REND. SEM. MAT. UNIV. PADOVA, Vol. 116 (2006)

A remark on a Theorem by Henkin and Tumanov on Separately CR Functions.

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ABSTRACT - By using the approximation of CR functions by polynomials, we prove that functions which are separately CR are jointly CR. We regain in this way a result by Henkin and Tumanov [2] by a simple and expressive proof.
I wish to thank Dr. Luca Baracco for fruitful discussions.

Our discussion starts from the following statement

Let U and V be open subsets of $\mathbb{C}^n$, with $U \subset V$ and V connected, let $(\gamma_\lambda)_{\lambda \in \Lambda}$ be complex curves foliating V, such that $\gamma_\lambda \cap U \neq \emptyset \; \forall \lambda \in \Lambda$. If f is a C^0 function defined on V, such that $f_{|_U}$ is holomorphic and $f_{|_{\gamma_\lambda}}$ is holomorphic $\forall \lambda \in \Lambda$, then f is holomorphic on V.

A key point in the above assumption is that f is supposed to be C^0 from the beginning; the sole assumption of separate analyticity along the curves γ_λ would not suffice for the conclusion, in general. According to Hartogs theorem, it suffices in presence of a holomorphic foliation. Otherwise, in full generality, the validity of the statement is an open question. There are many possible ways to prove the above result. For instance, if we suppose in addition that f is C^1 we have an immediate proof. Let the foliation be described by a mapping

$$\Phi : \mathbb{C} \times \Lambda \to \mathbb{C}^n, \quad (\tau, \lambda) \mapsto \Phi(\tau, \lambda),$$

smooth in both its arguments and holomorphic in τ, so that the leaves are defined by $\gamma_\lambda := \{\Phi(\tau, \lambda) : \; \forall \tau \in \mathbb{C}\}$.

Since the statement is local with respect to λ, we can assume that in a neighborhood of a fixed value of λ, the set U contains the image under Φ of a neighborhood of $\tau = 0$. We then consider the form $\Phi^*(df \wedge dz_1 \wedge \ldots \wedge dz_n)$;

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we have

- the coefficients of the above form are holomorphic with respect to τ for $\lambda \equiv$ const,
- the coefficients are 0 for $|\tau| < \varepsilon$.

Hence the above form is $\equiv 0$. The above argument can be put in a "weak sense" and in this way we can handle the case in which f is for instance C^0. We want to generalize the former statement in many directions: first, in replacing the pair $U \subset V$ by $N \subset M$ where N and M are CR manifolds and also in replacing the foliation $\{\gamma_\lambda\}$ by a foliation $\{L_\lambda\}$ by CR manifolds of CR dimension 1. Our eventual goal is to prove that if f is C^0 in M, CR on N, CR and C^1 along each leaf L_λ, then it is in fact CR on M. An idea of the proof could be in selecting a totally real manifold $E_o \subset N$, invariant under the foliation $\{L_\lambda \cap N\}$ and in defining an approximation of f by entire functions $\{f_\alpha\}$ defined by

$$f_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \int_{+E_o} f(\xi) e^{-\alpha(\xi - z)^2} d\xi_1 \wedge \ldots \wedge d\xi_n. \tag{0.1}$$

By deforming the manifold E_o we see that the above sequence provides in fact an uniform approximation of f in a neighborhood of E_o in M: by using different E_o's, we have in particular that f is CR over M. We will carry out in detail the above argument and get the proof of

THEOREM 1. *Let M be a CR manifold of $\mathbb{C}^n$, with boundary N ($codim_M N$=1) and let $(L_\lambda)_{\lambda \in \Lambda}$ be a foliation of M with the following properties: every L_λ is a CR manifold of CR-dimension* 1, *the intersection of L_λ with N is not empty and $T^{\mathbb{C}} L_\lambda|_{N \cap L_\lambda} + TN|_{N \cap L_\lambda} = TM|_{N \cap L_\lambda}$. If $f \in C^0(M) \cap CR(N) \cap CR(L_\lambda)$ and f is C^1 along each leaf L_λ, then f is, in a neighbourhood of each point of N, a limit of polynomials. In particular, it is CR in a neighbourhood of N in M.*

Note that M is not required to be compact.

