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The Extremal Ranks of $A_1 - B_1XC_1$ subject to a Pair of Matrix Equations.

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ABSTRACT - For a given linear matrix expression $A_1 - B_1XC_1$, where X is a variable matrix, this paper gives two formulas for the maximal and minimal ranks of $A_1 - B_1XC_1$ subject to a pair of consistent matrix equations $B_2XC_2 = A_2$ and $B_3XC_3 = A_3$. As a consequence, we give necessary and sufficient conditions for the triple matrix equations $B_1XC_1 = A_1$, $B_2XC_2 = A_2$ and $B_3XC_3 = A_3$ to have a common solution.

1. Introduction

Let

$$p(X) = A - BXC \tag{1.1}$$

be a linear matrix expression over an arbitrary field $\mathbb{F}$, where $A \in \mathbb{F}^{m \times n}$, $B \in \mathbb{F}^{m \times p}$ and $C \in \mathbb{F}^{q \times n}$ are given, and $X \in \mathbb{F}^{p \times q}$ is a variable matrix. In this case, the matrix $p(X)$ varies with respect to X. Of interest for us will be properties of $p(X)$ when X varies. One of the basic properties on $p(X)$ is the maximal and minimal possible ranks of $p(X)$ when X running over $\mathbb{F}^{p \times q}$, or a subset of $\mathbb{F}^{p \times q}$. Because the rank of a matrix is an integer between zero and the minimum of the row and column numbers of the matrix, the maximum and minimum of the rank of (1.1) with respect to X must exist. Theoretically, any matrix expression has the maximal and minimal ranks with respect to its variable entries. The extremal ranks of matrix expres-

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sions have close links with many problems in matrix theory and applications. For example,

(a) the matrix equation $BXC = A$ is consistent if and only if

$$\min_{X \in \mathbb{F}^{p\times q}} \operatorname{rank}(A - BXC) = 0;$$

(b) the matrix equation $B_1X_1C_1 + B_2X_2C_2 = A$ is consistent if and only if

$$\min_{X_1 \in \mathbb{F}^{p_1\times q_1},\, X_2 \in \mathbb{F}^{p_2\times q_2}} \operatorname{rank}(A - B_1X_1C_1 - B_2X_2C_2) = 0;$$

(c) the two consistent matrix equations $B_1X_1C_1 = A_1$ and $B_2X_2C_2 = A_2$, where X_1 and X_2 have the same size, have a common solution if and only if

$$\min_{\substack{B_1X_1C_1 = A_1\\ B_2X_2C_2 = A_2}} \operatorname{rank}(X_1 - X_2) = 0,$$

or equivalently, $\min_{B_2XC_2 = A_2} \operatorname{rank}(A_1 - B_1XC_1) = 0$; all solutions of $B_2XC_2 = A_2$ are solutions of $B_1XC_1 = A_1$ if and only if $\max_{B_2XC_2 = A_2} \operatorname{rank}(A_1 - B_1XC_1) = 0$;

(d) there is a matrix $X \in \mathbb{F}^{p\times q}$ such that the square block matrix $\begin{bmatrix} A & B \\ C & X \end{bmatrix}$ of order n is nonsingular if and only if $\max_{X \in \mathbb{F}^{p\times q}} \operatorname{rank}\begin{bmatrix} A & B \\ C & X \end{bmatrix} = n$; $\begin{bmatrix} A & B \\ C & X \end{bmatrix}$ is nonsingular for any $X \in \mathbb{F}^{p\times q}$ if and only if $\min_{X \in \mathbb{F}^{p\times q}} \operatorname{rank}\begin{bmatrix} A & B \\ C & X \end{bmatrix} = n$.

In general, for any two matrix expressions $p(X_1, \ldots, X_s)$ and $q(Y_1, \ldots, Y_t)$ of the same size, there are $X_1, \ldots, X_s$ and $Y_1, \ldots, Y_t$ such that $p(X_1, \ldots, X_s) = q(Y_1, \ldots, Y_t)$ if and only if

$$\min_{X_1, \ldots, X_s, Y_1, \ldots, Y_t} \operatorname{rank}[\, p(X_1, \ldots, X_s) - q(Y_1, \ldots, Y_t)\,] = 0;$$

$p(X_1, \ldots, X_s)$ and $q(Y_1, \ldots, Y_t)$ are identical if and only if

$$\max_{X_1, \ldots, X_s, Y_1, \ldots, Y_t} \operatorname{rank}[\, p(X_1, \ldots, X_s) - q(Y_1, \ldots, Y_t)\,] = 0.$$

Moreover, the rank invariance and the range invariance of matrix expressions with respect to variable matrices can also be derived through the matrix rank method. The essential part in solving these problems is to give explicit formulas for the extremal ranks of the matrix expressions with respect to their variable matrices. This topic was studied in the late 1980s in the investigation of matrix completion problems; see, e.g. [1, 2, 3, 4, 13]. Some recent work on extremal ranks of matrix expressions and their applications can be found in [8, 9, 10, 11, 12].

