REND. SEM. MAT. UNIV. PADOVA, Vol. 116 (2006)

Some Remarks on Global Dimensions for Cotorsion Pairs.

EDGAR E. ENOCHS (*) - HAE-SIK KIM (**)

ABSTRACT - We introduce several global dimensions associated with a cotorsion pair of classes of left R-modules and consider the relations between them.

Cotorsion theories were introduced by Salce in [10]. Recently several authors use the terminology cotorsion pair for the same notion(e.g. Hovey in [8]).

DEFINITION 1. Let R be a ring and let $\mathcal{F}$ and $\mathcal{C}$ be classes of (left) R-modules. We say that $(\mathcal{F}, \mathcal{C})$ is a *cotorsion pair* if for any R-module F, $F \in \mathcal{F}$ if and only if $Ext^1_R(F, C) = 0$ for all $C \in \mathcal{C}$ and similarly for any R-module C, $C \in \mathcal{C}$ if and only if $Ext^1_R(F, C) = 0$ for all $F \in \mathcal{F}$.

We note that if $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair, then both $\mathcal{F}$ and $\mathcal{C}$ are closed under extensions and summands. Also $\mathcal{F}$ contains all projective modules and $\mathcal{C}$ all injective modules. In fact a result of Eklof says that $\mathcal{F}$ is closed under transfinite extensions (see Eklof [2] for the result and Hovey [8, Section 6] for the definition of transfinite extensions).

Examples of cotorsion pairs are $(\mathcal{P}, \mathcal{M})$ and $(\mathcal{M}, \mathcal{E})$, where $\mathcal{P}$ is the class of projective modules, $\mathcal{E}$ is that of injective modules, and $\mathcal{M}$ is the class of all modules. A less trivial example is $(\mathcal{F}, \mathcal{C})$, where $\mathcal{F}$ is the class of flat modules and $\mathcal{C}$ is the class of cotorsion modules (see Xu [11], Definition 3.1.1 and Lemma 3.4.1).

PROPOSITION 2. *Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair of (left) R-modules and let $n = sup\ \{proj.dim\ F | F \in \mathcal{F}\}$ (so, $0 \le n \le \infty$). If $0 \le n < \infty$, then n is the least s with $0 \le s < \infty$ such that if* $0 \to N \to E^0 \to E^1 \to \cdots \to E^{s-1} \to$

(*) Indirizzo dell'A.: Department of Mathematics University of Kentucky Lexington, KY 40506-0027, USA; e-mail: enochs@ms.uky.edu

(**) Indirizzo dell'A.: Department of Mathematics Kyungpook National University Taegu 702-701, Korea; e-mail: hkim@dreamwiz.com

$\to C \to 0$ is a partial injective resolution of any R-module N then $C \in \mathcal{C}$. And $n = \infty$ precisely when there is no such s.

PROOF. We first note that we are using the convention that $0 \to N \to \to E^0 \to E^1 \to \cdots \to E^{s-1} \to C \to 0$ means an exact sequence $0 \to N \to \to C \to 0$ when $s = 0$. Now assume that $0 \le n < \infty$. Suppose that s with $0 \le s < \infty$ is such that for any R-module N and any partial injective resolution $0 \to N \to E^0 \to E^1 \to \cdots \to E^{s-1} \to C \to 0$ of N, we have $C \in \mathcal{C}$. Note that $Ext_R^{s+1}(F, N) = Ext_R^1(F, C)$ for all $F \in \mathcal{M}$. Then if $F \in \mathcal{F}$, we have $Ext_R^{s+1}(F, N) = Ext_R^1(F, C) = 0$ for all N. Hence proj.dim $F \le s$ for all $F \in \mathcal{F}$. This gives $n \le s$.

But also if $0 \to N \to E^0 \to E^1 \to \cdots \to E^{n-1} \to C \to 0$ is any partial injective resolution of any R-module N, then $Ext_R^{n+1}(F, N) = 0$ for all $F \in \mathcal{F}$ since proj.dim $F \le n$. So $Ext_R^1(F, C) = Ext_R^{n+1}(F, N) = 0$ for all $F \in \mathcal{F}$. Hence $C \in \mathcal{C}$. This gives that our n is the least such s as desired.

The same type argument gives that $n = \infty$ precisely when there is no such s with $0 \le s < \infty$. □

A dual argument gives us;

PROPOSITION 3. *Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair of (left) R-modules and let $p = sup\ \{\ inj.dim\ C | C \in \mathcal{C}\ \}$. If $0 \le p < \infty$, then p is the least t with $0 \le t < \infty$ such that if $0 \to F \to P_{t-1} \to P_{t-2} \to \cdots \to P_0 \to M \to 0$ is any partial projective resolution of any R-module M we have $F \in \mathcal{F}$. Furthermore, $p = \infty$ if and only if there is no such t.*

Now we let m be the left global dimension of R (with $m = \infty$ a possibility).

