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Generalized Almost Completely Decomposable Groups.

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1. Introduction.

Rank-one groups are torsion-free abelian groups isomorphic to some additive subgroup of the group of rational numbers. These groups have been classified by the so-called ***types*** that have a concrete description. However, here a type will simply be an isomorphism class of rank-one groups. ***Completely decomposable groups*** are direct sums of rank-one groups and have been classified up to isomorphism. ***Almost completely decomposable (*****acd*****-)groups*** are finite essential extensions of completely decomposable groups of finite rank. These groups have been studied extensively during the last fifteen years and although much is known about them, it is a class of groups whose complete understanding is beyond reach. Attempts have been made to extend the concepts, ideas and results of acd-groups to infinite rank. As an example, in [Arn81] and [MS00] **bcd-*****groups*** were studied that are, by definition, essential extensions of completely decomposable groups of arbitrary rank by bounded groups. The question remains what «generalized almost completely decomposable groups» should be. Now almost completely decomposable groups (of finite rank) are not only finite extensions of completely decomposable groups but are also contained in completely decomposable groups as subgroups of finite index. Hence another way of generalizing almost completely decomposable groups is to consider special subgroups of completely decomposable groups. Core features of almost completely decomposable groups are the existence of Butler decompositions and of completely decomposable regulating subgroups. Certainly these features should be present in «generalized almost completely decomposable groups».

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A group H is ***sharp*** in G if $H^\sharp(\boldsymbol{\tau}) = H \cap G^\sharp(\boldsymbol{\tau})$ for every type $\boldsymbol{\tau}$. A main result (Theorem 4.5) characterizes almost completely decomposable groups as exactly the sharp subgroups of completely decomposable groups of finite rank. This characterization makes sense for any rank and justifies our definition of ***generalized almost completely decomposable groups*** as the sharp subgroups of arbitrary completely decomposable groups. We show that sharp subgroups of completely decomposable groups have Butler decompositions and completely decomposable regulating subgroups (Corollary 4.1, Corollary 4.8) hence qualify to be called generalized almost completely decomposable groups. The earlier class of bcd-groups is a subclass of the new class.

We remark ([MS00]) that there are drastic differences between almost completely decomposable groups and the generalized almost completely decomposable groups. In contrast to the acd-groups in which the regulating subgroups are completely decomposable and «large» in the sense of having minimal (finite) index, the regulating subgroups of generalized acd-groups are not large in a similar sense. In fact, the quotient of a bcd-group modulo a regulating subgroup need not be torsion, and in some groups, the intersection of all regulating subgroups is the zero-subgroup. bcd-Groups with linearly ordered critical typeset need not be completely decomposable, in contrast to the finite rank case, but they always have non-trivial completely decomposable direct summands. The distinguished role that is due the regulating subgroups of generalized acd-groups remains a mystery.

A subgroup H of G is ***regular*** if $H(\tau) = H \cap G(\tau)$ for every type τ. Regularity is a weakening of purity. Butler groups are, by definition, pure subgroups of completely decomposable groups, and therefore pure subgroups of Butler groups are again Bulter groups. In Section 5 we show (Theorem 5.5) that regular subgroups of Butler groups are again Butler groups. We do not know whether the same is true for sharp subgroups. As there are many Butler groups that are not almost completely decomposable, not even pure subgroups of almost completely decomposable groups need to be almost completely decomposable; in contrast sharp subgroups of almost completely decomposable groups are almost completely decomposable (Theorem 4.5).

Section 6 contains applications to groups that are close to being completely decomposable. In particular, we reprove results due to Baer and Erdös (Theorem 6.7). Most of our applications (Corollary 4.11, Corollary 4.12, Corollary 4.13, Corollary 6.9, Corollary 6.10, Proposition 4.14) contain sufficient conditions for a group to be completely decomposable. The new proofs utilize systematically Butler decompositions and regulating subgroups, showing in each case that the group equals its (completely decomposable) regulating subgroup.

We conclude with a list of open problems.

2. Notation and Background.

We write maps on the right. Generally our notation is standard. Yet we mention that H_*^G denotes the pure hull of the subgroup H of G in G. A ***rational group*** is a subgroup of the additive group $\mathbb{Q}$ of rational numbers containing $\mathbb{Z}$. A ***type*** is the isomorphism class of all groups isomorphic to a rational group and we will frequently identify a rational group with its type. For a type (rational group) τ, the subgroups $G(\tau) = \sum\{\tau\phi : \phi \in \mathrm{Hom}(\tau, G)\}$ and $G[\tau] = \bigcap\{\mathrm{Ker}\,\psi : \psi \in \mathrm{Hom}(G, \tau)\}$ are pure in G and are called respectively the τ-***socle*** and the τ-***radical*** of G. Additional «type subgroups» are defined by $G^*(\sigma) = \sum_{\rho>\sigma} G(\rho)$, $G^\sharp(\sigma) = G^*(\sigma)_*^G$, and $G^\sharp[\sigma] = \bigcap_{\rho<\sigma} G[\rho]$. The ***typeset*** $\mathrm{Tst}(G)$ of a torsion-free group G is the set of all types of non-zero rank-one pure subgroups of G. A type τ of a group G is ***critical*** if $G(\tau)/G^\sharp(\tau) \neq 0$. The symbol $\mathrm{T_{cr}}(G)$ denotes the set of all critical types.

A completely decomposable group A has a «homogeneous decomposition»

$$A = \bigoplus_{\rho \in \mathrm{T_{cr}}(A)} A_\rho$$

where each A_τ is τ-homogeneous completely decomposable.

We use the letters A, B, C, D for completely decomposable groups, and X, Y, Z for almost completely decomposable groups and generalizations. If A is a subgroup of X such that X/A is bounded, then the ***exponent*** $\exp(X/A)$ is the least integer e such that $e(X/A) = 0$.

We recall a number of properties of the so-called type subgroups that can be found in [Mad00].

LEMMA 2.1. *Let G be a torsion-free group and σ, τ types. Then the following hold.*

1) $G^*(\sigma) \subseteq G^\sharp(\sigma) \subseteq G(\sigma)$ *and* $G[\sigma] \subseteq G^\sharp[\sigma]$.

2) *If* $\phi \in \mathrm{Hom}(G, H)$*, then* $G(\sigma)\phi \subseteq H(\sigma)$*,* $G^*(\sigma)\phi \subseteq H^*(\sigma)$*,* $G^\sharp(\sigma)\phi \subseteq H^\sharp(\sigma)$*, and* $G^\sharp[\sigma]\phi \subset H^\sharp[\sigma]$*, i.e., all type subgroups are functorial subgroups.*

3) *If* $G = G_1 \oplus G_2$, *then*

$$G(\tau) = G_1(\tau) \oplus G_2(\tau), \quad G^\sharp(\tau) = G_1^\sharp(\tau) \oplus G_2^\sharp(\tau),$$
$$G[\tau] = G_1[\tau] \oplus G_2[\tau], \quad G^\sharp[\tau] = G_1^\sharp[\tau] \oplus G_2^\sharp[\tau].$$

For a completely decomposable group $A = \bigoplus_{\rho \in \mathrm{T_{cr}}(A)} A_\rho$ we have explicitly

$$A(\tau) = \bigoplus \{A_\rho : \rho \geq \tau\}, \quad A^\sharp(\tau) = \bigoplus \{A_\rho : \rho > \tau\},$$
$$A[\tau] = \bigoplus \{A_\rho : \rho \not\leq \tau\}, \quad A^\sharp[\tau] = \bigoplus \{A_\rho : \rho \not< \tau\}$$

We will use the following properties of partially ordered sets.