Throughout this paper, the symbols A^T, $r(A)$ and $\mathscr{R}(A)$ stand for the transpose, the rank and the range (column space) of a matrix A, respectively. A matrix $X \in \mathbb{F}^{n\times m}$ is called a generalized inverse of a matrix $A \in \mathbb{F}^{m\times n}$, denoted by A^-, if it satisfies $AXA = A$. The symbols E_A and F_A stand for the two oblique projectors $E_A = I_m - AA^-$ and $F_A = I_n - A^-A$.

It was shown in [10, 12] that $p(X)$ in (1.1) satisfies the following two rank identities

$$(1.2)\quad r(A-BXC) = r[A,\, B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + r[\,E_{T_1}(X+TM^-S)F_{S_1}],$$

$$(1.3)\quad r(A-BXC) = r[A,\, B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + r(E_QAF_P - E_QBXCF_P),$$

where $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$, $T = [\,0,\, I_p\,]$, $S = \begin{bmatrix} 0 \\ I_q \end{bmatrix}$, $T_1 = TF_M$, $S_1 = E_MS$, $P = E_BA$ and $Q = AF_C$. It is easy to verify that the two matrix equations

$$E_{T_1}(X + TM^-S)F_{S_1} = 0 \text{ and } E_QBXCF_P = E_QAF_P$$

are solvable for X. Based on (1.2) and (1.3), it was shown in [10, 12] that

$$(1.4)\qquad \max_{X\in\mathbb{F}^{p\times q}} r(A - BXC) = \min\left\{ r[A,\, B\,],\quad r\begin{bmatrix} A \\ C \end{bmatrix} \right\},$$

$$(1.5)\qquad \min_{X\in\mathbb{F}^{p\times q}} r(A - BXC) = r[A,\, B\,] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}.$$

Moreover, Tian [11] showed that if the matrix equation $B_2XC_2 = A_2$ is consistent, then

$$(1.6)\quad \max_{B_2XC_2=A_2} r(A_1 - B_1XC_1)$$

$$= \min\left\{ r\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_2 & B_2 \\ C_1 & C_2 & 0 \end{bmatrix} - r(B_2) - r(C_2),\quad r\begin{bmatrix} A_1 \\ C_1 \end{bmatrix},\quad r[\,A_1,\, B_1\,] \right\},$$

$$\min_{B_2XC_2=A_2} r(A_1 - B_1XC_1) = r[\,A_1,\, B_1\,] + r\begin{bmatrix} A_1 \\ C_1 \end{bmatrix} - r\begin{bmatrix} A_1 & B_1 & 0 \\ C_1 & 0 & C_2 \end{bmatrix} - r\begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \\ 0 & B_2 \end{bmatrix} + r\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_2 & B_2 \\ C_1 & C_2 & 0 \end{bmatrix}. \tag{1.7}$$

A variety of consequences and applications of these two rank formulas are given in [11], for example, necessary and sufficient conditions for $B_1XC_1 = A_1$ and $B_2XC_2 = A_2$ to have a common solution, necessary and sufficient conditions for all solutions of $B_2XC_2 = A_2$ to be solutions of $B_1XC_1 = A_1$, and the extremal ranks of the generalized Schur complement $D - CA^-B$ with respect to A^-. As extensions of (1.4)–(1.7), we give in this paper two formulas for the maximal and minimal ranks of $A_1 - B_1XC_1$ subject to a pair of consistent matrix equations $B_2XC_2 = A_2$ and $B_3XC_3 = A_3$.

In order to simplify various matrices involving generalized inverses, we need the following rank equalities for partitioned matrices due to Marsaglia and Styan [5].

Lemma 1.1. *Let $A \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{m\times p}$ and $C \in \mathbb{F}^{q\times n}$. Then*

$$r[\,A,\, B\,] = r(A) + r(E_AB) = r(B) + r(E_BA), \tag{1.8}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{1.9}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_BAF_C), \tag{1.10}$$

where the ranks do not depend upon the particular choice of generalized inverses in E_A, F_A, E_B and F_C.

2. The extremal ranks of $A - B_1XC_1$ subject to $B_2XC_2 = A_2$ and $B_3XC_3 = A_3$

A direct motivation for finding the extremal ranks of $A - B_1XC_1$ subject to a pair of consistent matrix equations $B_2XC_2 = A_2$ and $B_3XC_3 = A_3$ arises from investigating common solutions of a triple matrix equations

$B_1XC_1 = A_1$, $B_2XC_2 = A_2$ and $B_3XC_3 = A_3$. To do so, it is necessary to know the general common solution to the pair of matrix equations $B_2XC_2 = A_2$ and $B_3XC_3 = A_3$.

