REND. SEM. MAT. UNIV. PADOVA, Vol. 113 (2005)

On even surfaces of general type with $K^2 = 8,\ p_g = 4,\ q = 0.$

P. A. OLIVERIO (*)

ABSTRACT - In this article we describe the moduli space of the minimal surfaces of general type S with $K^2 = 8,\ p_g = 4,\ q = 0$, K divisible by 2 and the canonical linear system $|K|$ base point free, showing that it forms an irreducible open set of the moduli space of minimal surfaces with $K^2 = 8,\ p_g = 4,\ q = 0$. By previous results of Ciliberto [Ci] and Ciliberto, Francia, Mendes Lopes [CFM], the above result shows that the moduli space of surfaces with $K^2 = 8,\ p_g = 4,\ q = 0$ has at least three connected components.

1. Introduction.

Surfaces with $K^2 = 8,\ p_g = 4,\ q = 0$ were considered by Enriques [En],Chap 8 pag 284 (see also [Ca] for a related discussion); his proposal was carried through by Ciliberto [Ci] who showed the existence of an algebraic family $\mathcal{K}(8)$ of surfaces of degree 8 in the projective space $\mathbb{P}^3$ such that the generic element F in $\mathcal{K}(8)$ is a canonical surface with $K^2 = 8,\ p_g = 4,\ q = 0$, with ordinary singularities without exceptional curves of the first kind; moreover any such surface is in $\mathcal{K}(8)$; $\mathcal{K}(8)$ is irreducible, unirational of dimension 49. Therefore the corresponding component of the coarse moduli space for the surfaces in $\mathcal{K}(8)$ is irreducible and unirational of dimension 34. Successively Ciliberto, Francia, Mendes Lopes [CFM] constructed another family of minimal surfaces with invariants $K^2 = 8,\ p_g = 4,\ q = 0$. Recall that a compact complex smooth surface is called an even surface if its second Stiefel-Withney class $w_2(S)$ vanishes, or equivalently, if the canonical line bundle K is 2-divisible, i.e. $K = 2L$ where L is a line bundle on S. In this article we show the existence of another connected component of the moduli space of even surfaces S with $K^2 = 8,\ p_g = 4,\ q = 0$. Since the condition $K = 2L$ implies that the

(*) Indirizzo dell'A.: Dipartimento di Matematica, Università della Calabria, I-87036 Arcavacata di Rende (Italy). E-mail address: oliverio@mat.unical.it

intersection form on S is even and in particular $K^2 = 4L^2$, it follows that among the minimal regular surfaces of general type which are even, ours are the ones with the smallest possible value of K^2. In order to introduce the main theorem, we consider the weighted projective space $\mathbb{P}(1,1,2,3,3)$ with weighted homogeneous coordinates (X_1, X_2, Y, Z_1, Z_2), the point $P = (0,0,1,0,0)$ and the line $H = \{X_1 = X_2 = Y = 0\}$ in $\mathbb{P}(1,1,2,3,3)$. Our main theorem is the following:

THEOREM 1.1. (Main Theorem) *The canonical model of an even surface of general type with $K^2 = 8$, $p_g = 4$, $q = 0$ and $|K|$ base point free is a weighted complete intersection X of type* $(6,6)$ *in the weighted projective space* $\mathbb{P}(1,1,2,3,3)$ *with $P \notin X$ and $H \cap X = \emptyset$. Conversely, a weighted complete intersection X of type* $(6,6)$ *with at most R.D.P.'s as singularities which does not intersect the line H and such that $P \notin X$, is the canonical model of an even surface S with $K^2 = 8$, $p_g = 4$, $q = 0$ and $|K|$ base point free.*

We have the following corollary:

COROLLARY 1.2. *The even surfaces of general type with $K^2 = 8$, $p_g = 4$, $q = 0$ and $|K|$ base point free form an irreducible, unirational open set of dimension 35 in their moduli space.*

Note that Horikawa [Ho2] and subsequently Konno [Ko] made a very deep study of some classes of even surfaces of general type.

Acknowledgments. The author is very grateful to Fabrizio Catanese for help and useful discussions. Finally the author would like to thank the referee for valuable suggestions.

Notation.

All varieties are projective and defined over the complex numbers. $H^i(Y,F)$ denotes the cohomology of a coherent sheaf F on the variety Y. We put $h^i(Y,F) = \dim H^i(Y,F)$. On a smooth variety, we do not distinguish between line bundles and divisors. If Y is a Gorenstein variety, then K_Y or ω_Y denotes the canonical divisor of Y. S will be a smooth surface with $K_S^2 = 8$, $p_g = 4$, $q = 0$ and $K_S = 2L_S$, where L_S is a line bundle on S, $p_g := h^0(S, K_S)$ is the geometric genus and $q := h^1(S, \mathcal{O}_S)$ is the irregularity of S. The line bundles K_S and L_S will be denoted by K and L, respectively. $\mathbb{P}(1,1,2,3,3)$ is the weighted projective space $Proj\mathbb{C}[X_1,X_2,Y,Z_1,Z_2]$, where $\deg X_1 = \deg X_2 = 1$, $\deg Y = 2$, $\deg Z_1 = \deg Z_2 = 3$.