DEFINITION 2.2. *Let T be a partially ordered set.*

1) *A poset T fulfills the* ***minimum condition*** *if every non-void subset contains a minimal element. The minimum condition is equivalent with the* ***descending chain condition*** *(DCC) that says that T contains no infinite descending chain $\tau_0 > \tau_1 > \cdots$.*

2) *A poset T fulfills the* ***maximum condition*** *if every non-void subset contains a maximal element. The maximum condition is equivalent with the* ***ascending chain condition*** *(ACC) that says that T contains no infinite chain $\tau_0 < \tau_1 < \cdots$.*

3) *A poset is* ***inversely well-ordered*** *if it is linearly ordered and satisfies the maximum condition.*

4) *The* ***depth*** $\mathrm{dp}(\tau)$ *of an element $\tau \in T$ is one less than the cardinality of the longest ascending chain in T beginning with τ if there are longest chains of this kind, otherwise* $\mathrm{dp}(\tau) = \infty$. *Thus $\tau \in T$ is maximal if and only if* $\mathrm{dp}(\tau) = 0$.

A poset T satisfying the maximum condition provides for a general induction principle as follows. To prove that a statement $p(\tau)$, $\tau \in T$, is valid for all τ, it suffices to show that

1) *$p(\tau)$ holds for maximal elements τ in T,*
2) *if $p(\sigma)$ holds for all $\sigma > \tau$, then $p(\tau)$ holds.*

The principle is easily established by showing that the set of elements τ for which $p(\tau)$ fails is empty.

A decomposition $G(\tau) = B_\tau \oplus G^\sharp(\tau)$ with B_τ τ-homogeneous completely decomposable is called a ***τ-Butler decomposition*** and the τ-homogeneous completely decomposable summand B_τ is a ***τ-Butler complement*** of G. A group G ***has Butler decompositions*** if G has a Butler decomposition for each type τ. If G has Butler decompositions and B_τ is a Butler complement for each critical type of G, then $B = \sum_{\rho\in \mathrm{T}_{\mathrm{cr}}(G)} B_\rho$ is a ***regulating subgroup*** of G. The following lemma can be found in [MS00] but Lemma 2.3.1 is an immediate consequence of the Baer Lemma and the rest is almost obvious.

LEMMA 2.3.

1) *A torsion-free group G has a τ-Butler decomposition if and only if the quotient group $G(\tau)/G^\sharp(\tau)$ is τ-homogeneous completely decomposable.*

2) *A direct summand of a group with Butler decompositions has Butler decompositions.*

3) *Direct sums of groups with Butler decompositions have Butler decompositions.*

3. Sharp Embeddings.

The way a subgroup is embedded in a group is of great importance. Being a direct summand or a pure subgroup are the most common types of embeddings, but there are several more. Arnold [Arn81] and Müller-Mutzbauer [MM92] introduced and studied embeddings that all have to do with the interplay of the type subgroups of subgroup and over-group. We are mainly concerned with two of these embeddings. The first is as follows.

DEFINITION 3.1. *Let H be a subgroup of the torsion-free group G. Then H is* ***sharp*** *in G if, for every type τ, $H^\sharp(\tau) = H \cap G^\sharp(\tau)$.*

The first lemma collects basic properties of sharp embeddings.

LEMMA 3.2. *Let $K \le H \le G$.*

1) *If K is sharp in H, and H is sharp in G, then K is sharp in G.*

2) *If K is sharp in G, then K is sharp in H.*

3) *If K is sharp in G, then K_*^G is sharp in G.*

4) (Müller-Mutzbauer) *For every type τ, $H(\tau)$, $H^*(\tau)$, and $H^\sharp(\tau)$ are sharp in H.*

5) *If H is sharp in G, then, for all types τ, $H(\tau)$ is sharp in $G(\tau)$, $H^\sharp(\tau)$ is sharp in $G^\sharp(\tau)$, and $H^*(\tau)$ is sharp in $G^*(\tau)$.*

PROOF. 1) Suppose that $K^\sharp(\tau) = K \cap H^\sharp(\tau)$ and $H^\sharp(\tau) = H \cap G^\sharp(\tau)$. Then $K^\sharp(\tau) = K \cap H^\sharp(\tau) = K \cap H \cap G^\sharp(\tau) = K \cap G^\sharp(\tau)$.

2) Suppose that $K^\sharp(\tau) = K \cap G^\sharp(\tau)$. Then

$$K^\sharp(\tau) \subseteq K \cap H^\sharp(\tau) \subseteq K \cap G^\sharp(\tau) = K^\sharp(\tau).$$

3) Suppose that $K^\sharp(\tau) = K \cap G^\sharp(\tau)$ and $H = K_*^G$. Then

$$H^\sharp(\tau) \subseteq H \cap G^\sharp(\tau) = K_*^G \cap G^\sharp(\tau) = \big(K \cap G^\sharp(\tau)\big)_*^G = K^\sharp(\tau)_*^G \subseteq H^\sharp(\tau).$$

4) [MM92, Satz 7]

5) In all cases it is a matter of combining (4) with (1). □

Lemma 3.2. 5) makes induction on the depth of the typeset possible.

We will be interested particularly in sharp embeddings. The following fundamental lemma shows that some important properties of groups are passed down to sharp subgroups.

LEMMA 3.3. *Let G be an arbitrary torsion-free group and H sharp in G. Then the following hold.*

1) $\mathrm{T}_{\mathrm{cr}}(H) \subseteq \mathrm{T}_{\mathrm{cr}}(G)$.

2) *If G has Butler decompositions and hence regulating subgroups, then H has Butler decompositions and regulating subgroups.*

3) *If* $\mathrm{Tst}(G)$ *is finite, then* $\mathrm{Tst}(H)$ *is finite and* $\mathrm{Tst}(H) \subseteq \mathrm{Tst}(G)$.

4) *If* $G^\sharp(\tau) = G(\tau) \cap G[\tau]$, *then* $H^\sharp(\tau) = H(\tau) \cap H[\tau]$.

PROOF. 1) Since $H^\sharp(\tau) = H \cap G^\sharp(\tau) = H(\tau) \cap G^\sharp(\tau)$ we have the natural injection

$$\frac{H(\tau)}{H^\sharp(\tau)} \rightarrowtail \frac{G(\tau)}{G^\sharp(\tau)}. \tag{3.4}$$

This establishes 1).

2) By assumption $G(\tau)/G^\sharp(\tau)$ is τ-homogeneous completely decomposable, therefore $H(\tau)/H^\sharp(\tau)$ is τ-homogeneous and by the Baer-Kolettis Theorem ([Fuc73, Theorem 86.6]) it is also completely decomposable. Now H has Butler decompositions by Lemma 2.3.

3) By 1) $\mathrm{T}_{\mathrm{cr}}(H) \subseteq \mathrm{T}_{\mathrm{cr}}(G) \subseteq \mathrm{Tst}(G)$ showing that $\mathrm{T}_{\mathrm{cr}}(H)$ is finite. Define

$$f : \mathrm{Tst}(H) \setminus \mathrm{T}_{\mathrm{cr}}(H) \to \{G^\sharp(\sigma) : \sigma \in \mathbb{T}\} : \rho f = G^\sharp(\rho).$$

We will show that f is injective. Suppose that $\sigma_1, \sigma_2 \in \mathrm{Tst}(H) \setminus \mathrm{T}_{\mathrm{cr}}(H)$ and $G^\sharp(\sigma_1) = G^\sharp(\sigma_2)$. Then

$$H(\sigma_1) = H^\sharp(\sigma_1) = H \cap G^\sharp(\sigma_1) = H \cap G^\sharp(\sigma_2) = H^\sharp(\sigma_2) = H(\sigma_2),$$

and hence $\sigma_1 = \sigma_2$. There are only finitely many different groups $G^\sharp(\sigma)$ since $\mathrm{Tst}(G)$ is finite and it follows that $\mathrm{Tst}(H)$ is finite. By [Mad00, Lemma 2.4.20] $\mathrm{Tst}(H)$ is the meet-closure of $\mathrm{T}_{\mathrm{cr}}(H) \subseteq \mathrm{T}_{\mathrm{cr}}(G)$ and hence $\mathrm{Tst}(H) \subseteq \mathrm{Tst}(G)$.