PROPOSITION 4. *If $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair of R-modules and if m, n and p are as above, then $m \le n + p$.*

PROOF. If either $n = \infty$ or $p = \infty$, this inequality trivially holds. So assume that $n, p < \infty$. Let M be an R-module and let $0 \to F \to \to P_{p-1} \to P_{p-2} \to \cdots \to P_0 \to M \to 0$ be a partial projective resolution of M. By Proposition 3 we have $F \in \mathcal{F}$. Also by the definition of n we have proj.dim $F \le n$. So proj.dim $M \le n + p$ for all R-module M. Hence $m \le n + p$. □

REMARK 5. There is a special case of Proposition 3 given in Xu ([11, Theorem 3.3.2]). So we thought that it is of interest to consider these dimensions in the framework of a cotorsion pair and then to deduce Proposition 4.

For a given cotorsion pair $(\mathcal{F}, \mathcal{C})$, it is of interest to find n and p. Also it is of interest to know whether all possibilities subject to $n, p \leq m \leq n+p$ occur for some cotorsion pair $(\mathcal{F}, \mathcal{C})$. We briefly indicate how all possibilities for m, n subject to $n \leq m$ do occur.

LEMMA 6. *Given such m, n with $n \leq m$, let R be any ring with left global dimension equal to m. Let $\mathcal{L}$ consist of all R-modules L with proj.dim $L \leq n$ and let $\mathcal{H}$ be the class of all R-module H such that $Ext^1_R(L, H) = 0$ for all $L \in \mathcal{L}$. Then $(\mathcal{L}, \mathcal{H})$ is a cotorsion pair.*

PROOF. If $n = \infty$, then we get $\mathcal{L} = \mathcal{M}$ and $\mathcal{H} = \mathcal{E}$ with $\mathcal{M}$ all modules and $\mathcal{E}$ the injective modules. So we just get the cotorsion pair $(\mathcal{L}, \mathcal{H})$. Now suppose that $n < \infty$. Then $\mathcal{L}$ is a Kaplansky class(see [5] for the definition and [1, Proposition 4.1] for the proof). A basic result of Auslander says that $\mathcal{L}$ is closed under transfinite extensions. Since $\mathcal{L}$ contains all projective modules and is closed under summands, we get $(\mathcal{L}, \mathcal{H})$ is a complete cotorsion pair. $\square$

This fact is a consequence of [6, Theorem 10] and of the proof of that theorem. Also note that in the statement of that theorem the cotorsion pair is said to have enough injectives and projectives. To say that the pair is complete means precisely this.

If R is commutative Noetherian and $(\mathcal{F}, \mathcal{C})$ is the cotorsion pair with $\mathcal{F}$ the class of flat modules, a result of Jensen says that proj.dim $F \leq$ K-dim R (the Krull dimension of R) for every $F \in \mathcal{F}$([9] or see [3, Proposition 3.1] for an alternate proof). Hence $n \leq$ K-dim R in this situation. There are easy examples with K-dim $R < \infty$ and global dimension $m = \infty$. So in this situation we have $p = \infty$.

For our next result we need the notion of a hereditary cotorsion pair.

DEFINITION 7. ([4]) A cotorsion pair $(\mathcal{F}, \mathcal{C})$ of R-modules is said to be *hereditary* if it satisfies one (so all) of the three equivalent conditions.

(1) If $0 \to F' \to F \to F'' \to 0$ is exact with $F, F'' \in \mathcal{F}$, then $F' \in \mathcal{F}$.
(2) If $0 \to C' \to C \to C'' \to 0$ is exact with $C', C \in \mathcal{C}$, then $C'' \in \mathcal{C}$.
(3) $Ext^i_R(F, C) = 0$ for all $F \in \mathcal{F}$, $C \in \mathcal{C}$ and $i \geq 1$.

To get (1) $\Rightarrow$ (3) let $F \in \mathcal{F}$ and let $0 \to F' \to P \to F \to 0$ be exact with P projective. Since $P \in \mathcal{F}$, we get $F' \in \mathcal{F}$. Consider the long exact sequence

$$\cdots \to Ext^1_R(F,C) \to Ext^1_R(P,C) \to Ext^1_R(F',C) \to \\ \to Ext^2_R(F,C) \to Ext^2_R(P,C) \to Ext^2_R(F',C) \to \cdots$$

with $C \in \mathcal{C}$ and use induction on $i \geq 1$.

To see (3) $\Rightarrow$ (1) note that for the given exact sequence $0 \to F' \to F \to \to F'' \to 0$, the long exact sequence $\cdots \to Ext^1_R(F,C) \to Ext^1_R(F',C) \to \to Ext^2_R(F'',C) \to \cdots$ with $C \in \mathcal{C}$ gives us $Ext^1_R(F',C) = 0$. Then $F' \in \mathcal{F}$.