2. The semicanonical divisor.

Let S be a minimal surface of general type defined over the complex numbers. We will suppose that S is even. Then there exists a line bundle L such that $K = 2L$. In particular we have that S is minimal and the intersection form is even. Since S is minimal of general type, the canonical divisor K is nef and big. By $K = 2L$, we have that also L is nef and big. Since L^2 is even and positive, $K^2 = 8$ gives the smallest possible value to $K^2 = 4L^2$. We shall assume in the sequel that S is an even surface with

$$K^2 = 8,\ p_g = 4,\ q = 0.$$

We recall some general facts about our surface S. The canonical ring of S is defined as the graded ring

$$R(S,K) := \bigoplus_{n\geq 0} H^0(S,nK).$$

We call semicanonical ring the following graded ring

$$R(S,L) := \bigoplus_{n\geq 0} H^0(S,nL).$$

Since $K = 2L$, one has $R(S,K) = R(S,L)^{(2)}$, where

$$R(S,L)^{(2)} := \bigoplus_{m\geq 0} H^0(S,2mL).$$

From [EGA], Proposition 2.4.7, there exists a canonical isomorphism

$$Proj\ R(S,K) \cong Proj\ R(S,L).$$

Let $X := Proj\,R(S,L) \cong Proj\,R(S,K)$ be the canonical model of S, and let $\boldsymbol{\pi} : S \longrightarrow X$ be the natural morphism. X is a normal surface with at most Rational Double Points as singularities and with an invertible dualizing sheaf ω_X. Let K_X be an associated Cartier divisor to ω_X, then $K = \boldsymbol{\pi}^* K_X$. Therefore, one has a natural isomorphism between the canonical graded rings $R(S,K)$ and $R(X,K_X)$. Our task is to calculate generators and relations of the graded ring

$$R(S,L) := \bigoplus_{n\geq 0} H^0(S,nL).$$

We start with the cohomology of L.

LEMMA 2.1. *We have $h^0(S,L)=2$, and $h^1(S,kL)=0$ for all $k\in\mathbb{Z}$.*

PROOF. The Riemann-Roch theorem implies $2h^0(L)-h^1(L)=4$, then one gets $2\leq h^0(L)$. Since $4=h^0(2L)\geq 2h^0(L)-1$, it follows that $h^0(L)=2$ and $h^1(L)=0$. Since S is regular, $h^1(S,K)=h^1(S,2L)=0$. Since L is nef and big, the Ramanujam's vanishing theorem [Ra2] gives that $h^1(S,(2-k)L)=0$ for all $k\geq 3$. Then, using Serre duality, it follows that $h^1(S,kL)=0$ for all $k\in\mathbb{Z}$. Q.E.D.

LEMMA 2.2. *If $k\geq 3$, then $h^0(S,kL)=k^2-2k+5$.*

PROOF. Riemann-Roch gives

$$h^0(S,\mathcal{O}_S(kL))=\frac{k^2L^2-2kL^2}{2}+(\mathcal{O}_S)-h^0((2-k)L)=$$

$$=k^2-2k+5+h^0((2-k)L).$$

This proves the lemma, since $h^0((2-k)L)=0$, for each $k\geq 3$. Q.E.D.

We recall the following definition

DEFINITION 2.3. *An effective divisor D is numerically k-connected if $D_1D_2\geq k$ for every decomposition $D=D_1+D_2$ with $D_1>0$, $D_2>0$.*

REMARK 2.4. From [Bo], we known that every divisor $D\in|K|$ is 2-connected. From $|K|=|2L|$, it follows that every $C\in|L|$ is 1-connected. For an effective 1-connected divisor D on S, Ramanujam's Lemma [Ra1] gives $h^0(D,\mathcal{O}_D)=1$.

LEMMA 2.5. *Let S be an even surface of general type, and suppose that it admits a morphism $f\colon S\to B$ onto a smooth curve B of genus b whose general fiber is a smooth curve of genus 2. Then we have the relation*

$$K^2=2(\mathcal{O}_S)-6(\mathcal{O}_B).$$

PROOF. See [Ko], Lemma 1.3, pag 17 Q.E.D.

LEMMA 2.6. *Let S be a even surface of general type with $K^2=8$, $p_g=4$ and $q=0$. Then S does not have a base point free pencil of curves of genus 2.*

PROOF. Since $q=0$, every pencil on S is rational. Then from Lemma 2.5, we should have $K^2=2p_g-4$, which gives a contradiction. Q.E.D.