4) $H^\sharp(\tau) \subseteq H(\tau) \cap H[\tau] \subseteq G(\tau) \cap G[\tau] \cap H = H \cap G^\sharp(\tau) = H^\sharp(\tau)$. □

If a group and a subgroup both have Butler decompositions, it may not be possible to choose Butler complements that are nested. The following lemma contains a substitute for nested Butler complements which will be needed in the prove of the main result Theorem 4.5.

LEMMA 3.5. *Let G be a finite rank Butler group and let H be a sharp subgroup of G. Then H has Butler decompositions, say $H(\tau) = A_\tau \oplus H^\sharp(\tau)$, and there are a positive integer m and subgroups $M_\tau \cong G(\tau)/G^\sharp(\tau)$, $\tau \in \mathrm{T}_{\mathrm{cr}}(G)$, of G such that $M_\tau \oplus G^\sharp(\tau) \subseteq G(\tau)$, $mA_\tau \subseteq M_\tau$ and $mG \subseteq M = \sum_{\rho \in \mathrm{T}_{\mathrm{cr}}(G)} M_\rho$.*

PROOF. The Butler group G has Butler decompositions $G(\tau) = D_\tau \oplus G^\sharp(\tau)$, and by Lemma 3.3 H has Butler decompositions $H(\tau) = A_\tau \oplus H^\sharp(\tau)$. Set $D = \sum_{\rho \in \mathrm{T}_{\mathrm{cr}}(G)} D_\rho$ and $A = \sum_{\rho \in \mathrm{T}_{\mathrm{cr}}(H)} A_\rho$. Let $\pi_\tau : G(\tau) \to D_\tau$ be the projection along $G^\sharp(\tau)$. Define

$$\psi_\tau : A_\tau \to D_\tau : x\psi_\tau = x\pi_\tau$$

The map ψ_τ is a monomorphism because $A_\tau \cap G^\sharp(\tau) = 0$. Let $B_\tau = A_\tau\psi_\tau = A_\tau\pi_\tau$. Then $B_\tau \cong A_\tau$. The purification $(B_\tau)_*^G$ is a direct summand of D_τ, say, $D_\tau = (B_\tau)_*^G \oplus C_\tau$. Moreover B_τ and $(B_\tau)_*^G$ are homogeneous completely decomposable groups of the same type and rank, hence isomorphic, and therefore the quotient $(B_\tau)_*^G/B_\tau$ is finite ([Mad00, Proposition 2.1.3]). Choose n_τ such that $n_\tau(B_\tau)_*^G \subseteq B_\tau$. We define a homomorphism as follows.

$$\varphi_\tau : D_\tau = (B_\tau)_*^G \oplus C_\tau \to G^\sharp(\tau) : (b + c)\varphi_\tau = (n_\tau b)\psi_\tau^{-1} - n_\tau b,$$

where $b \in (B_\tau)_*^G$, $c \in C_\tau$. The map φ_τ is well-defined because the image of φ_τ clearly is contained in $G(\tau)$ and $\big((n_\tau b)\psi_\tau^{-1} - n_\tau b\big)\pi_\tau = n_\tau b - n_\tau b = 0$ showing that φ_τ maps into $G^\sharp(\tau)$ as required. It is apparent that $M_\tau := D_\tau(n_\tau + \varphi_\tau)$ is disjoint from $G^\sharp(\tau)$, hence $M_\tau \oplus G^\sharp(\tau) \subseteq G(\tau)$. Also $M_\tau \cong D_\tau \cong G(\tau)/G^\sharp(\tau)$ since $n_\tau + \varphi_\tau : D_\tau \to M_\tau = D_\tau(n_\tau + \varphi_\tau)$ is a monomorphism. In fact, if $d \in D$ and $d(n_\tau + \varphi_\tau) = 0$, then $n_\tau d = -d\varphi_\tau \in$

$\in D_\tau \cap G^\sharp(\tau) = 0$, so $d = 0$. We claim that

$$n_\tau A_\tau \subseteq M_\tau. \tag{3.6}$$

Indeed, given $a \in A_\tau$, $n_\tau a = (a\psi_\tau)\varphi_\tau + n_\tau(a\psi_\tau) = (a\psi_\tau)(n_\tau + \varphi_\tau) \in M_\tau$.

It is well-known that $\sum_{\rho>\tau} D_\rho$ is a regulating subgroup of the Butler group $G^\sharp(\tau)$. Hence there are positive integers e_τ such that $e_\tau G^\sharp(\tau) \subseteq \subseteq \sum_{\rho>\tau} D_\rho$ ([Mad00, Proposition 4.1.6]). We claim that there is a positive integer n such that $nD \subseteq M := \sum_{\rho\in \mathrm{T}_{\mathrm{cr}}(G)} M_\rho$. It suffices to show that for every $\tau \in \mathrm{T}_{\mathrm{cr}}(G)$ there is k_τ such that $k_\tau D_\tau \subseteq M$. Assume first that τ is maximal in $\mathrm{T}_{\mathrm{cr}}(G)$. Then $G^\sharp(\tau) = 0$, hence $\varphi_\tau = 0$. Thus, for any $d \in D_\tau$, we have $n_\tau d = d(n_\tau + \varphi_\tau) \in M_\tau \subseteq M$. Suppose now that $\tau \in \mathrm{T}_{\mathrm{cr}}(G)$ is not maximal and assume by induction on depth that

$$\forall\, \sigma > \tau,\ \exists\, k_\sigma, k_\sigma D_\sigma \subseteq M.$$

Let $d \in D_\tau$. Then

$$n_\tau d = d(n_\tau + \varphi_\tau) - d\varphi_\tau.$$

Since $d\varphi_\tau \in G^\sharp(\tau)$, it follows that $e_\tau d\varphi_\tau \subseteq \sum_{\rho>\tau} D_\rho$. By induction hypothesis there is k such that $ke_\tau d\phi_\tau \in M$. Hence $ke_\tau n_\tau d = ke_\tau d(n_\tau + \varphi_\tau) - - ke_\tau d\varphi_\tau \in M$.

We have proved that $nD \subseteq M$ and since D has finite index in G there is a multiple m of n such that $mG \subseteq M$. Without loss of generality m is a multiple of each n_τ. Then, by (3.6), $mA_\tau \subseteq n_\tau A_\tau \subseteq M_\tau$. □

4. Generalized Almost Completely Decomposable Groups.

The theory of almost completely decomposable groups is largely based on the existence of a completely decomposable subgroup of finite index. However, it is not clearcut how «completely decomposable subgroup of finite index» should be extended to infinite rank. In Theorem 4.5 we characterize almost completely decomposable groups as the sharp subgroups of completely decomposable groups of finite rank. This makes sense for any rank and justifies our definition of «generalized almost completely decomposable group».

Some lemmas are needed in the proof. The first result is an application of Lemma 3.3.

COROLLARY 4.1. *Let D be a completely decomposable group and X a sharp subgroup. Then the following hold.*

1) $\mathrm{T}_{\mathrm{cr}}(X) \subseteq \mathrm{T}_{\mathrm{cr}}(D)$.

2) $X(\tau) = A_\tau \oplus X^\sharp(\tau)$ for a τ-homogeneous completely decomposable group A_τ.

3) $X^\sharp(\tau) = X(\tau) \cap X[\tau]$.

4) $A := \sum_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$ is a completely decomposable group $A = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$.