PROOF. We fix a point $z_o \in N$ and prove the conclusion in a neighbourhood of z_o. By a projection $\mathbb{C}^n \to T_{z_o}M + iT_{z_o}M$, which is a diffeomorphism when restricted to M, we can assume without loss of generality that M is generic. We choose E_o, a totally real maximal submanifold of N invariant under the foliation $\{L_\lambda \cap N\}$ and define a sequence $\{f_\alpha\}$ by means of the convolution (0.1) with the heat kernel. It is classical that $f_\alpha \rightrightarrows f$ as $\alpha \to \infty$ uniformly over (compact subsets of) E_o. We want to prove that in fact $f_\alpha \rightrightarrows f$ in a neighborhood of E_o on M. Let us see how to prove it.

Without loss of generality, we can assume, apart from a change of coordinates, that E_o is a small perturbation of $\mathbb{R}^n$. We insert E_o into a foliation $\{E\}$ by totally real maximal submanifolds on N invariant under the manifolds $\{L_\lambda \cap N\}$. We denote by $\{\Sigma\}$ the family of the unions of the L_λ's issued from each E; this provides a foliation of M. Given $z \in M \setminus E_o$, this belongs to a unique $\Sigma = \Sigma_z$. We take a deformation $\tilde{E}_o$ of E_o which contains z and of the type $\tilde{E}_o = E_1 \cup E_2 \cup E_3$ where $E_2 \subset N$, $E_3 \subset \Sigma_z$ and E_1 is the piece of E_o outside a neighborhood of z_o. We require that the deformation is small so that the following condition is fulfilled for some $k < 1$:

$$|\Im m(\xi - z_0)| \leqslant k |\Re e(\xi - z_0)| \quad \forall \xi \in \tilde{E}_o.$$

($\tilde{E}_o$ needs possibly to be shrunk here.) We denote by S a piecewise smooth manifold contained in $N \cup \Sigma_z$ with boundary $+\tilde{E}_o - E_o$. We define:

$$\widetilde{f_\alpha}(z) = \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \int\limits_{+\tilde{E}_o} f(\xi) e^{-\alpha(\xi - z)^2} d\xi_1 \wedge \ldots \wedge d\xi_n.$$

The previous estimate guarantees that $\widetilde{f_\alpha}(z)$ converges to $f(z)$ $\forall z \in \tilde{E}_o$. We show that $\widetilde{f_\alpha}(z) = f(z)$ $\forall z \in M$: E_o and $\tilde{E}_o$ delimit a submanifold of M, that we have denoted by S. By Stokes theorem:

$$f_\alpha(z) - \widetilde{f_\alpha}(z) = \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \int\limits_{+E_o} f(\xi) e^{-\alpha(\xi - z)^2} d\xi_1 \wedge \ldots \wedge d\xi_n -$$

$$-\left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \int\limits_{+\tilde{E}_o} f(\xi) e^{-\alpha(\xi - z)^2} d\xi_1 \wedge \ldots \wedge d\xi_n =$$

$$= \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} \int\limits_{+S} d_S\left(f(\xi) e^{-\alpha(\xi - z)^2} d\xi_1 \wedge \ldots \wedge d\xi_n\right).$$