We have the following:

PROPOSITION 2.7. *If S is an even surface with $p_g=4$, $K^2=8$, $q=0$, then the canonical system $|K|=|2L|$ is not composed with a pencil.*

PROOF. See [Ho3], Theorem 1.1. Q.E.D.

REMARK 2.8. We remark that since $h^0(S,L)=2$, the general element of the movable part $|D|$ of the pencil $|L|$ is irreducible.

3. $|L|$ has no fixed part.

We first consider the case where $|L|$ has no fixed part. Since $L^2=2$, the general element $C\in|L|$ is a nonsingular curve of genus 4. Moreover the linear system $|L|$ has two base points P_1,P_2, possibly with P_2 infinitely near to P_1.

LEMMA 3.1. *Let C be a general curve in the linear system $|L|$. Then C is a nonsingular nonhyperelliptic curve of genus 4.*

PROOF. The exact sequence

$$0\longrightarrow\mathcal{O}_S\longrightarrow\mathcal{O}_S(L)\longrightarrow\mathcal{O}_C(L)\longrightarrow 0$$

shows that $h^0(\mathcal{O}_C(L))=1$. If C is nonsingular, then $\mathcal{O}_C(L)=\mathcal{O}_C(P_1+P_2)$. Moreover we get $\omega_C=\mathcal{O}_C(3(P_1+P_2))$. Suppose by contradiction that C is hyperelliptic. Let P_1' be a point such that P_1+P_1' is a divisor of the canonical involution g_2^1 on C. We get $3P_1+3P_1'\in 3g_2^1$ and $3P_1+3P_2\in 3g_2^1$. Then $3P_2$ is linearly equivalent to $3P_1'$. Now Clifford Theorem yields $h^0(3P_2)=h^0(3P_1')\le 2$. Since $P_1'\ne P_2$(otherwise $h^0(C,\mathcal{O}_C(P_1+P_2)=2)$ we must have $h^0(3P_2)=h^0(3P_1')=2$. Then $P_1'=P_2$, and this is a contradiction. Q.E.D.

LEMMA 3.2. *If $|L|$ has no fixed part, the linear system $|3L|=|K+L|$ yields a birational morphism.*

PROOF. Let $x \neq y$ be points which belong to the set U which is the complement of the singular or hyperelliptic curves of $|L|$. If $\phi_{|3L|}(x) = \phi_{|3L|}(y)$, then there is a nonsingular nonhyperelliptic curve $C \in |L|$ such that $x, y \in C$. Now consider the long exact sequence of cohomology

$$0 \to H^0(S, \mathcal{O}_S(2L)) \to H^0(S, \mathcal{O}_S(3L)) \to H^0(C, \mathcal{O}_C(3L)) \to H^1(S, \mathcal{O}_S(2L)),$$

associated to the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_S(2L) \longrightarrow \mathcal{O}_S(3L) \longrightarrow \mathcal{O}_C(3L) = \omega_C \longrightarrow 0.$$

Since $H^1(S, \mathcal{O}_S(2L)) = 0$, one obtains that the map $H^0(S, \mathcal{O}_S(3L)) \longrightarrow$ $\longrightarrow H^0(C, \mathcal{O}_C(3L))$ is surjective. Then the morphism $\phi_{|\omega_C|}$ is the restriction of $\phi_{|3L|}$ to C. Since C is nonhyperelliptic, we have $\phi_{|\omega_C|}(x) \neq \phi_{|\omega_C|}(y)$, and this is contradiction. We have proved that $\phi_{|3L|}$ is birational. If $x \in S$ is a base point of $|3L|$, we have $x \in C$ since $3C \in |3L|$. Therefore x is a base point of $|\mathcal{O}_C(3L)| = |\omega_C|$, but $|\omega_C|$ is free from base points, thus $\phi_{|3L|}$ is a morphism. Q.E.D.

We return now to the study of the canonical linear system $|K_S|$. The exact sequence

$$0 \longrightarrow \mathcal{O}_S(L) \longrightarrow \mathcal{O}_S(2L) \longrightarrow \mathcal{O}_C(2L) \longrightarrow 0,$$

induces the exact sequence of cohomology

$$0 \to H^0(S, \mathcal{O}_S(L)) \to H^0(S, \mathcal{O}_S(2L)) \to H^0(C, \mathcal{O}_C(2L)) \to H^1(S, \mathcal{O}_S(L)) = 0.$$