LEMMA 4.2. *Let X be a torsion-free group having Butler decompositions and hence regulating subgroups. Assume that* $\mathrm{Tst}(X)$ *satisfies the maximum condition. Then X/A is torsion for every regulating subgroup A of X.*

PROOF. Let $A = \sum_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$ where $X(\tau) = A_\tau \oplus X^\sharp(\tau)$ for every (critical) type τ of X. We prove that X/A is torsion by induction on $\mathrm{Tst}(X)$. Let $x \in X$, $\tau = \mathrm{tp}^X(x)$. Suppose that τ is maximal in $\mathrm{Tst}(X)$. Then τ is critical and $x \in X(\tau) = A_\tau \subseteq A$. Now suppose that τ is not maximal in $\mathrm{Tst}(X)$ and for all $y \in X$ with $\mathrm{tp}^X(y) > \tau$, there is a positive integer m_y such that $m_y y \in A$. According to the Butler decomposition $X(\tau) = A_\tau \oplus X^\sharp(\tau)$, write $x = a + y$ with $a \in A_\tau$ and $y \in X^\sharp(\tau)$. Since $X^\sharp(\tau)/X^*(\tau)$ is a torsion group, there is a positive integer m such that $my = y_1 + \cdots + y_k$ where $y_i \in X$ and $\mathrm{tp}^X(y_i) > \tau$. By induction hypothesis there is a positive integer M such that $My_i \in A$ for every i. Then $mMx = mMa + M(my) \in A$. □

Recall that H is a ***quasi-summand*** of G if there is a subgroup K of G such that $H \cap K = 0$ and $nG \subseteq H \oplus K$ for some positive integer n.

LEMMA 4.3. *Let D be a completely decomposable group of finite rank and X a subgroup of D that has a regulating subgroup $A = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$ with $A_\tau \subseteq D_\tau$. If* $\mathrm{Tst}(X)$ *satisfies the maximum condition, then X is an almost completely decomposable group. Moreover, X is a quasi-summand of D.*

PROOF. For each $\tau \in \mathrm{Tst}(X)$ we have $A_\tau \cong (A_\tau)_*^D$ since both groups are τ-homogeneous completely decomposable of the same type and rank. Moreover, the group $(A_\tau)_*^D$ is a direct summand of D_τ. By [Mad00, Proposition 2.1.3] we obtain that $(A_\tau)_*^D$ is finite over A_τ. Thus, using Lemma 4.2,

$$
\begin{aligned}
\bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho \subseteq X &= \Big\langle \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho \Big\rangle_*^X \subseteq \Big\langle \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho \Big\rangle_*^D \\
&\subseteq \Big\langle \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} (A_\rho)_*^D \Big\rangle_*^D = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} (A_\rho)_*^D .
\end{aligned} \tag{4.4}
$$

Since $\bigoplus_\rho (A_\rho)_*^D$ is finite over $\bigoplus_\rho A_\rho$, we conclude that X is finite over $\bigoplus_\rho A_\rho$. We have established that X is almost completely decomposable. To show that X is a quasi-summand of D, choose complements $(A_\tau)_*^D \oplus C_\tau = D_\tau$. By (4.4) $\left(\bigoplus_{\rho\in \mathrm{T_{cr}}(X)} A_\rho\right)_*^D \Big/ X$ is finite and hence $D\Big/\left(X \oplus \bigoplus_{\rho\in \mathrm{T_{cr}}(X)} C_\rho\right)$ is finite.

We are now ready to prove the mentioned characterization of almost completely decomposable groups.

THEOREM 4.5. *A torsion-free group X of finite rank is almost completely decomposable if and only if it is a sharp subgroup of a completely decomposable group. Moreover, if X is sharp in the completely decomposable group D of finite rank, then X is a quasi-summand of D.*

PROOF. An almost completely decomposable group X, by definition, contains a completely decomposable subgroup A such that for some positive integer e, $eX \leq A$. Hence $X \leq e^{-1}A$ and X is sharp as $e^{-1}A/X$ is finite.

Conversely, suppose that X be sharp in the completely decomposable group $D = \bigoplus_{\rho\in \mathrm{T_{cr}}(D)} D_\rho$. Without loss of generality $\mathrm{rk}(D) < \infty$. Then $\mathrm{T_{cr}}(X) \subseteq \mathrm{T_{cr}}(D)$, and X inherits Butler decompositions, hence completely decomposable regulating subgroups from D (Lemma 3.3). It remains to show that for a regulating subgroup A of X, the quotient X/A is finite. Choose Butler decompositions $X(\tau) = A_\tau \oplus X^\sharp(\tau)$, and let $A = \bigoplus_{\rho\in \mathrm{T_{cr}}(D)} A_\rho$ be the corresponding regulating subgroup of X. By Lemma 3.5 there exist subgroups $M_\tau \cong D_\tau$ of D and a positive integer m such that $M_\tau \oplus D^\sharp(\tau) \subseteq D(\tau)$, $mA_\tau \subseteq M_\tau$ and $mD \subseteq M := \sum_{\rho\in \mathrm{T_{cr}}(D)} M_\rho$. We claim that M is the direct sum of its subgroups M_τ. The sum function $\Sigma : \bigoplus_{\rho\in \mathrm{T_{cr}}(D)} M_\rho \to M$ is surjective, and $\mathrm{rk}\left(\bigoplus_{\rho\in \mathrm{T_{cr}}(D)} M_\rho\right) = \sum_{\rho\in \mathrm{T_{cr}}(D)} \mathrm{rk}(D_\rho) = \mathrm{rk}D = \mathrm{rk}M$ which is possible only if $\mathrm{Ker}\Sigma = 0$ and $M = \bigoplus_{\rho\in \mathrm{T_{cr}}(D)} M_\rho$. Since mX has Butler decompositions with Butler complements mA_τ, we obtain by Lemma 4.3 that $mX/\bigoplus_{\rho\in \mathrm{T_{cr}}(D)} mA_\rho$ is finite, and therefore $X/\bigoplus_{\rho\in \mathrm{T_{cr}}(D)} A_\rho$ is finite.

It is left to show that X is a quasi-summand of D. By Lemma 3.3.3) $\mathrm{Tst}(X)$ is finite and by Lemma 4.3 it follows that mX is a quasi-summand of M. Since $mD \subseteq M$ we conclude that X is also a quasi-summand of D. □

COROLLARY 4.6. *A sharp subgroup of an almost completely decomposable group is again an almost completely decomposable group.*

PROOF. An almost completely decomposable group has finite index in some completely decomposable group, so is sharp in a completely decomposable group. The transitivity of sharp embeddings makes Theorem 4.5 applicable. □

We are ready to formally introduce infinite rank almost completely decomposable groups.

DEFINITION 4.7. *A group is* ***a generalized almost completely decomposable group*** *or an* ***gacd-group*** *for short, if it is a sharp subgroup of some completely decomposable group.*

As a consequence of Corollary 4.1, generalized almost completely decomposable groups have completely decomposable regulating subgroups that are the direct sums of Butler complements.

COROLLARY 4.8. *A generalized almost completely decomposable group has Butler decompositions and completely decomposable regulating subgroups.*

The following corollary is just a reformulation of Theorem 4.5.

COROLLARY 4.9. *The almost completely decomposable groups are exactly the generalized almost completely decomposable groups of finite rank.*

The following proposition shows that gacd-groups are not rare. E.g. Proposition 4.10.1 shows that for any completely decomposable group D and any set of positive integers n_τ, $\tau \in \mathbb{T}$, the subgroup $\sum_{\rho\in\mathbb{T}} n_\rho D^\sharp(\rho)$ is sharp in D.

PROPOSITION 4.10.