LEMMA 2.1. *Let $A_i \in \mathbb{F}^{m_i\times n_i}$, $B_i \in \mathbb{F}^{m_i\times p}$ and $C_i \in \mathbb{F}^{q\times n_i}$ be given for $i = 2, 3$, and suppose that each of the two matrix equations $B_2XC_2 = A_2$ and $B_3XC_3 = A_3$ is consistent, i.e., $\mathscr{R}(A_i) \subseteq \mathscr{R}(B_i)$ and $\mathscr{R}(A_i^T) \subseteq \mathscr{R}(C_i^T)$ for $i = 2, 3$. Then:*

(a) [6, 7] *The pair of matrix equations have a common solution if and only if*

$$r\begin{bmatrix} A_2 & 0 & B_2 \\ 0 & -A_3 & B_3 \\ C_2 & C_3 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} + r[\, C_2,\, C_3 \,]. \tag{2.1}$$

(b) [8] *Under* (2.1), *the general common solution of the pair of equations can be written as*

$$X = X_0 + F_B V_1 + V_2 E_C + F_{B_2} V_3 E_{C_3} + F_{B_3} V_4 E_{C_2}, \tag{2.2}$$

where X_0 is a special common solution to the pair of equations, $B = \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}$, $C = [\, C_2,\, C_3 \,]$, and $V_1, \ldots, V_4 \in \mathbb{F}^{p\times q}$ are arbitrary.

Substituting (2.2) into $A_1 - B_1XC_1$ gives

$$\begin{aligned} A_1 - B_1XC_1 = A_1 - B_1X_0C_1 - B_1F_BV_1C_1 - B_1V_2E_CC_1 \\ -\; B_1F_{B_2}V_3E_{C_3}C_1 - B_1F_{B_3}V_4E_{C_2}C_1, \end{aligned} \tag{2.3}$$

which is a linear matrix expression with four variable matrices $V_1, \ldots, V_4$. To find the extremal ranks of (2.3) with respect to the four variable matrices $V_1, \ldots, V_4$, we need the following result.

LEMMA 2.2. *Let*

$$p(X_1, X_2, X_3, X_4) = A - B_1X_1C_1 - B_2X_2C_2 - B_3X_3C_3 - B_4X_4C_4 \tag{2.4}$$

be a linear matrix expression, where $A \in \mathbb{F}^{m\times n}$, $B_i \in \mathbb{F}^{m\times p_i}$ and $C_i \in \mathbb{F}^{q_i\times n}$ are given, and $X_i \in \mathbb{F}^{p_i\times q_i}$ are variable matrices for $i = 1, \ldots, 4$. Also suppose

$$\mathscr{R}(B_i) \subseteq \mathscr{R}(B_2) \ \ and \ \ \mathscr{R}(C_j^T) \subseteq \mathscr{R}(C_1^T), \quad i = 1, 3, 4, \ \ j = 2, 3, 4. \tag{2.5}$$

Then the maximal and minimal ranks of $p(X_1, X_2, X_3, X_4)$ are given by

$$\max_{X_1,\ldots,X_4} r[\,p(X_1,\,X_2,\,X_3,\,X_4)\,]=\min\left\{ r[A,B_2\,],\; r\begin{bmatrix} A \\ C_1 \end{bmatrix},\; r\begin{bmatrix} A & B_1 \\ C_2 & 0 \\ C_3 & 0 \\ C_4 & 0 \end{bmatrix},\right. \tag{2.6}$$

$$\left. r\begin{bmatrix} A & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \end{bmatrix},\; r\begin{bmatrix} A & B_1 & B_3 \\ C_2 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix},\; r\begin{bmatrix} A & B_1 & B_4 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \end{bmatrix} \right\},$$

$$\min_{X_1,\ldots,X_4} r[\,p(X_1,\,X_2,\;X_3,\,X_4)\,] = r\begin{bmatrix} A & B_1 \\ C_2 & 0 \\ C_3 & 0 \\ C_4 & 0 \end{bmatrix} + r\begin{bmatrix} A & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \end{bmatrix} \tag{2.7}$$

$$+ r\begin{bmatrix} A \\ C_1 \end{bmatrix} + r[A,\,B_2\,] - r\begin{bmatrix} A & B_1 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_2 \\ C_2 & 0 \end{bmatrix} + \max\{s_1,\; s_2\},$$

where

$$s_1 = r\begin{bmatrix} A & B_1 & B_3 \\ C_2 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \\ C_4 & 0 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_3 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix},$$

$$s_2 = r\begin{bmatrix} A & B_1 & B_4 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \\ C_3 & 0 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_4 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix}.$$