Now $h^0(S, \mathcal{O}_S(L)) = 2$ and $h^0(S, \mathcal{O}_S(2L)) = 4$, so we can choose a basis $\{x_1, x_2\}$ of $H^0(S, \mathcal{O}_S(L))$ such that $\{x_1^2, x_1x_2, x_2^2, y\}$ is a basis of $H^0(S, \mathcal{O}_S(2L))$, with $y \notin Sym^2\, H^0(S, \mathcal{O}_S(L))$, where, if we set $C = div(x_1)$, the restrictions of x_2^2, y to C give a basis of $H^0(C, \mathcal{O}_C(2L))$. From the above basis of $H^0(S, \mathcal{O}_S(2L))$, it follows that the image of the rational map $\phi_{|K|}$ is a quadric cone in $\mathbb{P}^3$. Furthermore we can choose x_1 such that $div(x_1) = C$ is a nonsingular nonhyperelliptic curve. Now consider the canonical system $|K|$ of S. Let $\boldsymbol{\pi} : S_1 \longrightarrow S$ be a composition of blowing ups such that the movable part $|N|$ of $|\boldsymbol{\pi}^* K|$ is base point free. We assume that $\boldsymbol{\pi}$ is the minimal one among such compositions.

LEMMA 3.3. *We have $N^2 = 8$ or 6. If $N^2 = 8$, then $|K|$ is free of base points. While in the case $N^2 = 6$, the canonical system $|K|$ has two base points Q_1, Q_2 with Q_2 infinitely near to Q_1.*

PROOF. First note that, since the general element C of $|L|$ is irreducible and reduced with $2C \in |K|$, $|K|$ has no fixed part. Let E be the fixed part of $|\boldsymbol{\pi}^*K|$. Then

$$|\boldsymbol{\pi}^*K| = |N| + E,$$

where $\boldsymbol{\pi}^*KE = 0$. Thus $8 = \boldsymbol{\pi}^*K\boldsymbol{\pi}^*K = \boldsymbol{\pi}^*K(N+E) = \boldsymbol{\pi}^*KN = N^2 + NE = = N^2 - \sum m_i^2$, where m_i are the multiplicities of the base points in $|K|$. Since $|K|$ is not composed with a pencil, we have $4 \leq N^2 \leq 8$ (cf. [Hol]). The case $N^2 = 8$ corresponds to $|K|$ being base point free. The cases $N^2 = 5, 7$ are excluded, since the image of the canonical rational map $\phi_{|K|}$ is the quadric cone in $\mathbb{P}^3$. $N^2 = 4$ implies that the degree of the map $\phi_{|K|_{|C}} = \phi_{|K_{|C}|}$ is equal to two, and this is absurd since the curve C is nonhyperelliptic. If $N^2 = 6$, the only possible multiplicities are $m_1 = m_2 = 1$. Then $|K|$ has two base points Q_1, Q_2 with Q_2 infinitely near to Q_1, otherwise the moving part of $|K_{|C}|$ should give a g_2^1 on C, against the fact that C is nonhyperelliptic. Furthermore we note that $|L|$ has two base points P_1, P_2, where we necessarily have $P_1 = Q_1$ but $P_2 \neq Q_2$, again because C is nonhyperelliptic. Q.E.D.

4. $|L|$ has a fixed part.

We now suppose that $|L|$ has fixed part. Then

$$|L| = |D| + Z, \text{ with } D^2 = 0,\ DZ = 2,\ LZ = 0,\ Z^2 = -2.$$

THEOREM 4.1. *Let $|L| = |D| + Z$, with $Z \neq 0$. Then the canonical system $|K|$ is base point free.*

Before proving the above theorem, we make some remarks on the fixed part Z of $|L|$. First we note that the effective divisor Z is 1-connected. In fact suppose that $Z = Z_1 + Z_2$ with $Z_i > 0$, then $-2 = Z^2 = = Z_1^2 + Z_2^2 + 2Z_1Z_2$. On the other hand, by Hodge index theorem, $K^2 > 0$ and $KZ_1 = KZ_2 = 0$ imply that $Z_1^2 \leq -2$, $Z_2^2 \leq -2$, since the Z_i are positive and the intersection form is even. Then $Z_1Z_2 > 0$ and we have that Z is 1-connected. Using Ramanujam's Lemma [Ra1] it follows that $H^0(Z, \mathcal{O}_Z) = \mathbb{C}$. Further we note that the arithmetic genus of Z is

$$p_a(Z) = 1 + \frac{Z^2 + ZK}{2} = 0,$$

hence, plugging in the formula $(\mathcal{O}_Z) = 1 - p_a(Z)$, we get $h^1(Z, \mathcal{O}_Z) = 0$.