The equivalence (2) $\Leftrightarrow$ (3) can be proved in a similar manner.

We now give another characterization of the n, p defined earlier.

PROPOSITION 8. *Let $(\mathcal{F}, \mathcal{C})$ be a hereditary cotorsion pair of R-modules. Let n and p be as previously defined. If $p < \infty$, then p is the smallest t with $0 \leq t < \infty$ such that if $0 \to F_t \to F_{t-1} \to \cdots \to F_0 \to M \to 0$ is an exact sequence with $F_0, F_1, \cdots, F_{t-1} \in \mathcal{F}$ then $F_t \in \mathcal{F}$. Also $p = \infty$ if and only if there is no such t with $0 \leq t < \infty$.*

Dually, if $n < \infty$, then n is the smallest s with $0 \leq s < \infty$ such that if $0 \to N \to C^0 \to C^1 \to \cdots \to C^{s-1} \to C^s \to 0$ is an exact sequence with $C^0, C^1, \cdots, C^{s-1} \in \mathcal{C}$ then $C^s \in \mathcal{C}$. And $n = \infty$ precisely when there is no such s with $0 \leq s < \infty$.

PROOF. We prove the first claim. The proof of the second is dual to this proof. Given t with $0 \leq t < \infty$ let $0 \to F_t \to F_{t-1} \to \cdots \to F_0 \to M \to 0$ be any exact sequence with $F_0, F_1, \cdots, F_{t-1} \in \mathcal{F}$ and M any R-module and let $0 \to F \to P_{t-1} \to P_{t-2} \to \cdots \to P_0 \to M \to 0$ be any partial projective resolution of M. Then there is a commutative diagram (with exact rows) ;

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] \arrow[d] & P_{t-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F_t \arrow[r] & F_{t-1} \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

We think of the two rows as complexes and form the mapping cone (see [7, pg. 154]). So we get the complex $0 \to F \to F_t \oplus P_{t-1} \to F_{t-1} \oplus P_{t-2} \to \cdots \to F_1 \oplus P_0 \to F_0 \oplus M \to M \to 0$. This complex is exact since the two original complexes are exact. This complex has the exact sequence $0 \to M \to M \to 0$ with the identity map $M \to M$ as a subcomplex. So taking the quotient complex we get the exact complex $0 \to F \to F_t \oplus P_{t-1} \to F_{t-1} \oplus P_{t-2} \to \cdots \to F_1 \oplus P_0 \to F_0 \to 0$. Thus we have an exact sequence $0 \to F \to F_t \oplus P_{t-1} \to G \to 0$, where $G = \ker(F_{t-1} \oplus P_{t-2} \to F_{t-2} \oplus P_{t-3})$. Since $(\mathcal{F}, \mathcal{C})$ is hereditary, $G \in \mathcal{F}$. So $F \in \mathcal{F}$ if and only if $F_t \in \mathcal{F}$. With this observation we see that the result follows. $\square$

REMARK 9. If $\mathcal{F}$ is the class of flat modules, the first claim of this proposition is a familiar result. The usual argument in this case involves using the Tor functors.

REFERENCES

[1] S. T. ALDRICH - E. ENOCHS - O. M. G. JENDA - L. OYONARTE, *Envelopes and covers by modules of finite injective and projective dimensions*, J. Algebra, **242** (2001), pp. 447-459.

[2] P. EKLOF, *Homological algebra and set theory*, Trans. Amer. Math. Soc., **227** (1977), pp. 207-225.

[3] E. ENOCHS, *Minimal pure injective resolutions of flat modules*, J. Algebra, **105** (1987), pp. 351-364.

[4] E. ENOCHS - O. M. G. JENDA - B. TORRECILLAS - J. XU, *Torsion theories relative to Ext*, preprint.

[5] E. ENOCHS - J. A. LOPEZ-RAMOS, *Kaplansky classes*, Rend. Sem. Mat. Univ. Padova, **107** (2002), pp. 67-79.

[6] P. EKLOF - J. TRLIFAJ, *How to make Ext vanish*, Bull. London Math. Soc., **33** (2001), pp. 41-51.

[7] S.I. GELFAND - Y. I. MANIN, Methods of Homological Algebra, Springer, 1996.

[8] M. HOVEY, *Cotorsion pairs, model category structures, and representation theory*, Math. Z., **241** (2002), pp. 553-592.

[9] C. U. JENSEN, *On the vanishing of* $\varprojlim^{(i)}$, J. Algebra, **15** (1970), pp. 151-166.

[10] L. SALCE, *Cotorsion theories for abelian groups*, Sympos. Math., **23** (1979), pp. 11-32.

[11] J. XU, *Flat covers of modules*, Lecture Notes in Math. 1634, Springer-Verlag, 1996.

Manoscritto pervenuto in redazione il 14 luglio 2005