PROOF. The two rank equalities (2.6) and (2.7) are derived from the following two rank formulas in Tian [11]

$$\max_{X_1,X_2} r(A - B_1X_1C_1 - B_2X_2C_2\,) \tag{2.8}$$

$$= \min\left\{ r[A,\,B_1,\,B_2\,],\; r\begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix},\; r\begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix},\; r\begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} \right\},$$

$$\min_{X_1,X_2} r(A - B_1X_1C_1 - B_2X_2C_2\,) = r\begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} + r[A,B_1,B_2] + \max\{t_1,t_2\}, \tag{2.9}$$

where

$$t_1 = r\begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_2 \\ C_2 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix},$$

$$t_2 = r\begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}.$$

If $\mathscr{R}(B_1) \subseteq \mathscr{R}(B_2)$ and $\mathscr{R}(C_2^T) \subseteq \mathscr{R}(C_1^T)$, then (2.8) and (2.9) reduce to

$$\text{(2.10)} \quad \max_{X_1, X_2} r(A - B_1X_1C_1 - B_2X_2C_2) = \min\left\{ r[A, B_2], r\begin{bmatrix} A \\ C_1 \end{bmatrix}, r\begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} \right\},$$

$$\text{(2.11)} \quad \min_{X_1, X_2} r(A - B_1X_1C_1 - B_2X_2C_2) = r[A,\, B_2] + r\begin{bmatrix} A \\ C_1 \end{bmatrix} + r\begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_2 \\ C_2 & 0 \end{bmatrix}.$$

Recall that elementary matrix operations for a matrix do not change the rank of the matrix. Under (2.5), applying (2.11) to the two variable matrices X_1 and X_2 in (2.4) and simplifying by elementary matrix operations gives

$$\begin{aligned} \text{(2.12)} \quad & \min_{X_1, X_2} r[\, p(X_1,\, X_2,\, X_3,\, X_4)\,] = r[A - B_3X_3C_3 - B_4X_4C_4,\, B_2\,] \\ & + r\begin{bmatrix} A - B_3X_3C_3 - B_4X_4C_4 \\ C_1 \end{bmatrix} + r\begin{bmatrix} A - B_3X_3C_3 - B_4X_4C_4 & B_1 \\ C_2 & 0 \end{bmatrix} \\ & - r\begin{bmatrix} A - B_3X_3C_3 - B_4X_4C_4 & B_1 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A - B_3X_3C_3 - B_4X_4C_4 & B_2 \\ C_2 & 0 \end{bmatrix} \\ & = r[A,\, B_2\,] + r\begin{bmatrix} A \\ C_1 \end{bmatrix} + r\begin{bmatrix} A - B_3X_3C_3 - B_4X_4C_4 & B_1 \\ C_2 & 0 \end{bmatrix} \\ & - r\begin{bmatrix} A & B_1 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_2 \\ C_2 & 0 \end{bmatrix}. \end{aligned}$$

Notice that

$$\begin{bmatrix} A - B_3X_3C_3 - B_4X_4C_4 & B_1 \\ C_2 & 0 \end{bmatrix} = \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - \begin{bmatrix} B_3 \\ 0 \end{bmatrix} X_3[\,C_3,\; 0\,] - \begin{bmatrix} B_4 \\ 0 \end{bmatrix} X_4[C_4, 0\,].$$

Applying (2.9) (with $X_3, X_4, \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix}, \begin{bmatrix} B_3 \\ 0 \end{bmatrix}, \begin{bmatrix} B_4 \\ 0 \end{bmatrix}, [\, C_3,\ 0\,]$ and $[\, C_4,\ 0\,]$ in place of X_1, X_2, A, B_1, B_2, C_1 and C_2, respectively) to the expression and substituting the corresponding result into (2.12) yields (2.7). Formula (2.6) is derived by applying (2.8) and (2.10) to $p(X_1, X_2, X_3, X_4)$. The details are omitted. □

For convenience of representation, rewrite (2.3) as

$$(2.13) \qquad A_1 - B_1XC_1 = A - G_1V_1H_1 - G_2V_2H_2 - G_3V_3H_3 - G_4V_4H_4,$$

where

$$A = A_1 - B_1X_0C_1, \quad G_1 = B_1F_B, \quad G_2 = B_1, \quad G_3 = B_1F_{B_2}, \quad G_4 = B_1F_{B_3},$$

$$H_1 = C_1, \quad H_2 = E_CC_1, \quad H_3 = E_{C_3}C_1, \quad H_4 = E_{C_2}C_1.$$

It is easy to verify that the above matrices satisfy the conditions

$$(2.14)\ \mathscr{R}(G_1) \subseteq \mathscr{R}(G_i) \subseteq \mathscr{R}(G_2), \text{ and } \mathscr{R}(H_2^T) \subseteq \mathscr{R}(H_i^T) \subseteq \mathscr{R}(H_1^T), \quad i = 3,\ 4,$$

where the range inclusions do not depend upon the particular choice of generalized inverses in G_i and H_i. In this case, applying (2.6) and (2.7) to (2.13) yields the main results of this section.