Then Artin's criterion says that invertible sheaves on Z are classified by degree (see [Ba], Lemma 3.4). The invertible sheaf L has the property that $LZ_i = 0$ for every irreducible component Z_i of Z. Moreover, using a corollary of Artin's criterion (see [Ba], Corollary 3.10), one gets that the invertible sheaf L descends to an invertible sheaf on the canonical model X of S. That is there exists an invertible sheaf L_X on X such that $L = \boldsymbol{\pi}^* L_X$, hence $\boldsymbol{\pi}_* L = L_X$ and $K_X = 2L_X$.

PROOF OF THE THEOREM. We note that a base point p of $|K| = |2D + 2Z|$ is also a base point of $|L| = |D| + Z$, then $p \in Z$ since the pencil D is base point free. By Artin's criterion $\mathcal{O}_Z(K) = \mathcal{O}_Z$, therefore, the sections of K are constant on Z, and we get a contradiction if we prove that the restriction homomorphism

$$H^0(S, K) \longrightarrow H^0(Z, \mathcal{O}_Z)$$

is surjective. From the exact sequence

$$0 \to H^0(S, \mathcal{O}_S(2D + Z)) \to H^0(S, K) \to H^0(Z, \mathcal{O}_Z)$$

we are done, if $\dim H^0(S, \mathcal{O}_S(2D + Z)) = 3$. Suppose, by contradiction, that $\dim H^0(S, \mathcal{O}_S(2D + Z)) = 4$; then, using the exact sequence

$$0 \to H^0(S, \mathcal{O}_S(D + Z)) \to H^0(S, 2D + Z) \to H^0(D, \mathcal{O}_D(Z)) \to 0,$$

we obtain $h^0(D, \mathcal{O}_D(Z)) = 2$, i.e. D is a smooth hyperelliptic curve with canonical involution $|Z_{|D}| = g^1_2$. Since $L = \boldsymbol{\pi}^* L_X$ and $L_X = \boldsymbol{\pi}_* L$, one gets

$$L_X = \mathcal{O}_X(\boldsymbol{\pi}_* D)$$

A general element $C \in |L_X|$ is an irreducible curve with a double point at $x = \boldsymbol{\pi}(Z)$, and $\boldsymbol{\pi}_{|D} : D \longrightarrow C$ is the normalization morphism . The canonical involution $|Z_{|D}| = g^1_2$ on D corresponds to $|L_{X_{|C}}|$ on C. But this is an absurd from the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(L_X) \longrightarrow \mathcal{O}_C(L_X) \longrightarrow 0$$

and $H^1(X, \mathcal{O}_X) = H^1(S, \mathcal{O}_S) = 0$. Q.E.D.

REMARK 4.2. Notice that the proof of the theorem gives that the curve C is nonhyperelliptic, i.e. C has normalization D which is nonhyperelliptic. Arguments similar to those of Lemma 3.2 prove that the linear system $|3L_X|$ gives a birational morphism. In fact it is enough to observe that the linear system $|\omega_D|$ is base point free and it yields a birational morphism, since D is nonhyperelliptic.

5. Canonical Ring.

In this section, we are going to calculate generators and relations of the graded ring

$$R(S,L) := \bigoplus_{n\geq 0} H^0(S,nL).$$

Since $R(S,L) \cong R(X,L_X)$, we work on the canonical model X. Here a general member $C \in |L_X|$ is an irreducible curve of arithmetic genus 4. If $|L|$ has no fixed part, then C is a nonsingular nonhyperelliptic curve of genus 4, otherwise in the case $|L|$ has fixed component Z(and then $|K|$ will be base point free), the curve C has a double point at $x = \boldsymbol{\pi}(Z)$ and the normalization D of C is a non singular nonhyperelliptic curve of genus 3. In any case the canonical sheaf of C is $\omega_C = \mathcal{O}_C(3L_X)$. In these circumstances, we can apply the Theorem of Noether (see [Sch] Theorem 1.2) which says that the homomorphism

$$\theta^n : S^n H^0(C,\omega_C) \longrightarrow H^0(C,\omega_C^n) = H^0(C,\mathcal{O}_C(3nL_X))$$

is surjective for all $n \in \mathbb{N}$. By considering the exact sequence

$$0 \longrightarrow H^0(S,\mathcal{O}_X(2L_X)) \longrightarrow H^0(X,\mathcal{O}_X(3L_X)) \longrightarrow H^0(C,\mathcal{O}_C(3L_X)) \longrightarrow 0,$$

we see that $x_1^3, x_1^2x_2, x_1x_2^2, x_1y, x_2^3, x_2y$ are linearly independent. We complete this set to a basis of $H^0(X,\mathcal{O}_X(3L_X))$:

$$\{x_1^3, x_1^2x_2, x_1x_2^2, x_1y, x_2^3, x_2y, z_1, z_2\},$$

where the restrictions of the elements x_2^3, x_2y, z_1, z_2 to C give a basis of $H^0(C,\omega_C)$. Suppose from now on that the canonical system $|K|$ on S is base point free, thus $|K_X|$ is base point free too.

