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Divergence Measure Fields and Cauchy's Stress Theorem.

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ABSTRACT - Divergence measure fields are integrable vector fields whose distributional divergence is a measure. Some versions are derived of the divergence theorem for divergence measure fields on sets of finite perimeter. Using these results, it is shown that Cauchy fluxes from the theory of Cauchy's stress theorem can be extended to a class of surfaces that includes singular surfaces of continuum mechanics (shock waves and phase boundaries). On the singular surfaces, the divergence of the stress has a surface delta type singularity, with tractions on a surface and its opposite different from each other.

1. Introduction.

Cauchy's stress theorem asserts that the force $\boldsymbol{f}(S)$ exerted by one part of a continuous body on another part through a surface S of contact is expressed by

$$\boldsymbol{f}(S) = \int_S \boldsymbol{T}\boldsymbol{n}\, dA$$

where $\boldsymbol{n}$ is the normal to S, dA is the element of area of S, and $\boldsymbol{T}$, the main object of the stress theorem, is the stress tensor ([1]). Cauchy's derivation was heuristic, with unnecessary additional assumptions. Noll [19] raised the question of a rigorous derivation under minimal assumptions. Basic properties of interactions in a body were formalized in the concept of a Cauchy flux ([2]) in [13] and using this notion, the proof of the Cauchy stress

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([1]) Similarly, the flux of heat $F(S)$ through a surface S is given by the heat flux vector $\boldsymbol{q}$ via $F(S) = \int_S \boldsymbol{q} \cdot \boldsymbol{n}\, dA$.

([2]) See Section 5.

theorem was given under natural, albeit still restrictive assumptions (3). These ideas were then adapted to the context of sets of finite perimeter [3, 29, 24, 14, 20] (4). The assumptions of [13] lead to bounded stress fields, and in [29] it was shown that the distributional divergence div $\boldsymbol{T}$ of $\boldsymbol{T}$ is a bounded (integrable) function; see also [1]. Clearly, there are many situations where the boundedness of $\boldsymbol{T}$ and of div $\boldsymbol{T}$ are violated. Unbounded stress fields occur in fracture mechanics and in the existence theorems in nonlinear elasticity; div $\boldsymbol{T}$ has a (surface) δ type singularity on singular surfaces (shock waves and propagating phase boundaries). Cauchy fluxes leading to unbounded stress fields and with div $\boldsymbol{T}$ an (unbounded) integrable function are treated in [24-25]. In [25] it was shown that at this generality the Cauchy flux can be defined only for «almost all» surfaces S. This covers unbounded stresses but excludes singular surfaces. The extension to stress fields arising in the presence of singular surfaces is in [7, 15-18]. These works consider Cauchy fluxes for which the resulting stress field may be unbounded with distributional divergence a Radon measure. Following [7], such tensor/vector fields are called divergence measure fields in the subsequent treatment (5). However, the concept of almost every surface adopted in [7, 15-18] excludes surfaces where div $\boldsymbol{T}$ is not absolutely continuous with respect to Lebesgue's measure, in particular, the Cauchy flux is generally undefined on singular surfaces.

This paper (i) derives some new properties of the divergence measure fields including versions of the divergence theorem for them and (ii) uses (i) to extend Cauchy fluxes to a class of surfaces which includes the singular surfaces. To simplify the description, we switch from vector valued Cauchy fluxes (such as force) to scalar valued ones (such as the heat flux) (6); accordingly, the stress tensor $\boldsymbol{T}$ changes to the flux vector $\boldsymbol{q}$. The results may be extended to vector valued fluxes by components.

The divergence measure vector fields are locally integrable fields $\boldsymbol{q}:\boldsymbol{R}^n\to\boldsymbol{R}^n$ whose distributional divergence div $\boldsymbol{q}$ is a Radon measure. The paper first briefly addresses the question of the nature of the measure div $\boldsymbol{q}$ in Section 3. It is shown that if $\boldsymbol{q} \in L^p_{\mathrm{loc}}(\boldsymbol{R}^n)$ then for $1 \le p < n/(n-1)$

(3) See below.

(4) A variational approach to Cauchy's theorem is developed in [11], and an approach based on Whitney's geometric integration theory [28] is outlined in [21]. See also [23].

(5) The spaces of fields with integrable distributional divergence are treated in many works, e.g., [2, 12]; papers [4–6] consider divergence measure fields.

(6) See footnote (1) above.

any Radon measure can arise as div $\boldsymbol{q}$, while if $n/(n-1) \le p \le \infty$, then the singularities of div $\boldsymbol{q}$ are not be arbitrary: div $\boldsymbol{q}$ vanishes on sets of Hausdorff dimension $m \le n - p/(p-1)$ if p is finite and of dimension $m < n - 1$ if $p = \infty$; moreover, in the latter case div $\boldsymbol{q}$ also vanishes on each set of $n-1$ dimensional Hausdorff measure 0.

Next, several versions of the divergence theorem are given for divergence measure vector fields and sets of finite perimeter. If $\boldsymbol{q}$ is a smooth vector field and φ a smooth scalar field with compact support then the divergence theorem reads

$$\int_{\partial M} \varphi \boldsymbol{q} \cdot \boldsymbol{n}^M d\mathscr{H}^{n-1} = \int_M D\varphi \cdot \boldsymbol{q}\, d\mathscr{L}^n + \int_M \varphi \boldsymbol{q}\ \operatorname{div} \boldsymbol{q}\, d\mathscr{L}^n \tag{1.1}$$

for any normalized set of finite perimeter $M \subset \boldsymbol{R}^n$ with the measure theoretic boundary ∂M and the measure theoretic normal $\boldsymbol{n}^M$ (see Section 2 for definitions). For divergence measure vector fields the right hand side generalizes to

$$\int_M D\varphi \cdot \boldsymbol{q}\, d\mathscr{L}^n + \int_M \varphi \operatorname{div} \boldsymbol{q}$$

where the last integral is the integral of a continuous function with respect to the Radon measure div $\boldsymbol{q}$, but the left hand side of (1.1) does not have an immediate meaning. It turns out that for divergence measure vector fields the expression $\boldsymbol{q} \cdot \boldsymbol{n}^M$ cannot be interpreted pointwise, i.e., the left hand side of (1.1) must be interpreted as a functional on scalar fields φ on ∂M, [4-6]; this occurs even when the distributional divergence div $\boldsymbol{q}$ is an integrable function [26; Theorem 1.2, Chapter I]. It is shown that such a functional exists for every set of finite perimeter (Proposition 4.1, generalizing [6] to sets of finite perimeter); this functional is called the normal trace ([7]) of $\boldsymbol{q}$. If $\boldsymbol{q}$ is «bounded» near ∂M (see the definition of domination in Section 4), the normal trace has additional properties. Theorem 4.2 gives two conditions under which the normal trace is a measure; one of these guarantees a measure with support on the closure $\overline{\partial M}$ of the measure theoretic boundary; the other guarantees a measure supported on ∂M but requires $\boldsymbol{q} \in L^p_{\mathrm{loc}}(\boldsymbol{R}^n)$ with $p \ge n/(n-1)$ and regular boundary in some measure theoretic sense. Finally, Theorems 4.4 and 4.6 give conditions which guarantee that the

([7]) [6].

normal trace is of the form

$$\int_{\partial M} \varphi\, q^M d\mathscr{H}^{n-1} \tag{1.2}$$

where q^M is an $\mathscr{H}^{n-1}$ integrable function. Theorem 4.4 deals with bounded vector fields ($\boldsymbol{q} \in L^\infty(\boldsymbol{R}^n)$) and shows that then the normal trace is as in (1.2) with $q^M \in L^\infty(\partial M)$ for any set of finite perimeter; this has been proved in [2; Theorem 1.9] for open sets with Lipschitz boundary. Theorem 4.6 deals with a general $\boldsymbol{q}$ and proves (1.2) for sets M whose boundaries do not intersect some exceptional set of $n-1$ dimensional Hausdorff measure 0 where div $\boldsymbol{q}$ is too singular.

Using the divergence theorem 4.6, it is shown that every Cauchy flux that is defined for almost every surface in the sense of [7, 15-18] can be automatically extended to a class of surfaces where the measure div $\boldsymbol{q}$ has a surface type singularity, thus including the singular surfaces. Clearly, this extended Cauchy flux reflects more fully the properties of the interaction. However, generally the flux cannot be extended to all surfaces, for, firstly, the flux $\boldsymbol{q}$ vector has to satisfy the domination condition as mentioned above, and, secondly the surface cannot intersect the exceptional set where div $\boldsymbol{q}$ is too singular. If $\boldsymbol{q}$ is bounded then the domination condition is automatically satisfied, the exceptional set is void and the Cauchy flux can be extended to all surfaces.

After a brief recapitulation of the basic measure theoretic notions in Section 2, Section 3 states some properties of the divergence measure fields. The divergence theorems and Cauchy fluxes are discussed in Sections 4 and 5, respectively. The proofs are given in the rest of the paper.

2. Preliminaries.

For a set $M \subset \boldsymbol{R}^n$ we denote by $M^c := \boldsymbol{R}^n \backslash M$ the complement of M. If $\boldsymbol{x} \in \boldsymbol{R}^n$ and $r > 0$ then $\boldsymbol{B}(\boldsymbol{x}, r)$ is the open ball of radius r and center $\boldsymbol{x}$. $\mathscr{L}^n$ is the (outer) ***Lebesgue measure*** in $\boldsymbol{R}^n$; if $A \subset \boldsymbol{R}^n$ then $|A| \equiv \mathscr{L}^n(A)$ is the Lebesgue measure of A. If $0 \le m < \infty$, we denote by $\mathscr{H}^m$ the m ***dimensional Hausdorff measure*** [10; §§ 2.10.2-6]. Briefly, if $A \subset \boldsymbol{R}^n$ then

$$\mathscr{H}^m(A) := \lim_{\delta \to 0} \mathscr{H}^m_\delta(A)$$

where for any $\boldsymbol{\delta} > 0$ the ***size δ approximation*** $\mathscr{H}^m_\delta(A)$ is defined by

$$\mathscr{H}^m_\delta(A) = \boldsymbol{a}_m \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} C_i/2)^m : C_i \subset \boldsymbol{R}^n,\ A \subset \bigcup_i C_i, \operatorname{diam} C_i < \delta\right\},$$

where $\operatorname{diam} C = \sup\{|\boldsymbol{x}-\boldsymbol{y}| : \boldsymbol{x},\boldsymbol{y} \in C\}$ is the diameter of C,

$$\boldsymbol{a}_m := \boldsymbol{\Gamma}^m(1/2)/\boldsymbol{\Gamma}(m/2+1),$$

and $\boldsymbol{\Gamma}$ is the Euler gamma function. If m is an integer, then $\boldsymbol{a}_m$ is the volume of the unit ball in $\boldsymbol{R}^m$ and $\mathscr{H}^m$ coincides with the m dimensional area on m dimensional manifolds in $\boldsymbol{R}^n$; in particular $\mathscr{H}^n = \mathscr{L}^n$.

If $M \subset \boldsymbol{R}^n$ is $\mathscr{L}^n$ measurable and $\boldsymbol{x} \in \boldsymbol{R}^n$ we say that M has a ***density*** at $\boldsymbol{x}$ if the following limit exists:

$$D(\boldsymbol{x}, M) := \lim_{r\to 0}\frac{|M \cap \boldsymbol{B}(x, r)|}{\boldsymbol{a}_n r^n},$$

which is then called the ***density of*** M ***at*** $\boldsymbol{x}$. A point $\boldsymbol{x} \in \boldsymbol{R}^n$ is said to be a ***point of density*** of M if $D(\boldsymbol{x}, M) = 1$. For a measurable set M we denote by M_* the set of all points of density of M. By Lebesgue's differentiation theorem [27; Theorem (7.2)], M_* is a Borel set and the symmetric difference of M and M_* has $\mathscr{L}^n$ measure 0. We define the ***measure theoretic boundary*** ∂M of a measurable set M by

$$\partial M = \boldsymbol{R}^n \backslash (M_* \cup (M^c)_*),$$

cf. [10; § 4.5.12]. ∂M is a Borel set and $|\partial M| = 0$. We say that a measurable set $M \subset \boldsymbol{R}^n$ is ***normalized*** if $M = M_*$.