1) *Let D be a completely decomposable group. A group $X \subseteq D$ is sharp in D if $D^\sharp(\tau) = X^\sharp(\tau)_*^D$ for every type τ.*

2) *Every* bcd-*group is a generalized almost completely decomposable group.*

3) *Direct sums of generalized almost completely decomposable groups are generalized almost completely decomposable groups.*

4) *Direct summands of generalized almost completely decomposable groups are again generalized almost completely decomposable groups.*

PROOF. 1) We have $X^\sharp(\tau) = X \cap \left(X^\sharp(\tau)\right)_*^D = X \cap D^\sharp(\tau)$.

2) It is well-known ([Mad00, Lemma 2.3.10.2]) and easy to check that subgroups of bounded index are sharp.

3) Suppose that $X = \bigoplus_{i\in I} X_i$ where each X_i is contained in a completely decomposable group D_i such that $X_i^\sharp(\tau) = X \cap D_i^\sharp(\tau)$. Let $D = \bigoplus_{i\in I} D_i$. We have $X^\sharp(\tau) = \bigoplus_{i\in I} X_i^\sharp(\tau) = \bigoplus_{i\in I} X_i \cap D_i^\sharp(\tau) = (\bigoplus_{i\in I} X_i) \cap$ $\cap (\bigoplus_{i\in I} D_i^\sharp(\tau)) = X \cap D^\sharp(\tau)$.

4) Let $X = X_1 \oplus X_2$ be a gacd-group. Then $X^\sharp(\tau) = X \cap D^\sharp(\tau)$. We must show that $X_i^\sharp(\tau) = X_i \cap D^\sharp(\tau)$. In fact, $X_i^\sharp(\tau) \subseteq X_i \cap D^\sharp(\tau) \subseteq$ $\subseteq X_i \cap X \cap D^\sharp(\tau) \subseteq X_i \cap X^\sharp(\tau) = X_i^\sharp(\tau)$. □

We conclude this section by considering an aspect in which acd-groups and gacd-groups differ substantially.

Almost completely decomposable groups with linearly ordered critical typeset are necessarily completely decomposable. This is not true for gacd-groups .

Corollary 4.11. *Let X be a generalized almost completely decomposable group whose typeset* Tst(X) *satisfies the maximum condition. Then X is completely decomposable if and only if $X^\sharp(\tau) = X^*(\tau)$ for each* $\tau \in$ Tst(X).

Proof. It is clear that $X^\sharp(\tau) = X^*(\tau)$ if X is completely decomposable. For the converse we prove that $X = A$ where $A = \bigoplus_{\rho\in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$ is a regulating subgroup of X. It is sufficient to show that $X(\sigma) \subseteq A$ for each $\sigma \in \mathrm{Tst}(X)$. Suppose first that σ is maximal in $\mathrm{Tst}(X)$. Then $\sigma \in \mathrm{T}_{\mathrm{cr}}(X)$ and $X(\sigma) = A_\sigma \subseteq A$. Now suppose that $X(\sigma) \subseteq A$ for each $\sigma \in \mathrm{Tst}(X)$ with $\sigma > \tau$. Then $X(\tau) = A_\tau \oplus X^\sharp(\tau) = A_\tau \oplus X^*(\tau) \subseteq A$ by induction hypothesis. □

Corollary 4.12. *Let X be a generalized almost completely decomposable group such that* Tst(X) *is inversely well-ordered. Then X is completely decomposable.*

Proof. Since $\mathrm{Tst}(X)$ is linearly ordered we have $X^\sharp(\tau) = X^*(\tau)$. Hence Corollary 4.11 implies that X is completely decomposable. □

An immediate corollary is [Elt96, Satz 6.5] since bcd-groups are generalized almost completely decomposable groups.

Corollary 4.13 (Elter). *Any bcd-group with inversely well-ordered typeset is completely decomposable.*

Nongxa [Non87, Example p. 614] exhibits a sharp and pure subgroup of a completely decomposable group (hence of a generalized almost completely decomposable group) that is not completely decomposable. The typeset of the Example in [Non87] is of the form $\{\tau_0, \tau_1, \tau_2\}$ where τ_1 and τ_2 are incomparable and $\tau_0 = \inf\{\tau_1, \tau_2\}$. This shows that the assumptions in Corollary 4.12 are necessary. By [Arn81] (or [MS00]) there are bcd-groups (hence generalized almost completely decomposable groups) with linearly ordered typeset that are not completely decomposable.

Nevertheless, Corollary 4.12 can be modified in the following way.

PROPOSITION 4.14. *Let X be sharp in D such that $\mathrm{T_{cr}}(D)$ is inversely well-ordered. Then X is completely decomposable.*

PROOF. By Corollary 4.12 it suffices to show that $\mathrm{Tst}(X)$ is inversely well-ordered. Note that $\mathrm{Tst}(D) = \mathrm{T_{cr}}(D)$ is linearly ordered. First we show that $\mathrm{Tst}(X)$ is linearly ordered. Let $\sigma_1, \sigma_2 \in \mathrm{Tst}(X)$. Then $D(\sigma_1) \subseteq D(\sigma_2)$ or $D(\sigma_2) \subseteq D(\sigma_1)$ since $\mathrm{Tst}(D)$ is linearly ordered. Without loss of generality we assume that $D(\sigma_1) \subseteq D(\sigma_2)$. If $\sigma_1 \in \mathrm{Tst}(D)$, then $\sigma_1 \geq \sigma_2$ follows immediately. Hence assume that $\sigma_1 \notin \mathrm{Tst}(D)$. Then $D(\sigma_1) = D^\sharp(\sigma_1)$ and $\sigma_1 \notin \mathrm{T_{cr}}(X)$ since $\mathrm{T_{cr}}(X) \subseteq \mathrm{T_{cr}}(D)$ (Lemma 4.1). We have to distinguish two cases:

Case 1: $\sigma_2 \notin \mathrm{T_{cr}}(D)$. Then $D(\sigma_2) = D^\sharp(\sigma_2)$ and $X(\sigma_2) = X^\sharp(\sigma_2)$, hence

$$X(\sigma_1) = X^\sharp(\sigma_1) = X \cap D^\sharp(\sigma_1) \subseteq X \cap D^\sharp(\sigma_2) = X^\sharp(\sigma_2) = X(\sigma_2)$$

and therefore $\sigma_1 \geq \sigma_2$.

Case 2: $\sigma_2 \in \mathrm{T_{cr}}(D)$. We will show that either $\sigma_2 \geq \sigma_1$ or $X \cap D^\sharp(\sigma_1) \subseteq$ $\subseteq X \cap D^\sharp(\sigma_2)$. Let $x \in D^\sharp(\sigma_1) \subseteq D(\sigma_2) = D_{\sigma_2} \oplus D^\sharp(\sigma_2)$. Then x has a representation $x = y \oplus z$ where $y \in D_{\sigma_2}$, $z \in D^\sharp(\sigma_2)$. If $y \neq 0$, then $\mathrm{tp}(x) = \sigma_2$ and hence $\sigma_2 \geq \sigma_1$. Hence assume that $X \cap D^\sharp(\sigma_1) \subseteq X \cap D^\sharp(\sigma_2)$. Then

$$X(\sigma_1) = X^\sharp(\sigma_1) \subseteq X \cap D^\sharp(\sigma_1) \subseteq X \cap D^\sharp(\sigma_2) = X^\sharp(\sigma_2) \subseteq X(\sigma_2)$$

which implies $\sigma_1 \geq \sigma_2$.

Therefore $\mathrm{Tst}(X)$ is linearly ordered and since $\mathrm{T_{cr}}(X) \subseteq \mathrm{T_{cr}}(D)$ we conclude that $\mathrm{Tst}(X) = \mathrm{T_{cr}}(X)$ must also be inversely well-ordered. □

There is a kind of inverse to Proposition 4.14 as follows.