THEOREM 2.3. *Let $A_i \in \mathbb{F}^{m_i\times n_i}$, $B_i \in \mathbb{F}^{m_i\times p}$ and $C_i \in \mathbb{F}^{q\times n_i}$ be given for $i = 1, 2, 3$, and suppose that the pair of matrix equations $B_2XC_2 = A_2$ and $B_3XC_3 = A_3$ have a common solution. Then*

$$(2.15) \quad \max_{\substack{B_2XC_2=A_2\\ B_3XC_3=A_3}} r(A_1 - B_1XC_1) = \min\left\{ r[\, A_1,\ B_1\,],\ r\begin{bmatrix} A_1 \\ C_1 \end{bmatrix},\ u_1,\ u_2,\ u_3,\ u_4 \right\},$$

where

$$u_1 = r\begin{bmatrix} A_1 & 0 & 0 & B_1 \\ 0 & -A_2 & 0 & B_2 \\ 0 & 0 & -A_3 & B_3 \\ C_1 & C_2 & 0 & 0 \\ C_1 & 0 & C_3 & 0 \end{bmatrix} - r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} - r(C_2) - r(C_3),$$

$$u_2 = r\begin{bmatrix} A_1 & 0 & 0 & B_1 & B_1 \\ 0 & -A_2 & 0 & B_2 & 0 \\ 0 & 0 & -A_3 & 0 & B_3 \\ C_1 & C_2 & C_3 & 0 & 0 \end{bmatrix} - r[\, C_2,\ C_3\,] - r(B_2) - r(B_3),$$

$$u_3 = r\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_2 & B_2 \\ C_1 & C_2 & 0 \end{bmatrix} - r(B_2) - r(C_2),$$

$$u_4 = r\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_3 & B_3 \\ C_1 & C_3 & 0 \end{bmatrix} - r(B_3) - r(C_3).$$

PROOF. Under (2.14), by (2.6) we first find that

$$\begin{aligned}
\text{(2.16)}\qquad & \max_{\substack{B_2XC_2=A_2\\ B_3XC_3=A_3}} r(A_1 - B_1XC_1) \\
&= \max_{V_1,\dots,V_4} r(A - G_1V_1H_1 - G_2V_2H_2 - G_3V_3H_3 - G_4V_4H_4) \\
&= \min\left\{ r[A,\, G_2],\ r\begin{bmatrix} A \\ H_1 \end{bmatrix},\ r\begin{bmatrix} A & G_1 \\ H_3 & 0 \\ H_4 & 0 \end{bmatrix},\ r\begin{bmatrix} A & G_3 & G_4 \\ H_2 & 0 & 0 \end{bmatrix}, \right. \\
&\qquad\qquad \left. r\begin{bmatrix} A & G_3 \\ H_4 & 0 \end{bmatrix},\ r\begin{bmatrix} A & G_4 \\ H_3 & 0 \end{bmatrix} \right\}.
\end{aligned}$$

Simplifying the ranks of the block matrices in (2.16) by (1.8), (1.9) and (1.10), the conditions $B_2X_0C_2 = A_2$ and $B_3X_0C_3 = A_3$, and elementary matrix operations leads to

$$r[A,\, G_2] = r[A_1 - B_1X_0C_1,\, B_1] = r[A_1,\, B_1],$$

$$r\begin{bmatrix} A \\ H_1 \end{bmatrix} = r\begin{bmatrix} A_1 - B_1X_0C_1 \\ C_1 \end{bmatrix} = r\begin{bmatrix} A_1 \\ C_1 \end{bmatrix},$$

$$\begin{aligned}
r\begin{bmatrix} A & G_1 \\ H_3 & 0 \\ H_4 & 0 \end{bmatrix} &= r\begin{bmatrix} A_1 - B_1X_0C_1 & B_1F_B \\ E_{C_3}C_1 & 0 \\ E_{C_2}C_1 & 0 \end{bmatrix} \\
&= r\begin{bmatrix} A_1 - B_1X_0C_1 & B_1 & 0 & 0 \\ C_1 & 0 & C_3 & 0 \\ C_1 & 0 & 0 & C_2 \\ 0 & B_2 & 0 & 0 \\ 0 & B_3 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} - r(C_2) - r(C_3)
\end{aligned}$$

$$
= r\begin{bmatrix} A_1 & B_1 & 0 & 0 \\ C_1 & 0 & C_3 & 0 \\ C_1 & 0 & 0 & C_2 \\ 0 & B_2 & 0 & -A_2 \\ 0 & B_3 & -A_3 & 0 \end{bmatrix} - r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} - r(C_2) - r(C_3)
$$

$$
= r\begin{bmatrix} A_1 & 0 & 0 & B_1 \\ 0 & -A_2 & 0 & B_2 \\ 0 & 0 & -A_3 & B_3 \\ C_1 & C_2 & 0 & 0 \\ C_1 & 0 & C_3 & 0 \end{bmatrix} - r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} - r(C_2) - r(C_3).
$$