By a ***Borel measure*** in $\boldsymbol{R}^n$ we mean any σ additive function $\mu : \mathscr{A} \to [0, \infty]$ whose domain $\mathscr{A}$ is a σ algebra which *contains* all Borel sets in $\boldsymbol{R}^n$. Thus the restrictions of $\mathscr{L}^n$, $\mathscr{H}^m$ to their respective systems of measurable sets are Borel measures. By a (signed) ***Radon measure*** in $\boldsymbol{R}^n$ we mean any σ additive function $\mu : \mathscr{B} \to \boldsymbol{R}$ defined on the σ algebra $\mathscr{B}$ of all Borel sets in $\boldsymbol{R}^n$. By a ***measure*** we mean either a Borel or a Radon measure. If μ is a Radon measure we denote by $|\mu|$ the ***total variation measure,*** which is a nonnegative Radon measure. Then $\|\mu\| = |\mu|(\boldsymbol{R}^n) < \infty$ denotes the ***total variation*** of μ. We denote by $\mathscr{M}(\boldsymbol{R}^n)$ the set of all Radon measures on $\boldsymbol{R}^n$ and by $\mathscr{M}_+(\boldsymbol{R}^n)$ the subset of nonnegative Radon measures on $\boldsymbol{R}^n$.

If $A \subset \boldsymbol{R}^n$ is a Borel set and μ a measure, we denote by $\mu \llcorner A$ the ***restriction of*** μ ***to*** A, i.e., a measure given by

$$\mu \llcorner A(B) = \mu(A \cap B)$$

for any B from the domain of μ. We say that a measure μ is ***supported*** on a Borel set A if $\mu = \mu \llcorner A$. We say that a measure μ ***vanishes*** on a Borel set N if $\mu(A) = 0$ for each $A \subset N$ from the domain of μ. If μ is a measure and $f: \boldsymbol{R}^n \to \boldsymbol{R}$ a Borel function that is integrable with respect to μ, then $f\mu$ denotes the measure given by

$$(f\mu)(A) = \int_A f\, d\mu$$

for each A from the domain of μ. The reader is referred to [10; Chapter 2] for further details of the measure theory.

A measurable set $M \subset \boldsymbol{R}^n$ is said to be a set of ***finite perimeter*** (cf. [10; Theorem 4.5.6]) if the distributional partial derivatives $D_i 1_M$, $1 \le i \le n$, of the characteristic function 1_M of M are Radon measures. M is a set of finite perimeter $\Leftrightarrow$ $\mathscr{H}^{n-1}(\partial M) < \infty$ (cf. [10; Theorem 4.5.11]) $\Leftrightarrow$ there exists a Borel function $\boldsymbol{n}^M: \partial M \to \boldsymbol{S}^{n-1}$, where $\boldsymbol{S}^{n-1}$ is the unit sphere in $\boldsymbol{R}^n$ such that

$$\int_M D\varphi\, d\mathscr{L}^n = \int_{\partial M} \varphi\, \boldsymbol{n}^M d\mathscr{H}^{n-1} \tag{2.1}$$

for each $\varphi \in C_0^\infty(\boldsymbol{R}^n)$. Here $C_0^\infty(\boldsymbol{R}^n)$ is the set of all infinitely differentiable functions with compact support and $D\varphi$ is the gradient of φ. The function $\boldsymbol{n}^M$ is determined by (2.1) uniquely to within a change on an $\mathscr{H}^{n-1}$ negligible subset of ∂M, and is called the ***measure theoretic normal*** of M.

3. Divergence measure fields.

A $\boldsymbol{q} \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ (8) is said to be a ***divergence measure field*** if there exists a $\mu \in \mathscr{M}(\boldsymbol{R}^n)$ such that

$$\int_{\boldsymbol{R}^n} D\varphi \cdot \boldsymbol{q}\, d\mathscr{L}^n = -\int_{\boldsymbol{R}^n} \varphi\, d\mu \tag{3.1}$$

for every $\varphi \in C_0^\infty(\boldsymbol{R}^n)$. One then writes $\operatorname{div} \boldsymbol{q} := \mu$. The space of all diver-

(8) Here $L^p_{\mathrm{loc}}(\boldsymbol{R}^n), 1 \le p \le \infty$, stands for the set of all measurable maps $\boldsymbol{q} : \boldsymbol{R}^n \to \boldsymbol{R}^n$ that are locally integrable with pover p.

gence measure fields is denoted by $L^1_{\rm loc}\mathscr{M}^{\rm div}(\boldsymbol{R}^n)$ and we write $L^p_{\rm loc}\mathscr{M}^{\rm div}(\boldsymbol{R}^n) := L^1_{\rm loc}\mathscr{M}^{\rm div}(\boldsymbol{R}^n) \cap L^p_{\rm loc}(\boldsymbol{R}^n)$ whenever $1 \le p \le \infty$.

For $n = 1$ the space $L^1_{\rm loc}\mathscr{M}^{\rm div}(\boldsymbol{R}^n)$ reduces to the space of functions of bounded variation on $\boldsymbol{R}^1$; throughout the rest of the paper we assume that $n > 1$. The following are archetypical examples of divergence measure fields:

EXAMPLES 3.1 – (i) *(Transversal fields). Let* $f \in L^1_{\rm loc}(\boldsymbol{R}^{n-1})$ *and define* $\boldsymbol{q} \in L^1_{\rm loc}(\boldsymbol{R}^n)$ *by* $\boldsymbol{q}(x_1, \ldots, x_n) = (0, \ldots, 0, f(x_1, \ldots, x_{n-1})),\ x \in \boldsymbol{R}^n$. *Then* $\boldsymbol{q} \in L^1_{\rm loc}\mathscr{M}^{\rm div}(\boldsymbol{R}^n)$ *and* $\operatorname{div}\boldsymbol{q} = 0$.

(ii) *(Singular surface). Let* $\boldsymbol{e}_1$ *be the coordinate vector in the* x_1 *direction and* $\boldsymbol{q}(\boldsymbol{x}) = \boldsymbol{e}_1$ *if* $x_1 > 0$ *and* $\boldsymbol{q}(\boldsymbol{x}) = \boldsymbol{0}$ *if* $x_1 \le 0$, $\boldsymbol{x} \in \boldsymbol{R}^n$. *Then* $\boldsymbol{q} \in L^1_{\rm loc}\mathscr{M}^{\rm div}(\boldsymbol{R}^n)$ *and* $\operatorname{div}\boldsymbol{q} = \mathscr{H}^{n-1} \llcorner \{\boldsymbol{x} \in \boldsymbol{R}^n : x_1 = 0\}$.

If μ is a Radon measure and $0 \le m < \infty$, we say that μ is $\mathscr{H}^m$ ***absolutely continuous*** if $|\mu|(B) = 0$ for every Borel set $B \subset \boldsymbol{R}^n$ with $\mathscr{H}^m(B) = 0$. We say that a Borel set B has σ ***finite*** $\mathscr{H}^m$ ***measure*** if B is a union of countably many Borel sets of finite $\mathscr{H}^m$ measure.

THEOREM 3.2. *Let* $n/(n-1) \le p \le \infty$, $\boldsymbol{q} \in L^p_{\rm loc}\mathscr{M}^{\rm div}(\boldsymbol{R}^n)$, *and set*

$$d := \begin{cases} n - p/(p-1) & \text{if } p < \infty\,, \\ n - 1 & \text{if } p = \infty\,. \end{cases}$$

(i) *If* $p < \infty$, *then* $|\operatorname{div}\boldsymbol{q}|(B) = 0$ *for every Borel set* B *of* σ *finite* $\mathscr{H}^d$ *measure;*

(ii) *if* $p = \infty$ *then* $\operatorname{div}\boldsymbol{q}$ *is* $\mathscr{H}^{n-1}$ *absolutely continuous.*

For the given range of p, the value of d changes monotonically from 0 to $n-1$. The theorem imposes a restriction on the dimensionality of the measure $\operatorname{div}\boldsymbol{q}$: if $\boldsymbol{q} \in L^p_{\rm loc}(\boldsymbol{R}^n)$, then $\operatorname{div}\boldsymbol{q}$ cannot be concentrated on sets of dimension $\le d$. Thus, e.g., the Dirac δ cannot occur as the divergence of some $\boldsymbol{q} \in L^p_{\rm loc}(\boldsymbol{R}^n)$ with $p \ge n/(n-1)$; the improved integrability of $\boldsymbol{q}$ implies improved regularity of $\operatorname{div}\boldsymbol{q}$. In particular, for bounded vector fields $\operatorname{div}\boldsymbol{q}$ is absolutely continuous with respect to the $n-1$ dimensional Hausdorff measure. The bound d is optimal: if $1 \le p < n/(n-1)$, then essentially every measure can occur as a divergence of some $\boldsymbol{q} \in L^p_{\rm loc}(\boldsymbol{R}^n)$ while if $n/(n-1) \le p \le \infty$ then there are vector fields $\boldsymbol{q} \in L^p_{\rm loc}(\boldsymbol{R}^n)$ with divergences concentrated on sets of dimension s higher than but arbitrarily close to d:

EXAMPLE 3.3. – (i) *If* $1 \le p < n/(n-1)$ *then for any signed Radon measure* μ *with compact support there exists a* $\boldsymbol{q} \in L^p_{\mathrm{loc}}\mathscr{M}^{\mathrm{div}}(\boldsymbol{R}^n)$ *such that* $\operatorname{div}\boldsymbol{q} = \mu$.

(ii) *If* $n/(n-1) \le p \le \infty$ *then for any* $s > d$ *there exists a* $\boldsymbol{q} \in L^p_{\mathrm{loc}}\mathscr{M}^{\mathrm{div}}(\boldsymbol{R}^n)$ *such that* $\operatorname{div}\boldsymbol{q}$ *is not* $\mathscr{H}^s$ *absolutely continuous.*

The vector field $\boldsymbol{q}$ is constructed as the Newton force (the gradient of the Newton potential) for the uniform mass distribution on a compact set K with $0 < \mathscr{H}^m(K) < \infty$, see Proposition 6.1.

4. The divergence theorem.

Proposition 4.1 and Theorems 4.2, 4.4 and 4.6 discuss the divergence theorem for normalized sets of finite perimeter M and $\boldsymbol{q} \in L^1_{\mathrm{loc}}\mathscr{M}^{\mathrm{div}}(\boldsymbol{R}^n)$ in decreasing generality but with improving properties of the boundary term (see the discussion in Introduction). It is first noted in Proposition 4.1 that the boundary term is always a linear functional on the space $\mathrm{Lip}_0(\partial M)$ of Lipschitz continuous functions with compact support on ∂M. For any set $W \subset \boldsymbol{R}^n$ we denote $\mathrm{Lip}_0(W)$ the set of all real valued Lipschitz continuous functions φ on W such that $\{\boldsymbol{x} \in W : \varphi(\boldsymbol{x}) \ne 0\}$ is bounded, with norm

$$\|\varphi\|_{\mathrm{Lip}_0(W)} = \mathrm{Lip}\,(\varphi) + \|\varphi\|_{C(W)}$$

where $\mathrm{Lip}\,(\varphi)$ is the Lipschitz constant of φ and

$$\|\varphi\|_{C(W)} := \sup\,\{|\varphi(\boldsymbol{x})| : \boldsymbol{x} \in W\}.$$

PROPOSITION 4.1 (Normal trace as a functional). *If* M *is a normalized set with finite perimeter and* $\boldsymbol{q} \in L^1_{\mathrm{loc}}\mathscr{M}^{\mathrm{div}}(\boldsymbol{R}^n)$ *then there exists a linear functional* $\boldsymbol{N}^M(\boldsymbol{q}, \cdot) : \mathrm{Lip}_0(\partial M) \to \boldsymbol{R}$ *such that*

$$\boldsymbol{N}^M(\boldsymbol{q}, \varphi\,|_{\partial M}) = \int_M D\varphi \cdot \boldsymbol{q}\, d\mathscr{L}^n + \int_M \varphi \operatorname{div}\boldsymbol{q} \tag{4.1}$$

for every $\varphi \in \mathrm{Lip}_0(\boldsymbol{R}^n)$. *If* $N := (M^c)_*$ *then*

$$\boldsymbol{N}^M(\boldsymbol{q}, \cdot) + \boldsymbol{N}^N(\boldsymbol{q}, \cdot) = -\operatorname{div}\boldsymbol{q} \llcorner \partial M. \tag{4.2}$$

If M *or* M^c *is bounded then* $\boldsymbol{N}^M(\boldsymbol{q}, \cdot)$ *is continuous with respect to* $\|\cdot\|_{\mathrm{Lip}_0(\partial M)}$.