COROLLARY 4.15. *Let D be completely decomposable and suppose that for any two critical types there is a prime number p such that neither σ_1*

nor σ_2 is divisible by p. Then every sharp subgroup of D is completely decomposable if and only if $\mathrm{T_{cr}}(D)$ *is inversely well-ordered.*

PROOF. By Proposition 4.14 any sharp subgroup of D is completely decomposable if $\mathrm{T_{cr}}(D)$ is inversely well-ordered. Conversely, if every sharp subgroup of D is completely decomposable and there exist $\sigma_1, \sigma_2 \in \mathrm{T_{cr}}(D)$ which are incomparable, then $\sigma_1 \oplus \sigma_2$ is a direct summand of D. By hypothesis there is a prime p that divides neither σ_1 nor σ_2. But then there are well-known finite extensions of $\sigma_1 \oplus \sigma_2$ that are indecomposable. Hence $\sigma_1 \oplus \sigma_2$ contains an indecomposable gacd-subgroup which is trivially also an indecomposable sharp subgroup of D, a contradiction. Hence $\mathrm{T_{cr}}(D)$ is linearly ordered. Moreover, if $\mathrm{T_{cr}}(D)$ contains an infinite ascending chain of types σ_i, $i \in \omega$, then there is a bounded extension of $\bigoplus_{i\in\omega} \sigma_i$ by [Elt96] or [MS00] which is not completely decomposable. Hence $\bigoplus_{i\in\omega} \sigma_i$ and therefore D contains an gacd-subgroup that is not completely decomposable, again a contradiction. Thus $\mathrm{T_{cr}}(D)$ is inversely well-ordered.

5. Butler Groups.

Butler groups, according to one possible definition, are the pure subgroups of completely decomposable groups. This implies immediately that a pure subgroup of a Butler group is again a Butler group. We will show more generally that a regular subgroup of a Butler group is a Butler group where «regular» is the following weakening of «pure».

DEFINITION 5.1 (Arnold). *Let H be a subgroup of the torsion-free group G. Then H is **regular** in G if $H(\tau) = H \cap G(\tau)$ for all types τ.*

Pure subgroups are always regular but regular subgroups need not be pure. An example of Müller-Mutzbauer that also appears in generalized form as [Arn00, Exampl 3.1.11] shows that *a sharp subgroup need not be regular.*

The first lemma is a simple house keeping matter.

LEMMA 5.2. *Let $K \leq H \leq G$.*

1) *H is regular in G if and only if for every $x \in H$, $\mathrm{tp}^H(x) = \mathrm{tp}^G(x)$.*
2) *If K is regular in H, and H is regular in G, then K is regular in G.*
3) *If K is regular in G, then K is regular in H.*

PROOF. 1) Suppose that H is regular in G, and $x \in H$. Always $\mathrm{tp}^H(x) \leq \mathrm{tp}^G(x)$. Set $\tau = \mathrm{tp}^G(x)$. Then $x \in H \cap G(\tau) = H(\tau)$, so $\mathrm{tp}^H(x) \geq \tau$ and equality follows.

Suppose that for every $x \in H$ it is true that $\mathrm{tp}^H(x) = \mathrm{tp}^G(x)$. Always $H(\tau) \subseteq H \cap G(\tau)$. Let $x \in H \cap G(\tau)$. Then $\tau \leq \mathrm{tp}^G(x) = \mathrm{tp}^H(x)$, so $x \in H(\tau)$.

2) Suppose that $K(\tau) = K \cap H(\tau)$ and $H(\tau) = H \cap G(\tau)$. Then $K(\tau) = K \cap H(\tau) = K \cap H \cap G(\tau) = K \cap G(\tau)$.

3) Suppose that $K(\tau) = K \cap G(\tau)$. Then $K(\tau) \subseteq K \cap H(\tau) \subseteq K \cap G(\tau) = K(\tau)$. □

The following proposition characterizes Butler groups in terms of regulating subgroups.

PROPOSITION 5.3. *A torsion-free group of finite rank is a Butler group if and only if it has a finite critical typeset, Butler decompositions, and a regulating subgroup that has finite index in the group.*

PROOF. It is well-known that Butler groups have finite critical typesets, Butler decompositions and regulating subgroups of finite index ([Mad00, Proposition 4.1.6]). Let H be torsion-free of finite rank and assume that $A = \sum_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$ is a regulating subgroup of finite index in H. Then A has finite rank and is an epimorphic image of a completely decomposable group of finite rank, namely $\bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$. So A is a Butler group and so is H as it is the epimorphic image of $F \oplus \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(X)} A_\rho$ where F is a suitable free group of finite rank. □

We need a lemma in order to prove the advertised closure property of the class of Butler groups.

LEMMA 5.4. *Let H be a subgroup of the torsion-free group G. If H is regular in G and G/H is torsion, then H is sharp in G.*

PROOF. Trivially $H \cap G^\sharp(\tau) \supset H^\sharp(\tau)$. To prove equality, let $x \in H \cap G^\sharp(\tau)$. Using that G/H is torsion and the definition of $G^\sharp(\tau)$, there exists a natural number n such that $nx = h_1 + \cdots + h_m$ for elements $h_i \in H$ with $\mathrm{tp}^G h_i > \tau$. By regularity $\mathrm{tp}^H h_i > \tau$ and, x being in H, by definition of $H^\sharp(\tau)$, it follows that $x \in H^\sharp(\tau)$. □

We can now prove that regular subgroups of Butler groups are Butler groups.

THEOREM 5.5. *A regular subgroup of a Butler group of finite rank is a Butler group.*

PROOF. Let H be a regular subgroup of the Butler group G. Then H is regular in H^G_* and H^G_* is a Butler group as a pure subgroup of a Butler group. Hence without loss of generality G/H is torsion. By Lemma 5.4 H is sharp in G. Therefore Lemma 3.5 applies. Thus we have a subgroup $M = \sum_{\rho\in \mathrm{T}_{\mathrm{cr}}(G)} M_\rho$ of G with $M_\tau \cong G(\tau)/G^\sharp(\tau)$, a regulating subgroup $A = \sum_{\rho\in \mathrm{T}_{\mathrm{cr}}(A)} A_\rho$ of H and a positive integer m such that $mG \subseteq M$ and $mA_\tau \subseteq M_\tau$. Because H is sharp and regular in G, we have an exact sequence with natural map

$$\frac{H(\tau)}{H^\sharp(\tau)} \rightarrowtail \frac{G(\tau)}{G^\sharp(\tau)} \twoheadrightarrow \frac{H+G(\tau)}{H+G^\sharp(\tau)}.$$

The right hand group is a subgroup of an epimorphic image of the torsion group G/H. We conclude that $mA_\tau \cong A_\tau \cong \frac{H(\tau)}{H^\sharp(\tau)}$ and $M_\tau \cong \frac{G(\tau)}{G^\sharp(\tau)}$ are isomorphic τ-homogeneous completely decomposable groups. Hence mA_τ has finite index in M_τ ([Mad00, Proposition 2.1.3]), and it follows that $[M : mA]$ is finite and $[G : mA]$ is finite. Hence $[H : A] = [mH : mA]$ is finite and by Prpoposition 5.3 H is a Butler group. □

We showed earlier that sharp subgroups of almost completely decomposable groups are again almost completely decomposable groups (Corollary 4.6). We could not decide whether sharp subgroups of Butler groups are again Butler groups.

[MM92, Satz 10] follows from Theorem 4.5 as we will see. A subgroup H of G is ***critically regular*** in G if, for every type τ, $H^\sharp(\tau) = H(\tau) \cap G^\sharp(\tau)$. A ***strongly regular*** subgroup is one that is both regular and critically regular. *«strongly regular» is strictly stronger than «sharp»* ([MM92]).