We have the following exact sequence

$$0 \longrightarrow H^0(X,\mathcal{O}_X(3L_X)) \longrightarrow H^0(X,\mathcal{O}_X(4L_X)) \longrightarrow H^0(C,\mathcal{O}_C(4L_X)) \longrightarrow 0.$$

Since $\{x_1 = x_2 = y = 0\} = \emptyset$, we see that $x_2^4, x_2^2y, x_2z_1, x_2z_2, y^2$ are linearly independent in $H^0(C,\mathcal{O}_C(4L_X))$, then, since $h^0(C,\mathcal{O}_C(4L_X)) = 5$,

$$H^0(X,\mathcal{O}_X(4L_X)) = x_1H^0(X,\mathcal{O}_X(3L_X)) \oplus V_1,$$

where V_1 is the vector space with a basis given by

$$x_2^4, x_2^2y, x_2z_1, x_2z_2, y^2.$$

Using the following exact sequence

$$0 \longrightarrow H^0(X,\mathcal{O}_X(4L_X)) \longrightarrow H^0(X,\mathcal{O}_X(5L_X)) \longrightarrow H^0(C,\mathcal{O}_C(5L_X)) \longrightarrow 0.$$

and the surjectivity of the map

$$H^0(C,\mathcal{O}_C(3L_X))\otimes H^0(C,\mathcal{O}_C(2L_X))\to H^0(C,\mathcal{O}_C(5L_X))$$

we see that

$$H^0(X,\mathcal{O}_X(5L_X))=x_1H^0(X,\mathcal{O}_X(4L_X))\oplus V_2,$$

where V_2 is the vector space with a basis given by

$$x_2^5, x_3^2y, x_2^2z_1, x_2^2z_2, x_2y^2, yz_1, yz_2.$$

Then we have neither new generators, nor relations in degree ≤ 5. We now consider the exact sequence

$$0\longrightarrow H^0(X,\mathcal{O}_X(5L_X))\longrightarrow H^0(X,\mathcal{O}_X(6L_X))\longrightarrow H^0(C,\mathcal{O}_C(6L_X))\longrightarrow 0,$$

where $\mathcal{O}_C(6L_X)=\omega_C^{\otimes 2}$, and the set

$$\mathcal{B}=\{x_1^{i_1}x_2^{i_2}y^{i_3}z_1^{i_4}z_2^{i_5}|i_1+i_2+2i_3+3i_4+3i_5=6, i_j\geq 0, j=1,2,3,4,5\}.$$

Since $\mathcal{B}$ has 31 elements and $h^0(X,\mathcal{O}_X(6L_X))=29$, we see that there are two independent relations between elements of $\mathcal{B}$. To find these relations, we proceed as follows. First of all, one finds that the homomorphism

$$\theta^2: Sym^2H^0(C,\omega_C)\longrightarrow H^0(C,\omega_C^{\otimes 2})$$

has kernel of dimension 1. Thus we have that x_2, y, z_1, z_2 satisfy a quadratic relation $Q_2(x_2^3, x_2y, z_1, z_2)=0$ on C. Moreover we find another relation $y^3=Q_2'(x_2^3, x_2y, z_1, z_2)$ on C, since y^3 is in the image of θ^2. As the restriction maps

$$H^0(X,\mathcal{O}(nL_X))\longrightarrow H^0(C,\mathcal{O}_C(nL_X))$$

are surjective for all $n\geq 0$, it is possible to use the section principle of M. Reid [Re]. Therefore we obtain the two independent relations between x_1, x_2, y, z_1, z_2 on X :

$$F_6(x_1,x_2,y,z_1,z_2)=0,\ G_6(x_1,x_2,y,z_1,z_2)=0,$$

where

$$F_6(X_2^3,X_2Y,Z_1,Z_2)=Y^3-Q_2'(X_2^3,X_2Y,Z_1,Z_2)+X_1f_5(X_1,X_2,Y,Z_1,Z_2),$$

$$G_6(X_2^3,X_2Y,Z_1,Z_2)=Q_2(X_2^3,X_2Y,Z_1,Z_2)+X_1g_5(X_1,X_2,Y,Z_1,Z_2)$$

are two weighted homogeneous polynomials of degrees 6 in the variables X_1, X_2, Y, Z_1, Z_2 with $\deg X_1=1$, $\deg X_2=1$, $\deg Y=2$, $\deg Z_1=3$,

$\deg Z_2 = 3$. We consider the graded ring

$$Q = \mathbb{C}[X_1, X_2, Y, Z_1, Z_2],$$

and the homomorphism

$$\phi : Q \longrightarrow R(S, L)$$

defined by $\phi(X_1)=x_1$, $\phi(X_2) = x_2, \phi(Y)=y, \phi(Z_1)=z_1$, $\phi(Z_2)=z_2$. Obviously $Ker\phi$ contains F_6 and G_6.