Here $\varphi|_{\partial M}$ is the restriction of φ to ∂M and the assertion is that the right hand side of (4.1) depends only on the boundary values of φ. $\boldsymbol{N}^M(\boldsymbol{q}, \cdot)$ is called the ***normal trace*** of $\boldsymbol{q}$ on ∂M. Although the frameworks are not strictly comparable, (4.1) may be considered to be a generalization of the relevant results of [4-6]. Equation (4.2) says that the normal traces from the two sides of ∂M are different if $\operatorname{div} \boldsymbol{q}$ is concentrated on ∂M; see (4.11), below, for a more concrete form, cf. also Example 3.1(ii).

Next we are concerned with specific forms of the normal trace. If M is a normalized set of finite perimeter and $\boldsymbol{q} \in L^1_{\mathrm{loc}}\mathscr{M}^{\mathrm{div}}(\boldsymbol{R}^n)$ we say that the normal trace $\boldsymbol{N}^M(\boldsymbol{q}, \cdot)$ is

(i) a ***measure*** if there exists a $\nu^M = \nu^M(\boldsymbol{q}) \in \mathscr{M}(\overline{\partial M})$ such that

$$\boldsymbol{N}^M(\boldsymbol{q}, \varphi) = \int_{\boldsymbol{R}^n} \varphi \, d\nu^M$$

for every $\varphi \in \mathrm{Lip}_0(\partial M)$;

(ii) an ***integrable function*** if there exists $q^M = q^M(\boldsymbol{q}) \in$ $\in L^1(\partial M, \mathscr{H}^{n-1})$ such that

$$\boldsymbol{N}^M(\boldsymbol{q}, \varphi) = \int_{\partial M} \varphi q^M d\mathscr{H}^{n-1}$$

for every $\varphi \in \mathrm{Lip}_0(\partial M)$.

The following theorem discusses conditions under which the normal trace is a measure. If M is a normalized set of finite perimeter, we say that a $\boldsymbol{q} \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ is

(i) ***weakly dominated*** on ∂M if there exists a sequence $\rho_j > 0$, $\rho_j \to 0$, and a constant $C < \infty$ such that

$$\frac{1}{\boldsymbol{a}_n \rho_j^n} \int_{\partial M} \int_{\boldsymbol{B}(\boldsymbol{x}, \rho_j)} \left|\boldsymbol{q}(\boldsymbol{y}) \cdot \boldsymbol{n}^M(\boldsymbol{x})\right| d\mathscr{L}^n(\boldsymbol{y}) d\mathscr{H}^{n-1}(\boldsymbol{x}) \le C \tag{4.3}$$

for every $j \in \boldsymbol{N}$;

(ii) ***dominated*** on ∂M if there exists a function $g \in L^1(\partial M, \mathscr{H}^{n-1})$ and a sequence $\rho_j > 0$, $\rho_j \to 0$, such that

$$\frac{1}{\boldsymbol{a}_n \rho_j^n} \int_{\boldsymbol{B}(\boldsymbol{x}, \rho_j)} \left|\boldsymbol{q}(\boldsymbol{y}) \cdot \boldsymbol{n}^M(\boldsymbol{x})\right| d\mathscr{L}^n(\boldsymbol{y}) \le g(\boldsymbol{x}) \tag{4.4}$$

for $\mathscr{H}^{n-1}$ a.e. $\boldsymbol{x} \in \partial M$ and every $j \in \boldsymbol{N}$.

Clearly, if $\boldsymbol{q}$ is dominated on ∂M then it is weakly dominated on ∂M. If $\boldsymbol{q} \in L^\infty(\boldsymbol{R}^n)$ then q is dominated on any ∂M; if $\boldsymbol{q} \in L^\infty_{\text{loc}}(\boldsymbol{R}^n)$ then $\boldsymbol{q}$ is dominated on ∂M if M is bounded.

THEOREM 4.2. (Normal trace as a measure). *Let* $\boldsymbol{q} \in L^1_{\text{loc}} \mathscr{M}^{\text{div}}(\boldsymbol{R}^n)$ *and let* M *be a normalized set of finite perimeter. Then*

(i) *if* $\boldsymbol{q}$ *is weakly dominated on* ∂M, *the normal trace is a measure supported on* $\overline{\partial M}$;

(ii) *if* $\boldsymbol{q} \in L^p_{\text{loc}} \mathscr{M}^{\text{div}}(\boldsymbol{R}^n)$, *where* $n/(n-1) \le p < \infty$, d *is as in Theorem* 3.2, $\boldsymbol{q}$ *is dominated on* ∂M, *and*

$$\partial_s M \text{ has a } \sigma \text{ finite } \mathscr{H}^d \text{ measure} \tag{4.5}$$

where $\partial_s M = \{\boldsymbol{x} \in \partial M : D(\boldsymbol{x}, M) \text{ does not exist}\}$, *then the normal trace is a measure supported on* ∂M.

Assertion (i) guarantees a measure supported on $\overline{\partial M}$, which may be large, while (ii) guarantees a measure supported on ∂M, which is a set with $\mathscr{H}^{n-1}(\partial M) < \infty$. Condition (4.5) requires that the «singular part» $\partial_s M$ of the boundary be small. We note that $D(x, M)$ exists and is equal to 1/2 for $\mathscr{H}^{n-1}$ a.e. $\boldsymbol{x} \in \partial M$; however, since $d < n-1$, Condition (4.5) requires more. Theorem 4.2 holds also if $p = \infty$, in which case (4.5) can be omitted, but in this special case the normal trace is an integrable function, cf. Theorem 4.4, below.

Finally we consider situations when the normal trace is an integrable function. Let first $\boldsymbol{q} \in L^1_{\text{loc}} \mathscr{M}^{\text{div}}(\boldsymbol{R}^n) \cap L^\infty(\boldsymbol{R}^n)$; it will turn out that then the normal trace is an integrable function for every normalized set of finite perimeter; moreover, it is given by a function $\boldsymbol{q}^0(\boldsymbol{x}, n)$ which we shall now introduce. If $\boldsymbol{x} \in \boldsymbol{R}^n$, $\boldsymbol{n} \in \boldsymbol{S}^{n-1}$, and $r > 0$, let $\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{n}, r) := \boldsymbol{B}(\boldsymbol{x}, r) \cap \cap \{\boldsymbol{y} \in \boldsymbol{R}^n : (\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{n} < 0\}$. If $\boldsymbol{q} \in L^1_{\text{loc}}(\boldsymbol{R}^n)$, we define a function $q^0 : \boldsymbol{R}^n \times \times \boldsymbol{S}^{n-1} \to \boldsymbol{R}$ by

$$q^0(\boldsymbol{x}, \boldsymbol{n}) = \begin{cases} \lim\limits_{r \to 0} \dfrac{n}{\boldsymbol{a}_{n-1} r^n} \displaystyle\int_{\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{n}, r)} \boldsymbol{q}(\boldsymbol{y}) \cdot \dfrac{\boldsymbol{x} - \boldsymbol{y}}{|\boldsymbol{x} - \boldsymbol{y}|} d\mathscr{L}^n(\boldsymbol{y}) \\ \qquad \text{if the limit exists and is finite,} \\ 0 \quad \text{if the limit either does not exist or is infinite,} \end{cases}$$

$\boldsymbol{x} \in \boldsymbol{R}^n$, $\boldsymbol{n} \in \boldsymbol{S}^{n-1}$. The function $q^0(\boldsymbol{x}, \boldsymbol{n})$ is a generalization of the expression $\boldsymbol{q}(\boldsymbol{x}) \cdot \boldsymbol{n}$:

REMARK 4.3. *If $\boldsymbol{q} \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ and $\boldsymbol{x} \in \boldsymbol{R}^n$ is a Lebesgue point of $\boldsymbol{q}$ then*

$$q^0(\boldsymbol{x}, \boldsymbol{n}) = \boldsymbol{q}(\boldsymbol{x}) \cdot \boldsymbol{n}$$

for every $\boldsymbol{n} \in \boldsymbol{S}^{n-1}$.

Here $\boldsymbol{q}$ is any representation of the class $\boldsymbol{q}$. In Example 3.1(ii),

$$q^0(\boldsymbol{x}, \boldsymbol{n}) = \begin{cases} \boldsymbol{e}_1 \cdot \boldsymbol{n} & \text{if } \boldsymbol{e}_1 \cdot \boldsymbol{n} > 0, \\ 0 & \text{else,} \end{cases}$$

for any $\boldsymbol{x}$ with $x_1 = 0$. Thus q^0 is bound to be nonlinear in the presence of discontinuities in the normal component of $\boldsymbol{q}$.

THEOREM 4.4. (Normal trace as a function, special case).

(i) *If $\boldsymbol{q} \in L^1_{\mathrm{loc}}\mathscr{M}^{\mathrm{div}}(\boldsymbol{R}^n) \cap L^\infty(\boldsymbol{R}^n)$ then for every normalized set of finite perimeter M the normal trace is a bounded function, i.e., there exists a $q^M \in L^\infty(\partial M, \mathscr{H}^{n-1})$ such that*

$$\int_{\partial M} \varphi q^M \, d\mathscr{H}^{n-1} = \int_M D\varphi \cdot \boldsymbol{q} \, d\mathscr{L}^n + \int_M \varphi \, \mathrm{div}\, \boldsymbol{q} \tag{4.6}$$

for every $\varphi \in \mathrm{Lip}_0(\boldsymbol{R}^n)$; moreover, q^M is given by

$$q^M(\boldsymbol{x}) = q^0(\boldsymbol{x}, \boldsymbol{n}^M(\boldsymbol{x})) \tag{4.7}$$

for $\mathscr{H}^{n-1}$ a.e. $\boldsymbol{x} \in \partial M$, and $\|q^M\|_{L^\infty(\partial M, \mathscr{H}^{n-1})} \le \|\boldsymbol{q}\|_{L^\infty(\boldsymbol{R}^n)}$.

(ii) *If, more generally, $\boldsymbol{q} \in L^\infty_{\mathrm{loc}}\mathscr{M}^{\mathrm{div}}(\boldsymbol{R}^n)$ and $\boldsymbol{q}$ is dominated on ∂M, then the normal trace is an integrable function q^M which satisfies* (4.7) *for $\mathscr{H}^{n-1}$ a.e. $\boldsymbol{x} \in \partial M$.*

The existence of the normal trace as in (i) has been proved in [2; Theorem 1.9] for open sets with Lipschitz boundary. In addition, (4.7) shows that the normal trace depends on the shape of ∂M only through the normal $\boldsymbol{n}^M$. In the context of Cauchy fluxes, assertions of this type are called Cauchy's postulate.

To proceed to the general p, we need to isolate the part of the measure div $\boldsymbol{q}$ which is singular with respect to $\mathscr{H}^{n-1}$. The following proposition, a direct generalization of Lebesgue's decomposition, is a basis for that. If $0 \le m < \infty$, we say that an $\eta \in \mathscr{M}_+(\boldsymbol{R}^n)$ is $\mathscr{H}^m$ ***singular*** if it is supported on a Borel set B with $\mathscr{H}^m(B) = 0$.