THEOREM 5.6 (Müller-Mutzbauer). *The strongly regular subgroups of an almost completely decomposable group D of finite rank are exactly the quasi-summands of D.*

PROOF. Without loss of generality D may be assumed to be completely decomposable. A strongly regular subgroup clearly is sharp and hence a quasi-summand of D by Theorem 4.5. The converse is trivial. □

Müller and Mutzbauer prove in addition that a torsion-free group of finite rank all of whose strongly regular subgroups are quasi-summands, is necessarily almost completely decomposable.

COROLLARY 5.7. *A subgroup H of a completely decomposable group D of finite rank is sharp if and only if for every type τ, $H(\tau) = H \cap G(\tau)$ and $H^\sharp(\tau) = H(\tau) \cap G^\sharp(\tau)$.*

We could not find a more direct proof that sharp subgroups of completely decomposable groups of finite rank are regular.

6. Completely Decomposable Groups.

One of our main tools in the following are regulating subgroups. E.g., to show that a group is completely decomposable one can show that its regulating subgroups are completely decomposable and equal to the group. The following lemmas point in this direction.

LEMMA 6.1. *Let A be a torsion-free group whose typeset* $\mathrm{Tst}(A)$ *satisfies the maximum condition. Suppose that*

1) $A(\tau) = A_\tau \oplus A^\sharp(\tau)$ for each $\tau \in \mathrm{T}_{\mathrm{cr}}(A)$.
2) $A^*(\tau) = A^\sharp(\tau)$ *for each* $\tau \in \mathrm{T}_{\mathrm{cr}}(A)$.

Then each A_τ is τ-homogeneous and $A = \sum_{\rho \in \mathrm{T}_{\mathrm{cr}}(A)} A_\rho$.

PROOF. It is clear from 1) that A_τ is τ-homogeneous. Set $A' = \sum_{\rho \in \mathrm{T}_{\mathrm{cr}}(A)} A_\rho$.

We show that $A = A'$ by induction on depth in $\mathrm{Tst}(A)$. Let μ be maximal in $\mathrm{Tst}(A)$. Then $A^\sharp(\mu) = 0$, so $\mu \in \mathrm{T}_{\mathrm{cr}}(A)$ and $A(\mu) = A_\mu \subseteq A'$. Now let $\tau \in \mathrm{Tst}(A)$ be arbitrary and assume that $A(\sigma) \subseteq A'$ for every $\sigma \in \mathrm{Tst}(A)$ with $\sigma > \tau$. It is to show that $A(\tau) \subseteq A'$. In fact (justifications below),

$$A(\tau) = A_\tau \oplus A^\sharp(\tau) = A_\tau \oplus \textstyle\sum_{\rho \in \mathrm{T}, \rho > \tau} A(\rho) = A_\tau \oplus \sum_{\rho \in \mathrm{Tst}(A), \rho > \tau} A(\rho) \subseteq A'.$$

The first equality is 1), the second 2), the third is true because, in general, $G(\rho) = \sum_{\upsilon \in \mathrm{Tst}(G), \upsilon \geq \rho} G(\upsilon)$, and the last containment is by induction. □

An additional notion is needed. Let G be a torsion-free group and τ a type. Define $G^{**}(\tau) = \sum_{\rho \not\leq \tau} G(\rho)$. We clarify the relationship of $G^{**}(\tau)$ and the radical $G[\tau]$.

Recall Lady's formulas ([Arn00, Proposition 3.2.8(a)]) saying that, for a finite rank Butler group G,

(6.2) $$G[\tau] = \langle G(\sigma) : \sigma \not\leq \tau \rangle_*^G \quad \text{and} \quad G^\sharp[\tau] = \langle G(\sigma) : \sigma \not< \tau \rangle_*^G.$$

LEMMA 6.3. *Let G be a torsion-free group and τ a type.*

1) $G^{**}(\tau) \subseteq G[\tau]$.

2) *$G^{**}(\tau)$ need not be pure in G.*

3) *If G is a Butler group of finite rank, then $G^{**}(\tau)_*^G = G[\tau]$.*

4) *There exist torsion-free groups G of any finite rank such that $G^{**}(\tau) = 0$ and $G[\tau] = G$.*

PROOF. 1) This is obvious since homomorphisms cannot decrease types.

2) Let $\tau_1 = \mathbb{Z}[2^{-1}], \tau_2 = \mathbb{Z}[3^{-1}], \tau_3 = \mathbb{Z}[5^{-1}]$, $A = \tau_1 v_1 \oplus \tau_2 v_2 \oplus \tau_3 v_3$, and $X = A + \mathbb{Z}\,\frac{1}{7}\,(v_1 + v_2)$. Then $\operatorname{tp}^X(\frac{1}{7}(v_1 + v_2)) = \mathbb{Z}$, and $X^{**}(\tau_3) = \tau_1 v_1 \oplus \tau_2 v_2$ is not pure in X.

3) (6.2).

4) [Arn82, Example 2.7, p.26] exhibits a strongly indecomposable group G of prescribed rank r that is homogeneous of type $\mathbb{Z}$. Then $G^{**}(\mathbb{Z}) = 0$ while $G[\mathbb{Z}] = G$ since any non-zero homomorphism $G \to \mathbb{Z}$ would cause a non-trivial decomposition of G. □

LEMMA 6.4. *Let A be a torsion-free group. Suppose that*

1) *$A(\tau) = A_\tau \oplus A^\sharp(\tau)$ for each $\tau \in \mathrm{T}_{\mathrm{cr}}(A)$.*

2) *$A(\tau) \cap A^{**}(\tau) \subseteq A^\sharp(\tau)$ for each $\tau \in \mathrm{T}_{\mathrm{cr}}(A)$.*

Then each A_τ is τ-homogeneous and $\sum_{\rho \in \mathrm{T}_{\mathrm{cr}}(A)} A_\rho = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(A)} A_\rho$.

PROOF. It is clear that A_τ is τ-homogeneous. To show that $A' = \sum_{\rho \in \mathrm{T}_{\mathrm{cr}}(A)} A_\rho = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(A)} A_\rho$ we suppose to the contrary that there exist non-zero elements $a_i \in A_{\tau_i}$ such that $a_1 + \cdots + a_k = 0$. Without loss of generality we assume that τ_1 is minimal in the set $\{\tau_1, \ldots, \tau_k\}$. Then $a_1 = -a_2 - \cdots - a_k \in A(\tau_1) \cap A^{**}(\tau_1) \subseteq A^\sharp(\tau_1)$. This means that $a_1 \in A_{\tau_1} \cap A^\sharp(\tau_1) = 0$, a contradiction. □

We can now easily derive a result of Arnold and Vinsonhaler ([AV84, Theorem II]): Part (b) of [AV84, Theorem II] says that a Butler group contains a completely decomposable subgroup of finite index (i.e., is an almost completely decomposable group) if and only if the natural map

$\phi_\tau^G : G(\tau)/G^\sharp(\tau) \to G^\sharp[\tau]/G[\tau]$ is a monomorphism. Obviously ϕ_τ^G is a monomorphism if and only if $G(\tau) \cap G[\tau] = G^\sharp(\tau)$.