Similarly, we can find that

$$
\begin{aligned}
r\begin{bmatrix} A & G_3 & G_4 \\ H_2 & 0 & 0 \end{bmatrix} &= r\begin{bmatrix} A_1 - B_1X_0C_1 & B_1F_{B_2} & B_1F_{B_3} \\ F_CC_1 & 0 & 0 \end{bmatrix} \\
&= r\begin{bmatrix} A_1 & 0 & 0 & B_1 & B_1 \\ 0 & -A_2 & 0 & B_2 & 0 \\ 0 & 0 & -A_3 & 0 & B_3 \\ C_1 & C_2 & C_3 & 0 & 0 \end{bmatrix} - r[\, C_2,\, C_3 \,] - r(B_2) - r(B_3),
\end{aligned}
$$

$$
\begin{aligned}
r\begin{bmatrix} A & G_3 \\ H_4 & 0 \end{bmatrix} &= r\begin{bmatrix} A_1 - B_1X_0C_1 & B_1F_{B_2} \\ E_{C_2}C_1 & 0 \end{bmatrix} \\
&= r\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_2 & B_2 \\ C_1 & C_2 & 0 \end{bmatrix} - r(B_2) - r(C_2),
\end{aligned}
$$

$$
\begin{aligned}
r\begin{bmatrix} A & G_4 \\ H_3 & 0 \end{bmatrix} &= r\begin{bmatrix} A_1 - B_1X_0C_1 & B_1F_{B_3} \\ E_{C_3}C_1 & 0 \end{bmatrix} \\
&= r\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_3 & B_3 \\ C_1 & C_3 & 0 \end{bmatrix} - r(B_3) - r(C_3).
\end{aligned}
$$

Substituting these rank equalities into (2.16) yields (2.15). □

THEOREM 2.4. *Let* $A_i \in \mathbb{F}^{m_i \times n_i}$, $B_i \in \mathbb{F}^{m_i \times p}$ *and* $C_i \in \mathbb{F}^{q \times n_i}$ *be given for* $i = 1, 2, 3$, *and suppose that the pair of matrix equations* $B_2XC_2 = A_2$ *and*

$B_3XC_3 = A_3$ *have a common solution. Then*

$$(2.17)\quad \min_{\substack{B_2XC_2=A_2\\ B_3XC_3=A_3}} r(A_1 - B_1XC_1)$$

$$= r\begin{bmatrix} A_1 & 0 & 0 & B_1 \\ 0 & -A_2 & 0 & B_2 \\ 0 & 0 & -A_3 & B_3 \\ C_1 & C_2 & 0 & 0 \\ C_1 & 0 & C_3 & 0 \end{bmatrix} + r\begin{bmatrix} A_1 & 0 & 0 & B_1 & B_1 \\ 0 & -A_2 & 0 & B_2 & 0 \\ 0 & 0 & -A_3 & 0 & B_3 \\ C_1 & C_2 & C_3 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \\ 0 & B_2 \\ 0 & B_3 \end{bmatrix}$$

$$- \ r\begin{bmatrix} A_1 & B_1 & 0 & 0 \\ C_1 & 0 & C_2 & C_3 \end{bmatrix} + r\begin{bmatrix} A_1 \\ C_1 \end{bmatrix} + r[A_1, B_1] + \max\{v_1, v_2\},$$

where

$$v_1 = r\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_2 & B_2 \\ C_1 & C_2 & 0 \end{bmatrix} - r\begin{bmatrix} A_1 & 0 & B_1 & B_1 \\ 0 & -A_2 & B_2 & 0 \\ C_1 & C_2 & 0 & 0 \\ 0 & 0 & 0 & B_3 \end{bmatrix} - r\begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & -A_2 & B_2 & 0 \\ C_1 & C_2 & 0 & 0 \\ C_1 & 0 & 0 & C_3 \end{bmatrix},$$

$$v_2 = r\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_3 & B_3 \\ C_1 & C_3 & 0 \end{bmatrix} - r\begin{bmatrix} A_1 & 0 & B_1 & B_1 \\ 0 & -A_3 & B_3 & 0 \\ C_1 & C_3 & 0 & 0 \\ 0 & 0 & 0 & B_2 \end{bmatrix} - r\begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & -A_3 & B_3 & 0 \\ C_1 & C_3 & 0 & 0 \\ C_1 & 0 & 0 & C_2 \end{bmatrix}.$$