Proposition 4.5. *If $\eta \in \mathscr{M}_+(\boldsymbol{R}^n)$ and $0 \le m \le n$ then there exists a unique decomposition of η as*

$$\eta = \eta_{<m} + \eta_m + \eta_{>m} \tag{4.8}$$

where $\eta_{<m}, \eta_m, \eta_{>m} \in M_+(\boldsymbol{R}^n)$ have the properties

(i) *$\eta_{<m}$ is $\mathscr{H}^m$ singular;*

(ii) *η_m is $\mathscr{H}^m$ absolutely continuous and supported on a set of σ finite $\mathscr{H}^m$ measure;*

(iii) *$\eta_{>m}(B) = 0$ for every Borel set B of σ finite $\mathscr{H}^m$ measure.*

We may say that the dimensions of $\eta_{<m}, \eta_m, \eta_{>m}$ are less than, equal to, and bigger than m, respectively. The Lebesgue decomposition is the case $m = n$ by noting $\eta_{>n} = 0$ by (iii). By the Radon Nikodym theorem [27; Theorem (10.39)] we have

$$\eta_m(A) = f\,\mathscr{H}^m \llcorner S_0 \tag{4.9}$$

for some Borel set $S_0 \subset \boldsymbol{R}^n$ of s finite $\mathscr{H}^m$ measure and some nonnegative Borel function $f : \boldsymbol{R}^n \to \boldsymbol{R}$ with $f \in L^1(S_0, \mathscr{H}^m)$, $f(\boldsymbol{x}) > 0$ for every $\boldsymbol{x} \in S_0$ and $f(\boldsymbol{x}) = 0$ for every $\boldsymbol{x} \notin S_0$ (9). Then S_0, f are determined to within a change on a $\mathscr{H}^m$ null set. For a $\boldsymbol{q} \in L^1_{\text{loc}}\mathscr{M}^{\text{div}}(\boldsymbol{R}^n)$ we apply Proposition 4.5 with $m = n - 1$ to obtain

$$|\operatorname{div}\boldsymbol{q}| = |\operatorname{div}\boldsymbol{q}|_{<n-1} + |\operatorname{div}\boldsymbol{q}|_{n-1} + |\operatorname{div}\boldsymbol{q}|_{>n-1}.$$

Then $|\operatorname{div}\boldsymbol{q}|_{n-1} = f\mathscr{H}^{n-1} \llcorner S_0$ where f, S_0 are as above; moreover,

$$\operatorname{div}\boldsymbol{q} \llcorner S_0 = J\mathscr{H}^{n-1} \llcorner S_0$$

for some Borel function $J : \boldsymbol{R}^n \to \boldsymbol{R}$ such that $J \in L^1(S_0, \mathscr{H}^{n-1})$ and $J(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \notin S_0$. The set S_0 is the (analog of the) singular set of continuum mechanics and J is related to the jump of the normal trace of $\boldsymbol{q}$ across ∂M; see (4.11), below.

Theorem 4.6 (Normal trace as a function, general case). *Let $1 \le p \le \infty$, $q \in L^p_{\text{loc}}\mathscr{M}^{\text{div}}(\boldsymbol{R}^n)$ and let M be a normalized set M of finite perimeter for which $\boldsymbol{q}$ is dominated on ∂M and*

$$|\operatorname{div}\boldsymbol{q}|_{<n-1}(\partial M) = 0. \tag{4.10}$$

(9) On the contrary, the measure $\eta_{>m}$, despite of being $\mathscr{H}^m$ absolutely continuous, cannot be expressed as in (4.9), because it is supported on a set of non σ finite $\mathscr{H}^m$ measure.

Then the normal trace of $\boldsymbol{q}$ *on* ∂M *is an integrable function* $q^M \in L^1(\partial M, \mathscr{H}^{n-1})$. *If* $N := (M^c)_*$ *then also the normal trace of* $\boldsymbol{q}$ *on* ∂N *is an integrable function and*

$$q^M(\boldsymbol{x}) + q^N(\boldsymbol{x}) = -J(\boldsymbol{x}) \tag{4.11}$$

for $\mathscr{H}^{n-1}$ *a.e.* $x \in \partial M = \partial N$.

Equation (4.10) ensures that ∂M does not intersect the region where div $\boldsymbol{q}$ is too singular. In contrast to the situation $q \in L^1_{\mathrm{loc}} \mathscr{M}^{\mathrm{div}}(\boldsymbol{R}^n) \cap \cap L^\infty_{\mathrm{loc}}(\boldsymbol{R}^n)$, in the present context the normal trace q^M does not seem to be given by (4.7) generally. However, we have the following weak form of the local dependence of q^M on the shape of ∂M:

REMARK 4.7. *If* M_i, $i = 1, 2$, *are two normalized sets of finite perimeter which satisfy the hypothesis of Theorem* 4.6 *and if* $S := \{x \in \partial M_1 \cap \cap \partial M_2 : \boldsymbol{n}^{M_1}(\boldsymbol{x}) = \boldsymbol{n}^{M_2}(\boldsymbol{x})\}$ *then the normal traces* q^{M_i} *satisfy*

$$q^{M_1}(\boldsymbol{x}) = q^{M_2}(\boldsymbol{x}) \qquad \text{for } \mathscr{H}^{n-1} \text{ a.e.} \qquad \boldsymbol{x} \in S.$$

5. Cauchy fluxes.

Let $\mathscr{P}$ be the set of all bounded normalized sets of finite perimeter. An ***oriented surface*** is a pair $S = (\hat{S}, \boldsymbol{n}^S)$ such that is a Borel subset of ∂M of some $M \in \mathscr{P}$ and $\boldsymbol{n}^S(\boldsymbol{x}) = \boldsymbol{n}^M(\boldsymbol{x})$ for $\mathscr{H}^{n-1}$ a.e. $\boldsymbol{x} \in \hat{S}$. Let $\mathscr{S}$ be the set of all oriented surfaces. We say that the oriented surfaces $S = (\hat{S}, \boldsymbol{n}^S)$ and $T = (\hat{T}, \boldsymbol{n}^T)$ are ***compatible*** if there exists an oriented surface $U = (\hat{U}, \boldsymbol{n}^U) \in \mathscr{S}$ such that

$$\hat{S} \cup \hat{T} = \hat{U} \quad \text{and} \quad \boldsymbol{n}^U(\boldsymbol{x}) = \begin{cases} \boldsymbol{n}^S(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \hat{S}, \\ \boldsymbol{n}^T(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \hat{T} \end{cases}$$

for $\mathscr{H}^{n-1}$ a.e. $\boldsymbol{x} \in \hat{U}$. We then write $U = S \cup T$. If $M \in \mathscr{P}$ we interpret ∂M as the oriented surface $\partial M = (\partial M, \boldsymbol{n}^M)$.

We denote by $\mathscr{G}$ the set of all Borel functions (not classes of equivalence) $h : \boldsymbol{R}^n \to [0, \infty]$ such that $h \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$. If $h \in \mathscr{G}$ and $\eta \in \mathscr{M}_+(\boldsymbol{R}^n)$ we denote [7]

$$\mathscr{P}_{h\eta} := \{M \subset \boldsymbol{R}^n : M \in \mathscr{P}, \int_{\partial M} h \, d\mathscr{H}^{n-1} < \infty, \eta(\partial M) = 0\},$$

$$\mathscr{S}_{h\eta} := \{S \in \mathscr{S} : S \subset \partial M, M \in \mathscr{P}_{h\eta}\}.$$

We say that $N \subset \mathscr{S}$ is a null set if $\mathscr{N} \subset \mathscr{S} \setminus \mathscr{S}_{h\eta}$ for some $h \in \mathscr{G}$, $\eta \in \mathscr{M}_+(\boldsymbol{R}^n)$. We say that a set $\mathscr{D} \subset \mathscr{S}$ contains almost all of $\mathscr{S}$ if $\mathscr{S} \setminus \mathscr{D}$ is a null subset of $\mathscr{S}$. If π is a property associated with all surfaces $S \in \mathscr{S}$, we say that π holds for a.e. $S \in \mathscr{S}$ if there are $h \in \mathscr{G}$ and $\eta \in M_+(\boldsymbol{R}^n)$ such that $\pi(S)$ is true for all $S \in \mathscr{S}_{h\eta}$.

A ***Cauchy flux*** is any mapping $F: \mathscr{D} \to \boldsymbol{R}$, where $\mathscr{D} \subset \mathscr{S}$, such that for some $\mathscr{D}_0 \subset \mathscr{D}$ that contains almost all of $\mathscr{S}$ we have

(i) if $S, T \in \mathscr{D}_0$ are disjoint compatible material surfaces then $S \cup T \in \mathscr{D}_0$ and

$$F(S \cup T) = F(S) + F(T);$$

(ii) there exists $h \in \mathscr{G}$ such that

$$|F(S)| \le \int_S h \, d\mathscr{H}^{n-1} \tag{5.1}$$

for every $S \in \mathscr{D}_0$;

(iii) there exists $\eta \in M_+(\boldsymbol{R}^n)$ such that

$$|F(\partial P)| \le \eta(P)$$

for any $M \in \mathscr{P}$ with $\partial M \in \mathscr{D}_0$.

We say that F is a ***Cauchy flux of class*** L^p_{loc}, $1 \le p \le \infty$, if the function h as in (5.1) can be chosen in L^p_{loc}.

The above definition is equivalent to the one given in [7, 15-18]; the papers [13, 29] deal with $h \in L^\infty_{\mathrm{loc}}(\boldsymbol{R}^n)$, $\eta = c\mathscr{L}^n$, $c \in \boldsymbol{R}$, and [24-25] with $h \in L^p_{\mathrm{loc}}(\boldsymbol{R}^n)$, $\eta = f\mathscr{L}^n$, $f \in L^p_{\mathrm{loc}}(\boldsymbol{R}^n)$, $1 \le p \le \infty$.

THEOREM 5.1. *F is a Cauchy flux if and only if there exists a vector field $\boldsymbol{q} \in L^1_{\mathrm{loc}}\mathscr{M}^{\mathrm{div}}(\boldsymbol{R}^n)$ such that, for any representation $\boldsymbol{q}$ of the class $\boldsymbol{q}$,*

$$F(S) = \int_S \boldsymbol{q} \cdot \boldsymbol{n}^S \, d\mathscr{H}^{n-1} \tag{5.2}$$

for a.e. $S \in \mathscr{D}$. The correspondence $F \leftrightarrow \boldsymbol{q}$ is one to one if one identifies Cauchy fluxes that differ only on null subsets of $\mathscr{S}$ and interprets $\boldsymbol{q}$ as Lebesgue classes of equivalence. Moreover, F is of class L^p_{loc} if and only if $\boldsymbol{q} \in L^p_{\mathrm{loc}}\mathscr{M}^{\mathrm{div}}(\boldsymbol{R}^n)$, $1 \le p \le \infty$.

The field $\boldsymbol{q}$ is called the ***flux vector*** corresponding to F. Theorem 5.1 follows from the results of [7]; previous special cases are [13, 29, 24-25].

For a given Cauchy flux F with the flux vector $\boldsymbol{q}$ define

$$\mathscr{P}^* := \{M \in \mathscr{P}\colon \boldsymbol{q} \text{ is dominated on } \partial M \text{ and } |\operatorname{div} \boldsymbol{q}|_{<n-1}(\partial M) = 0\},$$

$$\mathscr{S}^* := \{S \in \mathscr{S}\colon S \subset \partial M \text{ for some } M \in \mathscr{P}^*\},$$

and $F^*\colon \mathscr{S}^* \to \boldsymbol{R}$ by

$$F * (S) = \int_S q^M \, d\mathscr{H}^{n-1} \tag{5.3}$$

for any $S \in \mathscr{S}^*$ where $M \in \mathscr{P}^*$, $S \subset \partial M$ and q^M is the normal trace of $\boldsymbol{q}$ on ∂M, which exists by Theorem 4.6. The function F^* is well defined by Remark 4.7.