PROPOSITION 5.1. *The induced homomorphism*

$$\tilde{\phi} : Q/(F_6, G_6) \longrightarrow R(X, L_X)$$

is an isomorphism.

PROOF. First we prove that $R(X, L_X)$ and $Q/(F_6, G_6)$ have the same Hilbert series. For $R(X, L_X)$ we have

$$H_{R(X,L_X)}(t) = 1 + 2t + 4t^2 + 8t^3 + \ldots + (n^2 - 2n + 5)\, t^n + \ldots$$

Note that F_6 and G_6 does not have common factor, thus they form a regular sequence in Q. Then we can consider the exact sequence(Koszul complex) associated to (F_6, G_6):

$$0 \to Q(-12) \to Q(-6) \oplus Q(-6) \to Q \to Q/(F_6, G_6) \to 0,$$

and we find that

$$H_{Q/(F_6,G_6)}(t) = \frac{(1-t^6)(1-t^6)}{(1-t)(1-t)(1-t^2)(1-t^3)(1-t^3)}.$$

By an easy calculation, it follows

$$H_{Q/(F_6,G_6)}(t) = H_{R(S,L)}(t) \quad \text{for every t.}$$

In order $\tilde{\phi}$ to be an isomorphism, it is now enough to show that $\tilde{\phi}$ is injective. The birational morphism $\phi_{|3L_X|}$ lifts to a morphism $\xi : X \longrightarrow \Sigma \subset \mathbb{P}(1,1,2,3,3)$, where Σ is defined by $F_6 = G_6 = 0$. Since $deg\xi(X) = L_X^2 = deg\Sigma$, then ξ is dominant and $\tilde{\phi}$ is injective. Q.E.D.

The proposition 5.1 shows that the canonical model $X = Proj(R(S,K))$ of S is isomorphic to the weighted complete intersection $\Sigma \subset \mathbb{P}(1,1,2,3,3)$ which is defined by the polynomials F_6 and G_6. In $\mathbb{P}(1,1,2,3,3)$, we consider the line $H = \{X_1 = X_2 = Y = 0\}$ and the point $P = (0,0,1,0,0)$ which give the singular locus of $\mathbb{P}(1,1,2,3,3)$ [IF]. We have the following theorem

THEOREM 5.2. *If S is an even surface with $K^2 = 8$, $p_g = 4$, $q = 0$ and $|K|$ base point free, then Σ does not intersect the line H and $P \notin \Sigma$. Conversely, if Σ is a weighted complete intersection of type $(6,6)$ with at most rational double points as singularities, which does not intersect the locus $H \cup P$, then Σ is the canonical model of a even surface S with $K^2 = 8$, $p_g = 4$, $q = 0$ and $|K|$ base point free.*

PROOF. Σ is a Gorenstein surface with at most rational double points, since it is isomorphic to the canonical model $ProjR(S,K)$. $F_6(P) \neq 0$ implies $P \notin \Sigma$, while $L \cap \Sigma = \emptyset$, since $|K|$ is base point free. Viceversa, let Σ be a weighted complete intersection of type $(6,6)$ in $\mathbb{P}(1,1,2,3,3)$, such that $H \cap \Sigma = \emptyset$ and $P \notin \Sigma$. Then, if Σ has at most rational double points as singularities(a sufficiently general weighted complete intersection of type $(6,6)$ has at most rational double points as singularities, see [IF]), $\omega_\Sigma = \mathcal{O}_\Sigma(2)$ is an invertible sheaf. Let $\pi\colon S \to \Sigma$ be the minimal resolution of singularities of Σ. By standard calculations $K_S^2 = 8$, $p_g(S) = 4$, $(\mathcal{O}_S) = 5$, whence $q(S) = 0$. Moreover $K_S = 2\pi^* L_X$, where $L_X = \mathcal{O}_\Sigma(1)$ is invertible since $\Sigma \subset \mathbb{P}(1,1,2,3,3) \setminus \{H \cup P\}$. Finally, the linear system $|K|$ is base point free since Σ does not intersect the complex line H. Q.E.D.