PROPOSITION 6.5. *Let G be a Butler group of finite rank. Then G is almost completely decomposable if and only if, for all τ, $G(\tau) \cap G[\tau] = G^\sharp(\tau)$.*

PROOF. Almost completely decomposable groups clearly have the stated property. Conversely, Butler groups have regulating subgroups and the regulating subgroups have finite index in the group (Proposition 5.3). The regulating subgroups are completely decomposable by Lemma 6.4 which applies because $G(\tau) \cap G^{**}(\tau) \subseteq G(\tau) \cap G[\tau] = G^\sharp(\tau)$. □

COROLLARY 6.6. *Let A be a torsion-free group whose typeset* $\mathrm{Tst}(A)$ *satisfies the maximum condition. Suppose that*

1) $A(\tau) = A_\tau \oplus A^\sharp(\tau)$ *for each* $\tau \in \mathrm{T_{cr}}(A)$.
2) $A^*(\tau) = A^\sharp(\tau)$ *for each* $\tau \in \mathrm{T_{cr}}(A)$.
3) $A(\tau) \cap A^{**}(\tau) \subseteq A^\sharp(\tau)$ *for each* $\tau \in \mathrm{T_{cr}}(A)$.

Then each A_τ is τ-homogeneous and $A = \bigoplus_{\rho \in \mathrm{T_{cr}}(A)} A_\rho$.

PROOF. Combine Lemma 6.1 and Lemma 6.4. □

The preceding result implies the non-trivial part of [Fuc73, Theorem 98.3] due to Baer and Erdös ([Bae37], [Erd54]).

THEOREM 6.7 (Baer, Erdös). *Let A be a torsion-free group whose typeset* $\mathrm{Tst}(A)$ *satisfies the maximum condition. Then A is the direct sum of homogeneous groups if and only if the following hold.*

1) *For each type τ, $A^*(\tau)$ is a direct summand of $A(\tau)$,*
2) *for each type τ, $A^*(\tau) = A(\tau) \cap A^{**}(\tau)$.*

PROOF. The assumption 1) implies that $A^*(\tau)$ is pure, hence $A^*(\tau) = A^\sharp(\tau)$ and $A(\tau) = A_\tau \oplus A^\sharp(\tau)$ for some A_τ. Thus the hypotheses of Corollary 6.6 are satisfied. □

A theorem of Prochazka ([Pro63]) is another easy consequence of our results. To derive it we need to consider groups with linearly ordered typesets.

LEMMA 6.8. *Let G be a torsion-free group with linearly ordered typeset. Then the following hold.*

1) *For any type τ, $G^*(\tau) = \bigcup\{G(\rho) : \rho \in \mathrm{Tst}(G), \rho > \tau\} = G^\sharp(\tau)$.*
2) *If H is regular in G, then H is sharp in G.*

PROOF. 1) Let τ be a type. Then $G^*(\tau) = \sum\{G(\rho) : \rho > \tau\} = = \sum\{G(\rho) : \rho > \tau, \rho \in \mathrm{Tst}(G)\} = \bigcup\{G(\rho) : \rho > \tau, \rho \in \mathrm{Tst}(G)\} = G^\sharp(\tau)$.

2) Let H be regular in G. Then the typeset of H is also linearly ordered. Further,

$$H \cap G^\sharp(\tau) = H \cap \bigcup\{G(\rho) : \rho > \tau, \rho \in \mathrm{Tst}(G)\} = \bigcup\{H \cap G(\rho) : \rho > \tau, \rho \in \mathrm{Tst}(G)\} =$$

$$= \bigcup\{H(\rho) : \rho > \tau, \rho \in \mathrm{Tst}(G)\} = \bigcup\{H(\rho) : \rho > \tau, \rho \in \mathrm{Tst}(H)\} = H^\sharp(\tau).$$

COROLLARY 6.9 (Prochazka). *Let A be a pure subgroup of the completely decomposable group D whose typeset* $\mathrm{Tst}(D)$ *is inversely well-ordered. Then A is completely decomposable.*

PROOF. By Lemma 6.8 A is sharp in D. Hence Corollary 4.12 applies and X is completely decomposable. □

Also [Non87, Theorem 1] follows from Corollary 4.12.

COROLLARY 6.10 (Nongxa). *A τ-homogeneous sharp and pure subgroup of a completely decomposable group is completely decomposable.*

Finally we obtain a variant of the well-known Baer-Kolettis Theorem.

COROLLARY 6.11. *Let X be sharp in D where D is τ-homogeneous. Then X is τ-homogeneous and completely decomposable.*

PROOF. By Lemma 3.3 $\mathrm{Tst}(X) \subseteq \mathrm{Tst}(D) = \{\tau\}$ and by Proposition 4.14 X is completely decomposable. □

7. Some open problems.

QUESTION 7.1. *Is a sharp subgroup of a finite rank Butler group again a Butler group? Let G be a Butler group and H a sharp subgroup.*

By Lemma 3.3 *H has a finite typeset and Butler decompositions. It remains to show that for every type,* $H^\sharp(\tau)/H^*(\tau)$ *is finite.*

1) *It is enough to check that* $H^\sharp(\tau)/H^*(\tau)$ *is finite for* $\tau \in \mathrm{Tst}(H)$ *as for other types* $H^\sharp(\tau) = H(\tau) = H^*(\tau)$.

2) *Without loss of generality* G/H *is a torsion group. If this is not the case replace G by* H_*^G *which is a Butler group, being a pure subgroup of a Butler group.*

QUESTION 7.2. *Considering the diminished role of the regulating subgroups in a gacd-group X, the* ***sigma regulator***

$$\Sigma(X) := \sum\{A : A \text{ is regulating in } X\} = \sum_{\rho\in \mathrm{T_{cr}}(X)} X(\rho)$$

may have increased significance. Investigate the sigma regulator of gacd-*groups.*

QUESTION 7.3. *Is there an intrinsic characterization of generalized almost completely decomposable groups? A generalized almost completely decomposable group X has Butler decompositions and completely decomposable regulating subgroups, also* $X(\tau) \cap X[\tau] = X^\sharp(\tau)$ *for each critical type* τ. *What more is needed to obtain an* gacd-*group?*

QUESTION 7.4. *Let X be a generalized almost completely decomposable group and let A be a regulating subgroup of X. Is A sharp in X?*

QUESTION 7.5. *If X is a generalized almost completely decomposable group which of the groups* $X(\tau)$, $X^\sharp(\tau)$, $X[\tau]$, $X^\sharp[\tau]$, $X/X(\tau)$, $X/X^\sharp(\tau)$, $X/X[\tau]$, $X/X^\sharp[\tau]$ *are again generalized almost completely decomposable groups? (See* [MM92, Theorem 7].*)*

A group G has ***co-Butler-decompositions*** if for all types τ

$$\frac{G}{G[\tau]} = \frac{G^\sharp[\tau]}{G[\tau]} \oplus \frac{H_\tau}{G[\tau]}$$

for some subgroup H_τ and $\frac{G^\sharp[\tau]}{G[\tau]}$ is τ-homogeneous completely decomposable.

QUESTION 7.6. *Do generalized almost completely decomposable groups have co-Butler-decompositions?*

QUESTION 7.7. *Which infinite rank Butler groups are generalized almost completely decomposable groups? Which generalized almost completely decomposable groups are Butler groups of various descriptions?*

QUESTION 7.8. *Is an* gacd*-group X with $X^\sharp(\tau)/X^*(\tau)$ bounded (uniformly bounded by e) necessarily a (e-)*bcd*-group?*

QUESTION 7.9. *There are many different ways in which subgroups of torsion-free Abelian groups can be embedded in the over-group. A scheme is as follows. A functorial subgroup is a functor Φ that assigns to each object G a sub-object $\Phi(G)$ such that for every morphism $f : H \to G$ it is true that $(\Phi(H))f \subseteq \Phi(G)$. The prominent examples of functorial subgroups in torsion-free Abelian groups are the various socles and radicals, including the identity functor as an extreme case. Let Φ and Ψ be two functorial subgroups with $\Phi \leq \Psi$ in the obvious sense. We obtain a special embedding of a subgroup H of G by requiring that $\Phi(H) = = \Psi(H) \cap \Phi(G)$. Which of the many variants are of significance in Abelian group theory?*

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