THEOREM 5.2. *If F is a Cauchy flux then F^* is a Cauchy flux and $F(S) = F^*(S)$ for a.e. $S \in \mathscr{S}$; if F is a Cauchy flux of class L^∞_{loc} then F^* is defined on $\mathscr{S}^* = \mathscr{S}$.*

The flux F^* is defined *naturally* as the densities from (5.3) satisfy the divergence theorem. Its domain S^* contains singular surfaces S (surfaces with $\operatorname{div} \boldsymbol{q} \llcorner S \neq 0$) provided $\boldsymbol{q}$ is dominated on S. By (4.11),

$$F^*(S) + F^*(-S) = -\operatorname{div} \boldsymbol{q}(S)$$

where $-S$ is the surface S with the opposite orientation.

6. Proof of Theorem 3.2 and Example 3.3.

PROOF OF THEOREM 3.2. (i): It suffices to prove that $|\operatorname{div} \boldsymbol{q}|(B) = 0$ for each Borel set B with $\mathscr{H}^d(B) < \infty$. Let B be a Borel set with $\mathscr{H}^d(B) < \infty$. From the Hahn decomposition [27; Theorem 10.36] we deduce that there exist Borel sets $B_\pm \subset B$ with $B_+ \cap B_- = \emptyset$, $B_+ = B_+ = B$ such that $\pm \operatorname{div} \boldsymbol{q} \llcorner B_\pm \geq 0$. Our goal is to prove that $\operatorname{div} \boldsymbol{q}(B_\pm) = 0$. It suffices to prove only $\operatorname{div} \boldsymbol{q}(B_+) = 0$ for which in turn by [10; § 2.2.5] it suffices to prove that $\operatorname{div} \boldsymbol{q}(K) = 0$ for any compact subset K of B_+. Thus let $K \subset B_+$ be compact. Let $\varphi\colon \boldsymbol{R}^n \to [0, 1]$ be given by

$$\varphi(\boldsymbol{x}) = \begin{cases} 1 & \text{if } |x| < 1, \\ 2 - |x| & \text{if } 1 \leq |x| \leq 2, \\ 0 & \text{if } |x| > 2 \end{cases}$$

and note that φ is a Lipschitz continuous function with $|D\varphi| \leq 1$ for $\mathscr{L}^n$ a.e. $x \in \boldsymbol{R}^n$. Let $\varepsilon > 0$. Since $d < n$ and $\mathscr{H}^d(K) < \infty$, we have $\mathscr{L}^n(K) = 0$. Using $\boldsymbol{q} \in L^p_{\text{loc}}(\boldsymbol{R}^n)$, $p < \infty$, we deduce that there exists a bounded open set U with

$K \subset U$ and $\|q\|_{L^p(U)} < \varepsilon$. Furthermore, by [10; Theorem 2.2.2(2)] this set U may be chosen so as to satisfy div $\boldsymbol{q}^{\pm}(U \backslash K) < \varepsilon$. Since K is compact, there exists an $\delta > 0$ such that for any ball $\boldsymbol{B}(\boldsymbol{x}, r)$ with $r < \delta$ and $K \cap \boldsymbol{B}(\boldsymbol{x}, r) \neq \emptyset$, we have $\boldsymbol{B}(\boldsymbol{x}, 2r) \subset U$. From the definition of $\mathscr{H}^d$, there exists a covering of K by a finite system of balls $\boldsymbol{B}(x_i, r_i)$, $i \in I$, with $r_i < \delta$ and $\boldsymbol{a}_d \sum_i r_i^d \mathscr{H}^d(B) + 1$ (10). Thus $\bigcup_{i \in I} \boldsymbol{B}(\boldsymbol{x}_i, 2r_i) \subset U$. Let $\varphi_i(\boldsymbol{x}) := \varphi(r_i^{-1}(\boldsymbol{x} - \boldsymbol{x}_i))$, $i \in I$, $\boldsymbol{x} \in \boldsymbol{R}^n$, and let

$$\omega(\boldsymbol{x}) = \max \ \{\varphi_i(\boldsymbol{x}) : i \in I\}$$

for every $\boldsymbol{x} \in \boldsymbol{R}^n$. Then $0 \le \omega \le 1$, ω is Lipschitz continuous, and $\omega = 1$ on K. Since the support of ω is in U, we obtain

$$\text{(6.1)} \qquad \operatorname{div} \boldsymbol{q}(K) = \int_K \omega \operatorname{div} \boldsymbol{q} = -\int_U D\omega \cdot \boldsymbol{q} \, d\mathscr{L}^n - \int_{U \backslash K} \omega \operatorname{div} \boldsymbol{q}$$

directly from the definition of div $\boldsymbol{q}$ (see (3.1)). We now estimate the right hand side. Since $0 \le \omega \le 1$ and div $\boldsymbol{q}^{\pm}(U \backslash K) \le \varepsilon$, we have

$$\text{(6.2)} \qquad \Big| \int_{U \backslash K} \omega \operatorname{div} \boldsymbol{q} \Big| \le 2\varepsilon.$$

Further,

$$\text{(6.3)} \qquad \Big| \int_U D\omega \cdot \boldsymbol{q} \, d\mathscr{L}^n \Big| \le L^{1/q} \|\boldsymbol{q}\|_{L^p(U)} \le L^{1/q} \varepsilon$$

where $\boldsymbol{q} := p/(p-1)$ is the conjugate Hölder exponent and $L = \int_U |D\omega|^q d\mathscr{L}^n$. One easily finds that for $\mathscr{L}^n$ a.e. $\boldsymbol{x} \in \boldsymbol{R}^n$ there exists at least one $i \in I$ such that $D\omega(\boldsymbol{x}) = D\varphi_i(\boldsymbol{x})$ and hence

$$\begin{aligned} L = \int_U |D\omega|^q d\mathscr{L}^n &\le \sum_{i \in I} \int_{\boldsymbol{R}^n} |D\varphi_i|^q d\mathscr{L}^n \\ &= \sum_{i \in I} \int_{\boldsymbol{B}(x_i, 2r_i)} |D\varphi_i|^q d\mathscr{L}^n \\ &\le 2^n \boldsymbol{a}_n \sum_{i \in I} r_i^{n-q} \\ &\le 2^n \boldsymbol{a}_n / \boldsymbol{a}_d (\mathscr{H}^d(B) + 1) \end{aligned}$$

(10) In this proof we switch without change in notation from the Hausdorff measure $\mathscr{H}^d$ to the spherical measure $\mathscr{S}^d$, [10; §§ 2.10.2–6], which is possible by the inequalities $\mathscr{H}^d \le \mathscr{S}^d \le [(2n/(n+1)]^{d/2} \mathscr{H}^d$, see [10; § 2.10.6], which show that $\mathscr{H}^d$ and $\mathscr{S}^d$ have the same system of null sets and sets of finite measure.

where we note that $n-q=d$ and $|D\varphi_i| \le r_i^{-1}$ on $\boldsymbol{B}(x_i,\, 2r_i)$. Thus

$$0 \le \operatorname{div} \boldsymbol{q}(K) \le (2^n \boldsymbol{a}_n / \boldsymbol{a}_d (\mathscr{H}^d(B)+1))^{1/q} \varepsilon + 2\varepsilon$$

by (6.1), (6.2) and (6.3). The arbitrariness of $\varepsilon > 0$ gives $\operatorname{div} \boldsymbol{q}(K) = 0$. Thus $|\operatorname{div} \boldsymbol{q}|(B) = 0$.

(ii): We have to prove that if $B \subset \boldsymbol{R}^n$ is a Borel set with $\mathscr{H}^{n-1}(B) = 0$ then $|\operatorname{div} \boldsymbol{q}|(B) = 0$. Thus let $\mathscr{H}^{n-1}(B) = 0,$ let $B_\pm$ have the same meaning as in the proof of part (i), let K be a compact subset of B_+, and prove that $\operatorname{div} \boldsymbol{q}(K) = 0$. Let $\varepsilon > 0,$ let U be an open set with $K \subset U$ such that $\operatorname{div} \boldsymbol{q}^\pm(U \backslash K) < \varepsilon$; corresponding to this U we find an $\delta > 0$ as in the proof of part (i). Since $\mathscr{H}^{n-1}(K) = 0,$ there exists a covering of K by a finite system of balls $\boldsymbol{B}(\boldsymbol{x}_i,\, r_i),\ i \in I,$ with $r_i < \boldsymbol{\delta}$ and $\sum_i r_i^{n-1} < \varepsilon$. If ω is as above, then (6.2) holds while (6.3) is replaced by

$$\left| \int_U D\omega \cdot \boldsymbol{q}\, d\mathscr{L}^n \right| \le L \|\boldsymbol{q}\|_{L^\infty(U)} \tag{6.4}$$

where $L = \int_U |D\omega| d\mathscr{L}^n$. As above,

$$L \le \sum_{i \in I} \int_{\boldsymbol{R}^n} |D\varphi_i| d\mathscr{L}^n = \sum_{i \in I} \int_{\boldsymbol{B}(\boldsymbol{x}_i,\, 2r_i)} |D\varphi_i| d\mathscr{L}^n \le 2^n \boldsymbol{a}_n \sum_{i \in I} r_i^{n-1} \le 2^n \boldsymbol{a}_n \varepsilon.$$

Thus we have

$$0 \le \operatorname{div} \boldsymbol{q}(K) \le 2^n \boldsymbol{a}_n \varepsilon \|\boldsymbol{q}\|_{L^\infty(U)} + 2\varepsilon$$

by (6.1), (6.2) and (6.4). Hence $\operatorname{div} \boldsymbol{q}(K) = 0$ and consequently $\operatorname{div} \boldsymbol{q}(B_+) = 0$. □

PROPOSITION 6.1. *Let μ is a signed Radon measure on $\boldsymbol{R}^n$ with compact support and*

$$\boldsymbol{q}(\boldsymbol{x}) := \frac{1}{n \boldsymbol{a}_n} \int_{\boldsymbol{R}^n} \frac{(\boldsymbol{x} - \boldsymbol{y}) d\mu(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^n} \tag{6.5}$$

for every $\boldsymbol{x} \in \boldsymbol{R}^n$ for which $\int_{\boldsymbol{R}^n} |\boldsymbol{x} - \boldsymbol{y}|^{1-n} d|\mu|(\boldsymbol{y}) < \infty$. Then

(i) *$\boldsymbol{q} \in L^1_{\mathrm{loc}} \mathscr{M}^{\mathrm{div}}(\boldsymbol{R}^n)$ and*

$$\operatorname{div} \boldsymbol{q} = \mu; \tag{6.6}$$

(ii) *if $1 \le p < n/(n-1)$ then $\boldsymbol{q} \in L^p_{\mathrm{loc}}(\boldsymbol{R}^n)$;*

(iii) *if $n/(n-1) \le p \le \infty$ then $\boldsymbol{q} \in L^p_{\mathrm{loc}}(\boldsymbol{R}^n)$ provided $|\mu|(\boldsymbol{B}(\boldsymbol{x},\, \boldsymbol{r})) \le$ $\le cr^m$ for all $\boldsymbol{x} \in \boldsymbol{R}^n$ and all $0 < r < a$, where $m > d,\ a > 0,\ c > 0$ are constants and d is as in Theorem* 3.2.