As a corollary we have the following

COROLLARY 5.3. *The even surfaces of general type with $K^2 = 8$, $p_g = 4$, $q = 0$ and $|K|$ base point free form an irreducible, unirational open set of dimension 35 in their moduli space $\mathcal{M}_{ev}$. This non empty open set gives rice to a new connected component of the moduli space of minimal surfaces of general type with $K^2 = 8$, $p_g = 4$, $q = 0$.*

PROOF. In fact for S to be even is equivalent to have the second Stiefel-Withney class $w_2(S)$ equal to zero. Since the condition $w_2(S) = 0$ is a topological condition, $\mathcal{M}_{ev}$ is a union of connected components of the moduli space of the minimal surfaces of general type with $K^2 = 8$, $p_g = 4$, $q = 0$. Consider now the set U of weighted complete intersections Σ of type $(6,6)$ in the weighted projective space $\mathbb{P}(1,1,2,3,3)$ with at most rational double points as singularities and such that $\Sigma \cap H = \emptyset$ and $P \notin \Sigma$. U is a Zariski open subset of the Grassmannian $\mathbb{G}(2, H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(6))$, since for Σ to have at most rational double points as singularities is an open condition, the condition $\Sigma \cap H = \emptyset$ means that two quadratic forms on the projective line $\mathbb{P}^1$ do not have common roots, which is an open condition too and, of course, the condition $P \notin \Sigma$ is open. It is clear that a Zariski open subset of the moduli space of even surfaces with $K^2 = 8$, $p_g = 4$, $q = 0$ is obtained taking the quotient of U with respect to the natural action of the automorphism group of $\mathbb{P}(1,1,2,3,3)$. Therefore we get an irreducible, unirational set. Making a count of constants, we find that the dimension of this irreducible component is equal to

$$\dim \mathbb{G}(2, H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(6)) - \dim \operatorname{Aut} \mathbb{P}(1,1,2,3,3) = 58 - 23 = 35.$$

(Note that the automorphism group of a surface of general type is finite). Q.E.D.

REMARK 5.4. We note that the moduli space of minimal surfaces of general type with $K^2 = 8$, $p_g = 4$, $q = 0$ has at least three connected components. One and only one contains the surfaces with the second Stiefel-Withney class $w_2(S) = 0$, which is a topological condition, another contains only the surfaces with torsion $\mathbb{Z}/2$, the component of Ciliberto, Francia, Mendes Lopes [CFM], and finally there is the component of Ciliberto [Ci], which is different from the above components since a canonical surface cannot have the canonical line bundle K divisible by 2.

REFERENCES

[Ba] L. BADESCU, *Algebraic Surfaces*, UTX, Springer-Verlag 2001.

[Bo] E. BOMBIERI, *Canonical models of surfaces of general type*, Publ. Math. IHES, **42** (1973), pp. 171-219.

[Ca] F. CATANESE, *Homological Algebra and Algebraic Surfaces*, A.M.S. Proc. Symp. Pure Math., **62**, 1 (1997), pp. 3-56.

[Ci] C. CILIBERTO, *Canonical surfaces with $p_g = p_a = 4$ and $K^2 = 5, \ldots, 10$*, Duke Math. J. 48, (1981), pp. 1-37.

[CFM] C. CILIBERTO - P. FRANCIA - M. MENDES LOPES, *Remarks on the bicanonical map for surfaces of general type*, Math. Zeit **224** (1997), pp. 137-166.

[En] F. ENRIQUES, *Le Superficie Algebriche*, Zanichelli, Bologna (1949).

[EGA] A. GROTHENDIECK, *Elements de géométrie algébrique*, Chap II. Publ. Math. de l'IHES 8 (1961).

[Ho1] E. HORIKAWA, *On deformations of quintic surfaces*, Invent. Math, **31** (1975), pp. 43-85.

[Ho2] E. HORIKAWA, *Deformations of Sextic Surfaces*, Topology, **32** (1993), pp. 757-772.

[Ho3] E. HORIKAWA, *Algebraic surfaces of general type with small c_1^2. III*, Invent. Math, **47** (1978), pp. 209-248.

[IF] A.R. IANO-FLECHTER, *Working with weighted complete intersections*, Explicit birational geometry of 3-folds, A. Corti and M. Reid (editors) Cambridge Univ. Press, 2000.

[Ko] K. KONNO, *Even surfaces with* $p_g = 7, q = 0$ *and* $K^2 = 16$, Math. Rep. Kyushu University, **18-1** (1991), pp. 15-41.

[Ra1] C. P. RAMANUJAM, *Remarks on the Kodaira vanishing theorem*, Journal of the Indian Math. Soc., **36** (1972), pp. 41-51.

[Ra2] C. P. RAMANUJAM, *Supplement to the article «Remarks on the Kodaira vanishing theorem»*, Journal of the Indian Math. Soc., **38** (1974), pp. 121-124.

[Re] M. REID, *Infinitesimal view of extending a hyperplane section, in Hyperplane sections and related topics* (L'Aquila 1988), LNM 1417 (1990).

[Sch] F. O. SCHREYER, *A standard basis approach to syzygies of canonical curves*, J. Reine Angew. Math., **421** (1991), pp. 83-123.

Manoscritto pervenuto in redazione il 15 ottobre 2003
e nella versione definitiva il 15 settembre 2004.