PROOF. Under (2.14), applying (2.7) to (2.13) yields

$$(2.18)\quad \min_{\substack{B_2XC_2=A_2\\ B_3XC_3=A_3}} r(A_1 - B_1XC_1)$$

$$= \min_{V_1,\ldots,V_4} r(A - G_1V_1H_1 - G_2V_2H_2 - G_3V_3H_3 - G_4V_4H_4)$$

$$= r\begin{bmatrix} A & G_1 \\ H_3 & 0 \\ H_4 & 0 \end{bmatrix} + r\begin{bmatrix} A & G_3 & G_4 \\ H_2 & 0 & 0 \end{bmatrix} + r[A, G_2] + r\begin{bmatrix} A \\ H_1 \end{bmatrix}$$

$$- \ r\begin{bmatrix} A & G_1 \\ H_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & G_2 \\ H_2 & 0 \end{bmatrix} + \max\{v_3, v_4\},$$

where

$$v_3 = r\begin{bmatrix} A & G_3 \\ H_4 & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 & G_4 \\ H_4 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 \\ H_3 & 0 \\ H_4 & 0 \end{bmatrix},$$

$$v_4 = r\begin{bmatrix} A & G_4 \\ H_3 & 0 \end{bmatrix} - r\begin{bmatrix} A & G_3 & G_4 \\ H_3 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & G_4 \\ H_3 & 0 \\ H_4 & 0 \end{bmatrix}.$$

Simplifying the ranks of the block matrices in (2.18) by (1.8), (1.9), (1.10), the conditions $B_2X_0C_2 = A_2$ and $B_3X_0C_3 = A_3$, and elementary matrix operations leads to (2.17). The details are omitted. □

COROLLARY 2.5. *Let $A_i \in \mathbb{F}^{m_i \times n_i}$, $B_i \in \mathbb{F}^{m_i \times p}$ and $C_i \in \mathbb{F}^{q \times n_i}$ be given for $i = 1, 2, 3$, and suppose that each of the three matrix equations $B_1XC_1 = A_1$, $B_2XC_2 = A_2$ and $B_3XC_3 = A_3$ is consistent and that any two of the three equations have a common solution. Then*

$$(2.19)\quad \min_{\substack{B_2XC_2=A_2 \\ B_3XC_3=A_3}} r(A_1 - B_1XC_1)$$

$$= r\begin{bmatrix} A_1 & 0 & 0 & B_1 \\ 0 & -A_2 & 0 & B_2 \\ 0 & 0 & -A_3 & B_3 \\ C_1 & C_2 & 0 & 0 \\ C_1 & 0 & C_3 & 0 \end{bmatrix} + r\begin{bmatrix} A_1 & 0 & 0 & B_1 & B_1 \\ 0 & -A_2 & 0 & B_2 & 0 \\ 0 & 0 & -A_3 & 0 & B_3 \\ C_1 & C_2 & C_3 & 0 & 0 \end{bmatrix}$$

$$- \; r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} - r[\,C_1,\, C_2,\, C_3\,] - r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix} - r\begin{bmatrix} C_1 & C_2 & 0 \\ C_1 & 0 & C_3 \end{bmatrix}.$$

PROOF. Since any two of the three matrix equations have a common solution, it turns out from Lemma 2.1 that

$$\mathscr{R}(A_i) \subseteq \mathscr{R}(B_i), \quad \mathscr{R}(A_i^T) \subseteq \mathscr{R}(C_i^T), \quad i = 1, 2, 3,$$

$$r\begin{bmatrix} A_i & 0 & B_i \\ 0 & -A_j & B_j \\ C_i & C_j & 0 \end{bmatrix} = r\begin{bmatrix} B_i \\ B_j \end{bmatrix} + r[\,C_i,\, C_j\,], \quad 1 \leqslant i < j \leqslant 3.$$

In this case, the ranks of the block matrices in (2.17) reduce to

$$r\begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \\ 0 & B_2 \\ 0 & B_3 \end{bmatrix} = r(C_1) + r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, \; r\begin{bmatrix} A_1 & B_1 & 0 & 0 \\ C_1 & 0 & C_2 & C_3 \end{bmatrix} = r(B_1) + r[\, C_1,\, C_1,\, C_3\,],$$

$$r\begin{bmatrix} A_1 \\ C_1 \end{bmatrix} = r(C_1),\, r[A_1,\, B_1] = r(B_1),\, v_1 = v_2 = -r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix} - r\begin{bmatrix} C_1 & C_2 & 0 \\ C_1 & 0 & C_3 \end{bmatrix}.$$

Substituting these results into (2.17) leads to (2.19). □

Under the assumption of Corollary 2.5, it can be derived from (2.19) that

$$\begin{aligned} \min_{\substack{B_2XC_2=A_2 \\ B_3XC_3=A_3}} r(A_1 - B_1XC_1) &= \min_{\substack{B_1XC_1=A_1 \\ B_3XC_3=A_3}} r(A_2 - B_2XC_2) \\ &= \min_{\substack{B_1XC_1=A_1 \\ B_2XC_2=A_2}} r(A_3 - B_3XC_3). \end{aligned}$$

A direct consequence of (2.19) is given below.