PROOF. Write $\eta := |\mu|$, let

$$\phi(\boldsymbol{x}) := \int_{\boldsymbol{R}^n} |\boldsymbol{x} - \boldsymbol{y}|^{1-n} d\eta(\boldsymbol{y})$$

for every $\boldsymbol{x} \in \boldsymbol{R}^n$ so that $\phi : \boldsymbol{R}^n \to [0, \infty]$ and prove that $\phi \in L^p_{\mathrm{loc}}(\boldsymbol{R}^n)$ for every p satisfying $1 \le p < n/(n-1)$. By Hölder's inequality, with q the conjugate exponent,

$$\phi^p(\boldsymbol{x}) \le \|\eta\|^{p/q} \int_{\boldsymbol{R}^n} |\boldsymbol{x} - \boldsymbol{y}|^{-p(n-1)} d\eta(\boldsymbol{y}).$$

Therefore, if $\boldsymbol{z} \in \boldsymbol{R}^n$ and $r > 0$,

$$\int_{\boldsymbol{B}(z,r)} \phi^p(\boldsymbol{x}) d\mathscr{L}^n(\boldsymbol{x}) \le \|\eta\|^{p/q} \int_{\boldsymbol{R}^n} \int_{\boldsymbol{B}(\boldsymbol{z},r)} |\boldsymbol{x} - \boldsymbol{y}|^{-p(n-1)} d\mathscr{L}^n(\boldsymbol{x}) d\eta(\boldsymbol{y}).$$

For any $\boldsymbol{y} \in \boldsymbol{R}^n$ we have $\boldsymbol{B}(\boldsymbol{z}, r) \subset B(\boldsymbol{y}, |\boldsymbol{z} - \boldsymbol{y}| + r)$ and therefore

$$\int_{\boldsymbol{B}(z,r)} |\boldsymbol{x} - \boldsymbol{y}|^{-p(n-1)} d\mathscr{L}^n(\boldsymbol{x}) \le \int_{\boldsymbol{B}(\boldsymbol{y},|\boldsymbol{z}-\boldsymbol{y}|+r)} |\boldsymbol{x} - \boldsymbol{y}|^{-} p(n-1) d\mathscr{L}^n(\boldsymbol{x})$$

$$= C(|\boldsymbol{z} - \boldsymbol{y}|) + r)^{n-p(n-1)}$$

where $C := n\boldsymbol{a}_n/(n - p(n-1))$. Therefore

$$\int_{\boldsymbol{B}(\boldsymbol{z},r)} \phi^p(\boldsymbol{x})\, d\mathscr{L}^n(\boldsymbol{x}) \le C\|\eta\|^{p/q} \int_{\boldsymbol{R}^n} (|\boldsymbol{z} - \boldsymbol{y}| + r)^{n-p(n-1)} d\eta(\boldsymbol{y}).$$

The last integrand is a bounded function of $\boldsymbol{y}$ on the compact support of η and thus the integral is finite. Hence $\phi \in L^p_{\mathrm{loc}}(\boldsymbol{R}^n)$; thus in particular, ϕ is finite for $\mathscr{L}^n$ a.e. $\boldsymbol{x} \in \boldsymbol{R}^n$ and hence $\boldsymbol{q}$ is defined $\mathscr{L}^n$ a.e. on $\boldsymbol{R}^n$. (i): By $|\boldsymbol{q}(\boldsymbol{x})| \le n^{-1}\boldsymbol{a}_n^{-1}(\boldsymbol{x})$ we see that $\boldsymbol{q}$ is locally integrable. Equation (6.6) is standard by noting that $\boldsymbol{q}$ is the derivative of the Newton potential corresponding to the mass distribution μ. (ii): Has been proved above. (iii): Let μ satisfy the hypothesis of (iii) and let s be any number such that $d < s < m$. Assume first that $p < \infty$ and denote by q the conjugate exponent. Writing $|\boldsymbol{x} - \boldsymbol{y}|^{1-n} = |\boldsymbol{x} - \boldsymbol{y}|^{-s/q} |\boldsymbol{x} - \boldsymbol{y}|^{s/q+1-n}$, we obtain by Hölder's inequality

$$\phi^p(\boldsymbol{x}) \le \Big(\int |\boldsymbol{x} - \boldsymbol{y}|^{-s} d\eta(\boldsymbol{y})\Big)^{p/q} \int |\boldsymbol{x} - \boldsymbol{y}|^{p(s/q-n+1)} d\eta(\boldsymbol{y}).$$

Prove that there exists a $C < \infty$ such that

$$\int |\boldsymbol{x} - \boldsymbol{y}|^{-s} d\eta(\boldsymbol{y}) < C \tag{6.7}$$

for every $\boldsymbol{x} \in \boldsymbol{R}^n$. Let $\boldsymbol{x} \in \boldsymbol{R}^n$ and write $M(r) = \eta(\boldsymbol{B}(\boldsymbol{x}, r))$ for $r > 0$ so that $M(r) \le cr^m$. Then, standardly,

$$\begin{aligned}
\int |\boldsymbol{x} - \boldsymbol{y}|^{-s} d\eta(\boldsymbol{y}) &= \int_{\boldsymbol{B}(\boldsymbol{x}, a)} |\boldsymbol{x} - \boldsymbol{y}|^{-s} d\eta(\boldsymbol{y}) + \int_{\boldsymbol{B}(\boldsymbol{x}, a)^c} |\boldsymbol{x} - \boldsymbol{y}|^{-s} d\eta(\boldsymbol{y}) \\
&\le \int_0^a r^{-s} dM(r) + \int_{\boldsymbol{B}(x, a)^c} a^{-s} d\eta(\boldsymbol{y}) \\
&\le r^{-s} M(r)|_0^a + s \int_0^a r^{-s-1} M(r)\, dr + a^{-s}\|\eta\| \\
&\le \frac{cma^{m-s}}{m-s} + a^{-s}\|\eta\| =: C
\end{aligned}$$

which proves (6.7). Hence,

$$\phi^p(\boldsymbol{x}) \le C^{p/q} \int |\boldsymbol{x} - \boldsymbol{y}|^{p(s/q-n+1)} d\eta(\boldsymbol{y}).$$

Thus if $z \in \boldsymbol{R}^n$ and $r > 0$, we have, using $\boldsymbol{B}(\boldsymbol{z}, r) \subset \boldsymbol{B}(\boldsymbol{y}, |\boldsymbol{z} - \boldsymbol{y}| + r)$,

$$\int_{\boldsymbol{B}(z, r)} \phi^p(\boldsymbol{x})\, d\mathscr{L}^n(\boldsymbol{x}) \le C^{p/q} \int_{\boldsymbol{R}^n} \int_{\boldsymbol{B}(\boldsymbol{y}, |\boldsymbol{z}-\boldsymbol{y}|+r)} |\boldsymbol{x} - \boldsymbol{y}|^{p(s/q-n+1)} d\mathscr{L}^n(\boldsymbol{x})\, d\eta(\boldsymbol{y}). \tag{6.8}$$

By $s > d$ we have $n + p(s/q - n + 1) > 0$; thus the inner integral is finite and equal to $n\boldsymbol{a}_n(n + p(s/q - n + 1))^{-1}(|\boldsymbol{z} - \boldsymbol{y}| + r)^{n+p(s/q-n+1)}$, which is a bounded function of $\boldsymbol{y}$ on the compact support of η. Thus the right hand side of (6.8) is finite. The case $p = \infty$ is similar. □

Proof of Example 3.3. (i): This follows from Proposition 6.1 (i), (ii). Proof of (ii): If $0 \le m \le n$ then there exists a compact set K such that $0 < \mathscr{H}^m(K) < \infty$ and for some constant c,

$$\mathscr{H}^m(K \cap \boldsymbol{B}(\boldsymbol{x}, r)) \le cr^m$$

for all $\boldsymbol{x} \in \boldsymbol{R}^n$ and $r > 0$ [9; Corollary 4.12]. Choose any m such that $d < m < s$. Let $\mu := \mathscr{H}^m \llcorner K$ and let $\boldsymbol{q}$ be given by (6.5). By Item (i) of Proposition 6.1, $\boldsymbol{q} \in L^1_{\mathrm{loc}}\mathscr{M}^{\mathrm{div}}(\boldsymbol{R}^n)$ and $\operatorname{div} \boldsymbol{q} = \mu$. The measure μ satisfies the hypothesis of Item (iii) of Proposition 6.1 and thus $\boldsymbol{q} \in L^p_{\mathrm{loc}}(\boldsymbol{R}^n)$. On the other hand, since $m < s$ and $\mathscr{H}^m(K) < \infty,$ we have $\mathscr{H}^s(K) = 0$. □

7. Proof of Proposition 4.1 and Theorem 4.2.

Let ξ be a radially symmetric mollifier on $\boldsymbol{R}^n$; for any $\rho > 0$ let $\xi_\rho(\boldsymbol{z}) = \rho^{-n}\xi(\boldsymbol{z}/\rho),\ \boldsymbol{z} \in \boldsymbol{R}^n$. For any $f \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ denote by $f_\rho \in C^\infty(\boldsymbol{R}^n)$ the ρ mollification of f,

$$f_\rho(\boldsymbol{x}) = \int\limits_{\boldsymbol{R}^n} \xi_\rho(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\, d\mathscr{L}^n(\boldsymbol{y}),$$

$\boldsymbol{x} \in \boldsymbol{R}^n$. If $\boldsymbol{q} \in L^1_{\mathrm{loc}}\mathscr{M}^{\mathrm{div}}(\boldsymbol{R}^n)$ then

$$\operatorname{div} \boldsymbol{q}_\rho(\boldsymbol{x}) = \int\limits_{\boldsymbol{R}^n} \xi_\rho(\boldsymbol{x}-\boldsymbol{y}) \operatorname{div} \boldsymbol{q}(\boldsymbol{y})$$

for every $\boldsymbol{x} \in \boldsymbol{R}^n$. For any $M \subset \boldsymbol{R}^n$ let 1_M be the characteristic function of M.

Lemma 7.1. *Let M be a normalized set of finite perimeter and $\varphi \in \mathrm{Lip}_0(\boldsymbol{R}^n)$ such that $\varphi = 0$ on ∂M. Then the function $\theta := 1_M\varphi$ is in $\mathrm{Lip}_0(\boldsymbol{R}^n)$ and $D\boldsymbol{\theta} = 1_M D\varphi$ for $\mathscr{L}^n$ a.e. point in $\boldsymbol{R}^n$.*

Proof. Let $\omega \in C^\infty_0(\boldsymbol{R}^n)$; by the Gauss-Green theorem for the set M and the function $\omega\varphi$,

$$\int\limits_{\boldsymbol{R}^n} \boldsymbol{\theta} D\omega\, d\mathscr{L}^n \equiv \int\limits_M \varphi D\omega\, d\mathscr{L}^n = -\int\limits_M \omega D\varphi\, d\mathscr{L}^n \equiv -\int\limits_{\boldsymbol{R}^n} \omega 1_M D\varphi\, d\mathscr{L}^n$$

since $\omega\varphi = 0$ on ∂M. Thus the weak derivative of $\boldsymbol{\theta}$ is $D\boldsymbol{\theta} = 1_M D\varphi$ and satisfies $|D\boldsymbol{\theta}| \le \mathrm{Lip}\,(\varphi)$ for L^n a.e. point of $\boldsymbol{R}^n$. The mollifications $\boldsymbol{\theta}_\rho$ of $\boldsymbol{\theta}$ satisfy

$$D\boldsymbol{\theta}_\rho(\boldsymbol{x}) = \int\limits_{\boldsymbol{R}^n} \xi_\rho(\boldsymbol{x}-\boldsymbol{y}) D\boldsymbol{\theta}(\boldsymbol{y})\, d\mathscr{L}^n;$$

hence $\left|D\boldsymbol{\theta}_\rho(\boldsymbol{x})\right| \le \mathrm{Lip}\,(\varphi)$ and consequently $\left|\boldsymbol{\theta}_\rho(\boldsymbol{x}) - \boldsymbol{\theta}_\rho(\boldsymbol{y})\right| \le \mathrm{Lip}\,(\varphi)|\boldsymbol{x}-\boldsymbol{y}|$ for every $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^n$. Furthermore, $\boldsymbol{\theta}_\rho(\boldsymbol{x}) \to \boldsymbol{\theta}(\boldsymbol{x})$ for every $\boldsymbol{x} \in \boldsymbol{R}^n$. Indeed, if $\boldsymbol{x} \in M$ it suffices to use that $\boldsymbol{\theta}$ is continuous on M, vanishes on $M^c,$ and $D(\boldsymbol{x}, M) = 1$; if $\boldsymbol{x} \in (M^c)_*$ then $\boldsymbol{\theta}$ vanishes on $M^c,$ is bounded on M, and