(Note here that if f is only C^0 and not C^1, as "formally" required by Stokes Theorem, nonetheless the form $d_S(\cdot)$ has C^0 coefficients and the above conclusion holds.) To prove it, let us use the notation $d_S = \partial_S + \overline{\partial}_S$. Consider the term on the right: since $e^{-\alpha(\xi - z)^2}$ is holomorphic (in $\mathbb{C}^n$) and f is CR on S, we have that the product is a CR function on S, so $\overline{\partial}_S(f(\xi) e^{-\alpha(\xi - z)^2} d\xi_1 \wedge \ldots \wedge d\xi_n) = 0$. On the other way, $\partial_S(f(\xi) e^{-\alpha(\xi - z)^2} \cdot$ $\cdot\, d\xi_1 \wedge \ldots \wedge d\xi_n) = 0$, because the expression is full in the holomorphic differentials. Then f_α coincides with $\widetilde{f_\alpha}$, and it follows that f_α converges to f on any perturbation $\tilde{E}_o$ of E_o, so f_α converges to f in a neighbourhood of E_o on M. □

The same method of polynomial approximation, as the one used in the above theorem, was first exploited by Tumanov in [3], in proving that a given function is CR.

A repeated use of Theorem 1 and a connectedness argument yield the following global Theorem. This is substantially due to Henkin-Tumanov [2] but it allows general foliations by manifolds of CR dimension 1 instead of complex curves. The proof is also far different.

THEOREM 2. *Let M be a CR connected manifold with boundary N, foliated by a family $\{L_\lambda\}$ of CR manifolds of CR dimension 1 issued from N, with $T^{\mathbb{C}}L_\lambda$ transversal to $T^{\mathbb{C}}N$ at any common point of $L_\lambda \cap N$. Let f be a C^0 function on M, which is CR along N, CR and C^1 along each L_λ. Then f is CR all over M.*

PROOF. According to Theorem 1, f is CR in a neighbourhood U of N in M. Let x be a point of M; we consider a family of domains $\Omega_\upsilon \subset M$, $\upsilon \in [0\ \ 1]$, with C^1 and CR boundary $M_\upsilon = b\Omega_\upsilon$, such that:

- $\Omega_o \subset U$
- $T^{\mathbb{C}}b\Omega_\upsilon$ is transversal to $T^{\mathbb{C}}L_\lambda$ at any point of $b\Omega_\upsilon \cap L_\lambda$
- $b\Omega_\upsilon \setminus U \subset\subset M$
- $\Omega_\upsilon \subset \Omega_\mu$ if $\mu > \upsilon$
- $\bigcup_{\upsilon<\mu} \Omega_\upsilon = \Omega_\mu$
- $\overline{\Omega_\mu} = \bigcap_{\upsilon>\mu} \Omega_\upsilon$
- $\Omega_1 \ni x$.

We claim that f is CR on Ω_1, which concludes the proof. In fact, if υ_o is the maximal index for which f is CR on Ω_{υ_o}, we can apply Theorem 1, for N replaced by $b\Omega_{\upsilon_o}$; we note here that since f is CR on Ω_{υ_o}, then its boundary value on $b\Omega_{\upsilon_o}$ is also CR.

We conclude that f is CR in a neighbourhood of $b\Omega_{\upsilon_o}$. Since $b\Omega_{\upsilon_o} \setminus U$ is compact, we conclude, by a finite covering argument, that f is CR in some Ω_μ for $\mu > \upsilon_o$. A contradiction. □

REMARK 1. As we have already noticed in the course of the proof of Theorem 1, by slicing S by means of the leaves L_λ and by applying Fubini's Theorem, we need not to assume that f is C^1 and can just require that the restrictions $f|_{L_\lambda}$ are C^1. In particular, when the L_λ's are replaced by complex curves, this comes as a consequence of the hypothesis that f is holomorphic along these curves.

Theorem 2 applies in particular if we replace M by an open subset V of $\mathbb{C}^n$ and the manifolds L_λ by complex curves γ_λ, and yields the proof of the following

COROLLARY 1. *Let V be an open domain of $\mathbb{C}^n$, N a part of its boundary, $\{\gamma_\lambda\}$ a foliation of V transversal to N and let f be continuous on V, CR on N and holomorphic on each γ_λ. Then f is holomorphic all over V.*

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Manoscritto pervenuto in redazione il 20 settembre 2005 e in forma riveduta il 29 settembre 2006.