– if $q = 1$ and $\gamma < 1/2 - 1/k$, then (3.5) is satisfied with $h = \dfrac{1/2 - (\gamma + 1/k)}{1-\gamma}$; clearly $h < 1/\sigma$, since we are supposing (3.2);

– if $q > 1$ and $\gamma < 1/2 - 1/k$, then we get (3.5) with

$$h = \max\left\{1 - \frac{1-\varepsilon}{q}, \frac{1/2 - (\gamma + 1/k)}{1-\gamma}\right\}, \tag{3.6}$$

and condition (3.2) provides $h < 1/\sigma$. To make precise estimate (3.6), we compute the crucial exponent

$$\gamma^* = 1 - \frac{q}{1-\varepsilon}\left(\frac{1}{2} + \frac{1}{k}\right),$$

and notice that $\gamma^* \le 0$ if $q \ge 2k/(k+2)$. This leads us to consider three cases:

– if $1 \le q < 2k/(k+2)$ and $0 \le \gamma < 1 - q(1/2 + 1/k)$, then $h = [1/2 - (\gamma + 1/k)](1-\gamma)$, and we get $\sigma_0 = (1-\gamma)/[1/2 - (\gamma + 1/k)]$;

– if $1 \le q < 2k/(k+2)$ and $1 - q(1/2 + 1/k) \le \gamma \le 1/2$, then $h = 1 - (1-\varepsilon)/q$ and $\sigma_0 = q/(q-1)$;

– if $q \ge 2k/(k+2)$, again $h = 1 - (1-\varepsilon)/q$, and $\sigma_0 = q/(q-1)$.

The proof of Theorem (3.1) is complete via Theorem 3.3.

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