$D(\boldsymbol{x}, M^c) = 1$. If $\boldsymbol{x} \in \partial M$ then $|\boldsymbol{\theta}| \le \mathrm{Lip}\,(\varphi)\boldsymbol{r}$ on $\boldsymbol{B}(\boldsymbol{x}, r)$. The limit in $\left|\boldsymbol{\theta}_\rho(\boldsymbol{x}) - \boldsymbol{\theta}_\rho(\boldsymbol{y})\right| \le \mathrm{Lip}\,(\varphi)|\boldsymbol{x} - \boldsymbol{y}|$ gives $|\boldsymbol{\theta}(\boldsymbol{x}) - \boldsymbol{\theta}(\boldsymbol{y})| \le \mathrm{Lip}\,(\varphi)|\boldsymbol{x} - \boldsymbol{y}|$ for each $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^n$. □

PROOF OF PROPOSITION 4.1. Let M be a normalized set of finite perimeter. Firstly, note that (3.1) holds for every $\varphi \in \mathrm{Lip}_0(\boldsymbol{R}^n)$. Indeed, it suffices to apply (3.1) to the sequence of mollifications φ_ρ of φ and let $\rho \to 0$. Secondly, note that if $\varphi \in \mathrm{Lip}_0(\boldsymbol{R}^n)$ and $\varphi = 0$ on ∂M then

$$\int_M D\varphi \cdot \boldsymbol{q}\, d\mathscr{L}^n + \int_M \varphi \,\mathrm{div}\,\boldsymbol{q} = 0. \tag{7.1}$$

Indeed, the function $\theta = 1_M\varphi$ is Lipschitz continuous by Lemma 7.1; the application of (3.1) to θ gives (7.1). For any $\varphi \in \mathrm{Lip}_0(\partial M)$, define $\boldsymbol{N}^M(\boldsymbol{q}, \varphi)$ by

$$\boldsymbol{N}^M(\boldsymbol{q}, \varphi) = \int_M D\tilde{\varphi} \cdot \boldsymbol{q}\, d\mathscr{L}^n + \int_M \tilde{\varphi}\,\mathrm{div}\,\boldsymbol{q} \tag{7.2}$$

where $\tilde{\varphi}$ is any Lipschitz extension of φ to $\boldsymbol{R}^n$ with compact support. The existence $\tilde{\varphi}$ is easily deduced from the existence of an extension with the same Lipschitz constant ([10; Theorem 2.10.43]) and the fact that $\{\boldsymbol{x} \in \partial M, \varphi(\boldsymbol{x}) \ne 0\}$ is bounded. We note that the value of the right hand side of (7.2) is independent of the extension $\tilde{\varphi}$ by (7.1). Moreover, $\boldsymbol{N}^M(\boldsymbol{q}, \cdot)$ is linear since if $\boldsymbol{\lambda} \in \boldsymbol{R}$, one can choose the extension corresponding to $\boldsymbol{\lambda}\varphi$ to be $\boldsymbol{\lambda}\tilde{\varphi}$ and similarly for the sum. This completes the proof of the existence of $\boldsymbol{N}^M(\boldsymbol{q}, \cdot)$. To prove (4.2), it suffices to write (4.1) for M and for N, to add the results, and subtract (3.1). Next, let M be bounded and prove that

$$|\boldsymbol{N}^M(\boldsymbol{q}, \varphi)| \le C\|\varphi\|_{Lip_0(\partial M)} \tag{7.3}$$

for some C and all $\varphi \in \mathrm{Lip}_0(\partial M)$. Using a suitable cutoff function that is equal to 1 on the (bounded) closure of M, one can show that the extension $\tilde{\varphi}$ of φ can be chosen as to satisfy

$$\|\tilde{\varphi}\|_{\mathrm{Lip}_0(\boldsymbol{R}^n)} \le D\|\varphi\|_{\mathrm{Lip}_0(\partial M)} \tag{7.4}$$

where D is a constant independent of φ. But then from (7.2) and (7.4) we obtain (7.3) where $C = D(\int_M |\boldsymbol{q}|d\mathscr{L}^n + \|\mathrm{div}\,\boldsymbol{q}\|)$. Finally, if M^c is bounded, then $\boldsymbol{N}^N(\boldsymbol{q}, \cdot)$ is continuous and (4.2) establishes the continuity of $\boldsymbol{N}^M(\boldsymbol{q}, \cdot)$. □

LEMMA 7.2. *Let $\boldsymbol{q} \in L^1_{\mathrm{loc}}\mathscr{M}^{\mathrm{div}}(\boldsymbol{R}^n)$ and let M be a normalized set of finite perimeter.*

(i) *If* $\boldsymbol{q}$ *is weakly dominated on* ∂M *then the normal trace is represented by a measure* $\nu^M = \pi^M - \sigma^M$ *where*

$$\pi^M = w^* - \lim_{\rho \to 0} \boldsymbol{q}_\rho\big|_{\partial M} \cdot \boldsymbol{n}^M \mathscr{H}^{n-1} \llcorner \partial M \quad \text{in } \mathscr{M}(\boldsymbol{R}^n), \tag{7.5}$$

$$\sigma^M = w^* - \lim_{\rho \to 0} \mu_\rho \quad \text{in } \mathscr{M}(\boldsymbol{R}^n), \tag{7.6}$$

where $\boldsymbol{q}_\rho$ *is the mollification of* $\boldsymbol{q}$*,* $\mu_\rho \in \mathscr{M}(\boldsymbol{R}^n)$ *is given by*

$$\langle \mu_\rho, \varphi \rangle = \int_{\partial M} \int_M \varphi(\boldsymbol{y}) \xi_\rho(\boldsymbol{y} - \boldsymbol{x}) \, d\mathscr{L}^n(\boldsymbol{y}) \, d \operatorname{div} \boldsymbol{q}(\boldsymbol{x})$$

for every $\varphi \in C_0(\boldsymbol{R}^n)$*, and* w^* *denotes the weak* convergence, understood along an appropriate sequence of* ρ *tending to* 0*; the support of* ν^M *is in* $\overline{\partial M}$*.*

(ii) *If* $\boldsymbol{q}$ *is dominated on* ∂M *then* $\pi^M = q_0 \mathscr{H}^{n-1} \llcorner \partial M$ *where*

$$q_0 = w - \lim_{\rho \to 0} \boldsymbol{q}_\rho\big|_{\partial M} \cdot \boldsymbol{n}^M \quad \text{in } L^1(\partial M, \mathscr{H}^{n-1})$$

where w *denotes the weak convergence.*

Proof (i): Since $\boldsymbol{q}$ is dominated on ∂M, there exists a sequence $\rho_j \to 0$ and a $C < \infty$ such that (4.3) holds. Since for each $x \in \partial M$ and $\rho > 0$,

$$\left|\boldsymbol{q}_\rho(\boldsymbol{x}) \cdot \boldsymbol{n}^M(\boldsymbol{x})\right| \le D \int_{\boldsymbol{B}(\boldsymbol{x}, \rho)} \left|\boldsymbol{q}(\boldsymbol{y}) \cdot \boldsymbol{n}^M(\boldsymbol{x})\right| d\mathscr{L}^n(\boldsymbol{y})$$

where D is the maximum of ξ, we deduce from (4.3) that

$$\int_{\partial M} \left|\boldsymbol{q}_\rho(\boldsymbol{x}) \cdot \boldsymbol{n}^M(\boldsymbol{x})\right| d\mathscr{H}^{n-1}(\boldsymbol{x}) \le E$$

for some $E < \infty$ and all $\rho = \rho_j$, $j \in \boldsymbol{N}$. Thus the total variation of $\boldsymbol{q}_\rho|_{\partial M} \cdot \boldsymbol{n}^M \mathscr{H}^{n-1} \llcorner \partial M$ is bounded and hence, for some subsequence of ρ_j, still denoted by ρ, the limit in (7.5) exists. Further, one easily finds that $|\langle \mu_\rho, \varphi \rangle| \le \operatorname{div} \boldsymbol{q}|(\partial M) \|\varphi\|_{L^\infty(\boldsymbol{R}^n)}$ for each $\varphi \in C_0(\boldsymbol{R}^n)$ and each ρ, i.e., $\|\mu_\rho\| \le \operatorname{div} \boldsymbol{q}|(\partial M)$. Thus for some subsequence of ρ_j, still denoted by ρ, the limit in (7.5) exists. Let $\varphi \in \operatorname{Lip}_0(\boldsymbol{R}^n)$. The divergence theorem for smooth vector fields and sets of finite perimeter reads

$$\int_{\partial M} \varphi \boldsymbol{q}_\rho \cdot \boldsymbol{n}^M d\mathscr{H}^{n-1} = \int_M D\varphi \cdot \boldsymbol{q}_\rho d\mathscr{L}^n + \int_M \varphi \operatorname{div} \boldsymbol{q}_\rho d\mathscr{L}^n, \tag{7.7}$$

and

$$\int_M D\varphi \cdot \boldsymbol{q}_\rho d\mathscr{L}^n \to \int_M D\varphi \cdot \boldsymbol{q}\, d\mathscr{L}^n, \qquad \int_{\partial M} \varphi \boldsymbol{q}_\rho \cdot \boldsymbol{n}^M d\mathscr{H}^{n-1} \to \int_{\boldsymbol{R}^n} \varphi\, d\boldsymbol{\pi}^M$$

since $\boldsymbol{q}_\rho \to \boldsymbol{q}$ in $L^1_{\text{loc}}(\boldsymbol{R}^n)$ and by (7.5). Next note that

$$\int_M \varphi \operatorname{div} \boldsymbol{q}_\rho\, d\mathscr{L}^n = \int_{\boldsymbol{R}^n} \varphi_\rho \operatorname{div} \boldsymbol{q}$$

where

$$\varphi_\rho(\boldsymbol{x}) = \int_M \varphi(\boldsymbol{y})\, \xi_\rho(\boldsymbol{x} - \boldsymbol{y})\, d\mathscr{L}^n(\boldsymbol{y}), \tag{7.8}$$

and write

$$\int_{\boldsymbol{R}^n} \varphi_\rho \operatorname{div} \boldsymbol{q} = \int_M \varphi_\rho \operatorname{div} \boldsymbol{q} + \int_{(M^c)_*} \varphi_\rho \operatorname{div} \boldsymbol{q} + \int_{\partial M} \varphi_\rho \operatorname{div} \boldsymbol{q}. \tag{7.9}$$

The three terms on the right hand side of (7.9) converge, respectively, to

$$\int_M \varphi \operatorname{div} \boldsymbol{q}, \quad 0, \quad \int_{\boldsymbol{R}^n} \varphi\, d\sigma^M.$$

The first two limits follow from [11]

$$\varphi_\rho(\boldsymbol{x}) \to \begin{cases} \varphi(\boldsymbol{x}) & \text{for every } \boldsymbol{x} \in M, \\ 0 & \text{for every } \boldsymbol{x} \in (M^c)_* \end{cases}$$

by the dominated convergence theorem, while the last limit is (7.6) by observing that $\int_{\partial M} \varphi_\rho \operatorname{div} \boldsymbol{q} = \langle \mu_\rho, \varphi \rangle$. To summarize, the limit in (7.7) gives

$$\int_{\boldsymbol{R}^n} \varphi\, d\boldsymbol{\pi}^M = \int_M D\varphi \cdot \boldsymbol{q}\, d\mathscr{L}^n + \int_{\boldsymbol{R}^n} \varphi\, d\sigma^M + \int_M \varphi \operatorname{div} \boldsymbol{q}.$$

Thus the normal trace $\boldsymbol{N}^M(\boldsymbol{q}, \cdot)$ is represented by a measure ν^M. The support of ν^M is in $\overline{\partial M}$. Indeed, the support of $\boldsymbol{q}_\rho|_{\partial M} \cdot \boldsymbol{n}^M \mathscr{H}^{n-1} \llcorner \partial M$ is in $\overline{\partial M}$ and to show that the support of σ^M is in $\overline{\partial M}$, we note that for each $\rho < r$, the support of μ_ρ is in $\{x \in \boldsymbol{R}^n : \operatorname{dist}(\boldsymbol{x}, \partial M) \le r\}$.