Corollary 2.6. [8] *Let $A_i \in \mathbb{F}^{m_i \times n_i}$, $B_i \in \mathbb{F}^{m_i \times p}$ and $C_i \in \mathbb{F}^{q \times n_i}$ be given for $i = 1, 2, 3$. Then the triple matrix equations $B_1XC_1 = A_1$, $B_2XC_2 = A_2$ and $B_3XC_3 = A_3$ have a common solution if and only if any pair of the triple equations have a common solution, meanwhile the following two rank equalities hold*

$$r\begin{bmatrix} A_1 & 0 & 0 & B_1 & B_1 \\ 0 & -A_2 & 0 & B_2 & 0 \\ 0 & 0 & -A_3 & 0 & B_3 \\ C_1 & C_2 & C_3 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 & B_1 \\ B_2 & 0 \\ 0 & B_3 \end{bmatrix} + r[\, C_1,\, C_2,\, C_3\,],$$

$$r\begin{bmatrix} A_1 & 0 & 0 & B_1 \\ 0 & -A_2 & 0 & B_2 \\ 0 & 0 & -A_3 & B_3 \\ C_1 & C_2 & 0 & 0 \\ C_1 & 0 & C_3 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} + r\begin{bmatrix} C_1 & C_2 & 0 \\ C_1 & 0 & C_3 \end{bmatrix}.$$

Letting $A_1 = 0$, $B_1 = I_p$ and $C_1 = I_q$ in (2.15) and (2.17), we obtain the following result.

Corollary 2.7. *Suppose that the pair of matrix equations* $B_2XC_2 = A_2$ *and* $B_3XC_3 = A_3$ *have a common solution, where* $X \in \mathbb{F}^{p\times q}$. *Then:*

(a) *The maximal rank of the common solution to the pair of equations is given by*

$$\max_{\substack{B_2XC_2=A_2\\ B_3XC_3=A_3}} r(X) = \min\{\, p, \quad q, \quad w_1, \quad w_2, \quad w_3, \quad w_4 \,\},$$

where

$$w_1 = r\begin{bmatrix} A_2 & 0 \\ 0 & A_3 \\ C_2 & C_3 \end{bmatrix} - r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} - r(C_2) - r(C_3) + p + q,$$

$$w_2 = r\begin{bmatrix} A_2 & 0 & B_2 \\ 0 & A_3 & B_3 \end{bmatrix} - r[\, C_2, \, C_3 \,] - r(B_2) - r(B_3) + p + q,$$

$$w_3 = r(A_2) - r(B_2) - r(C_2) + p + q,$$

$$w_4 = r(A_3) - r(B_3) - r(C_3) + p + q.$$

(b) *The minimal rank of the common solution to this pair of equations is given by*

$$\min_{\substack{B_2XC_2=A_2\\ B_3XC_3=A_3}} r(X) = r\begin{bmatrix} A_2 & 0 \\ 0 & A_3 \\ C_2 & C_3 \end{bmatrix} + r\begin{bmatrix} A_2 & 0 & B_2 \\ 0 & A_3 & B_3 \end{bmatrix} + \max\{\, w_5, \, w_6 \,\},$$

where

$$w_5 = r(A_2) - r\begin{bmatrix} A_2 & B_2 \\ 0 & B_3 \end{bmatrix} - r\begin{bmatrix} A_2 & 0 \\ C_2 & C_3 \end{bmatrix},$$

$$w_6 = r(A_3) - r\begin{bmatrix} B_2 & 0 \\ B_3 & A_3 \end{bmatrix} - r\begin{bmatrix} C_2 & C_3 \\ 0 & A_3 \end{bmatrix}.$$

In the theory of generalized inverses, it is of interest to consider common generalized inverses of two matrices of the same size. Some previous work can be found in [6]. From Corollary 2.7, we obtain the following result on the extremal ranks of common generalized inverses of two matrices.

Corollary 2.8. *Let* $A, B \in \mathbb{F}^{q\times p}$ *be given, and suppose that* $AXA = A$ *and* $BXB = B$ *have a common solution* $X \in \mathbb{F}^{p\times q}$, *i.e.,* A *and*

B have a common generalized inverse. Then

$$\max_{\substack{AXA=A\\BXB=B}} r(X) = \min\{\, p, q\}, \quad \min_{\substack{AXA=A\\BXB=B}} r(X) = \max\{\, r(A), \quad r(B)\,\}.$$

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