([11]) Here we use that M is a *normalized* set of finite perimeter.

(ii): If $\boldsymbol{q}$ is dominated on ∂M, there exists a sequence $\rho_j \to 0$ and a $g \in L^1(\partial M, \mathscr{H}^{n-1})$ such that (4.4) holds for $\mathscr{H}^{n-1}$ a.e. $\boldsymbol{x} \in \partial M$. If D is the maximum of ξ, then

$$|\boldsymbol{q}_\rho(\boldsymbol{x}) \cdot \boldsymbol{n}^M(\boldsymbol{x})| \le Dg(\boldsymbol{x})$$

for $\mathscr{H}^{n-1}$ a.e. $\boldsymbol{x} \in \partial M$ and every $\rho = \rho_j$, $j \in \boldsymbol{N}$. The sequence of functions $\boldsymbol{q}_\rho \cdot \boldsymbol{n}^M$ is thus $\mathscr{H}^{n-1}$ equiintegrable on ∂M and hence [8; Corollary 11, § IV.8] there exists a subsequence of ρ_j, still denoted ρ, and a $q_0 \in$ $\in L^1(\partial M, \mathscr{H}^{n-1})$ such that

$$\boldsymbol{q}_\rho \cdot \boldsymbol{n}^M \rightharpoonup q_0 \quad \text{in} \quad L^1(\partial M, \mathscr{H}^{n-1})$$

The limit in (7.5) is $\boldsymbol{\pi}^M = q_0 H^{n-1} \llcorner \partial M$. □

Proof of Theorem 4.2. (i): Follows from Lemma 7.2(i).

(ii): We only have to prove that the measure ν^M of (i) is supported on ∂M. By Lemma 7.2(ii) we have $\pi^M = q_0 \mathscr{H}^{n-1} \llcorner \partial M$. The measure σ^M satisfies

$$\langle \sigma^M, \varphi \rangle = \lim_{\rho \to 0} \int_{\partial M} \varphi_\rho \operatorname{div} \boldsymbol{q}$$

where φ_ρ is given by (7.8). At every $\boldsymbol{x} \in \partial M$ where $D(\boldsymbol{x}, M)$ exists we have $\varphi_\rho(\boldsymbol{x}) \to D(\boldsymbol{x}, M)\, \varphi(\boldsymbol{x})$. By (4.5) and Theorem 3.2, $|\operatorname{div} \boldsymbol{q}|(\partial_s M) = 0$. Thus we have that $\varphi_\rho \to D(\cdot, M)\, \varphi$ for $|\operatorname{div} \boldsymbol{q}| \llcorner \partial M$ a.e. $\boldsymbol{x} \in \boldsymbol{R}^n$. Hence

$$\lim_{\rho \to 0} \int_{\partial M} \varphi_\rho \operatorname{div} \boldsymbol{q} \to \int_{\partial M} \varphi(\boldsymbol{x})\, D(\boldsymbol{x}, M) \operatorname{div} \boldsymbol{q}(\boldsymbol{x}).$$

Thus $\sigma^M = D(\cdot, M) \operatorname{div} \boldsymbol{q} \llcorner \partial M$ and consequently

$$\nu^M = q_0 \mathscr{H}^{n-1} \llcorner \partial M - D(\cdot, M) \operatorname{div} \boldsymbol{q} \llcorner \partial M,$$

which is a measure supported on ∂M. □

8. Proof of Theorems 4.4 and 4.6.

If $\eta \in M_+(\boldsymbol{R}^n)$ and $0 \le m < \infty$, the m dimensional upper density of η by

$$\theta^m(\boldsymbol{x}, \eta) = \lim_{r \to 0} \sup \left\{ \frac{\eta(B)}{\boldsymbol{\alpha}_m \rho^m} : B \text{ an open ball of radius } \rho \le \boldsymbol{r},\ \boldsymbol{x} \in B \right\}, \quad x \in \boldsymbol{R}^n.$$

PROPOSITION 8.1. ([10; §2.10.19(1)(3)]). *If η is a nonnegative Radon measure, $F \subset \boldsymbol{R}^n$ a Borel set and $0 < c < \infty$ then*

(i) *if $\theta^m(\boldsymbol{x}, \eta) < c$ for all $\boldsymbol{x} \in F$ then $\mathscr{H}^m(F) \geq 2^{-m}\eta(F)/c$;*

(ii) *if $\theta^m(\boldsymbol{x}, \eta) > c$ for all $\boldsymbol{x} \in F$ then $\mathscr{H}^m(F) \leq \|\eta\|/c$.*

PROOF OF PROPOSITION 4.5. Prove that the decomposition (4.8) is unique. Thus let $\bar{\eta}_{<m}, \bar{\eta}_m, \bar{\eta}_{>m}$, with the Properties (i)–(iii), be such that

$$\eta = \bar{\eta}_{<m} + \bar{\eta}_m + \bar{\eta}_{>m}. \tag{8.1}$$

The measures $\eta_{<m}$ and $\bar{\eta}_{<m}$ are supported on sets $A, \bar{A}$ of $\mathscr{H}^m$ measure 0 with $\mathscr{H}^m(A) = \mathscr{H}^m(\bar{A}) = 0$. Let $\hat{A} := A \cup \bar{A}$ and let C be any Borel set. Noting that $\mathscr{H}^m(C \cap \hat{A}) = 0$, referring to the properties (ii), (iii), and applying (4.8), (8.1) to $C \cap B$ we obtain

$$\eta(C \cap \hat{A}) = \eta_{<m}(C \cap \hat{A}) = \bar{\eta}_{<m}(C \cap \hat{A}).$$

Since $\eta_{<m}, \bar{\eta}_{<m}$ are supported on A and $\bar{A}$, respectively, we have $\eta_{<m}(C \cap \hat{A}) = \eta_{<m}(C)$, $\bar{\eta}_{<m}(C \cap \hat{A}) = \bar{\eta}_{<m}(C)$. Thus $\eta_{<m} = \bar{\eta}_{<m}$ and we have

$$\eta = \eta_{>m} + \bar{\eta}_m + \bar{\eta}_{>m}. \tag{8.2}$$

The measures $\eta_m, \bar{\eta}_m$ are supported on sets $B, \bar{B}$ of σ finite $\mathscr{H}^m$ measure. Set $\hat{B} = B \cup \bar{B}$. Let C be any Borel set; the application of (8.2) to $\hat{B} \cap C$ and the property (iii) gives $\eta_m(\hat{B} \cap C) = \bar{\eta}_m(\hat{B} \cap C)$ which again reduces to $\eta_m(C) = \bar{\eta}_m(C)$. This completes the proof of the uniqueness. The existence. (Cf. [22; Theorem 67] for $n = 1$.) Define $W_{<m}, W_m, W_{>m}$ by

$$W_{<m} = \{\boldsymbol{x} \in \boldsymbol{R}^n : \boldsymbol{\theta}^m(\boldsymbol{x}, \eta) = \infty\}, \quad W_m := \{\boldsymbol{x} \in \boldsymbol{R}^n : 0 < \boldsymbol{\theta}^m(\boldsymbol{x}, \eta) < \infty\},$$
$$W_{>m} = \{\boldsymbol{x} \in \boldsymbol{R}^n : \boldsymbol{\theta}^m(\boldsymbol{x}, \eta) = 0\}$$

and let $\eta_{<m}, \eta_m, \eta_{>m}$ be the restrictions of η to $W_{<m}, W_m, W_{>m}$, respectively. On $W_{<m}$ one has $\boldsymbol{\theta}^m(x, \eta) > c$ for any $c > 0$ and thus Proposition 8.1(ii) gives $\mathscr{H}^m(W_{<m}) = 0$; thus $\eta_{<m}$ satisfies (i). We have $W_m = \bigcup_i W_i$ where $W_i = \{\boldsymbol{x} \in \boldsymbol{R}^n : \infty > \boldsymbol{\theta}^m(x, \eta) > 1/i\}$; Proposition 8.1(ii) says that $\mathscr{H}^m(W_i) \leq \|\eta\| i < \infty$; thus W_m has σ finite measure. To prove the H^m absolute continuity of η_m, let B be a Borel set with $\mathscr{H}^m(B) = 0$. We have $\eta_m(B) = \eta(B \cap W_m)$ and prove that $\eta(B \cap W_m) = 0$. Let $W^j = \{\boldsymbol{x} \in \boldsymbol{R}^n : 0 < \boldsymbol{\theta}^m(\boldsymbol{x}, \eta) < j\}$, $j \in \boldsymbol{N}$. Then $W_m = \bigcup_j W^j$ and $\eta(B \cap W_m) = \lim_{j \to \infty} \eta(B \cap W^j)$. Thus it suffices to prove that $\eta(B \cap W^j) = 0$. Proposition 8.1(i) says $0 = 2^m j \mathscr{H}^m(B \cap W^j) \geq \eta(B \cap W^j)$. This completes the proof of (ii). To prove (iii), let B be a Borel set of σ finite $\mathscr{H}^m$ measure and prove that $\eta_{>m}(B) = 0$. It suffices to assume that $\mathscr{H}^m(B) < \infty$. We have

$\eta_{>m}(B) = \eta(W_{>m} \cap B)$. On $W_{>m}$ we have $\theta^m(\boldsymbol{x}, \eta) < c$ for any $c > 0$ and thus by Proposition 8.1(i), $\eta(W_{>m} \cap B) \le 2^m c \mathscr{H}^m(B)$. Hence $\eta(W_{>m} \cap B) = 0$. □

LEMMA 8.2. *Let* $m \ge 0$. *If* $B \subset \boldsymbol{R}^n$ *is a Borel set and* $\eta \in M_+(\boldsymbol{R}^n)$ *a measure with* $\eta(B) = 0$ *then*

$$\lim_{r\to 0} \frac{\eta(\boldsymbol{B}(\boldsymbol{x}, r))}{r^m} = 0$$

for $\mathscr{H}^m$ *a.e.* $\boldsymbol{x} \in B$.

PROOF. Since $\eta(B) = 0$, for each $\delta > 0$ there exists an open set $U \supset B$ such that $\eta(U) < \delta$. One has $\theta^m(\boldsymbol{x}, \eta) = \boldsymbol{\theta}^m(\boldsymbol{x}, \eta \llcorner U)$ for each $\boldsymbol{x} \in B$. Let $B_e := \{\boldsymbol{x} \in B : \boldsymbol{\theta}^m(\boldsymbol{x}, \eta) > 0\}$ so that $B_e = \bigcup_i B_i$ where

$$B_i := \{\boldsymbol{x} \in B : \boldsymbol{\theta}^m(\boldsymbol{x}, \eta) > 1/i\}.$$

By Proposition 8.1(ii), $\mathscr{H}^m(B_i) \le 2^m i\eta(U) 2^m i\boldsymbol{\delta}$. Since $\boldsymbol{\delta} > 0$ is arbitrary, we have $\mathscr{H}^m(B_i) = 0$. □

PROOF OF REMARK 4.3. Let $\boldsymbol{x}$ be a Lebesgue point of $\boldsymbol{q}$ [27; (7.14)-(7.15)] and $\boldsymbol{n} \in \boldsymbol{S}^{n-1}$. Abbreviating

$$G_r(\boldsymbol{y}) := \frac{n(\boldsymbol{x} - \boldsymbol{y})}{\boldsymbol{a}_{n-1} r^n |\boldsymbol{x} - \boldsymbol{y}|}, \qquad F(r) := \int_{\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{n}, r)} \boldsymbol{q}(\boldsymbol{y}) \cdot G_r \, d\mathscr{L}^n,$$

and noting that $\boldsymbol{n} = \int_{\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{n}, r)} G_r \, d\mathscr{L}^n$, we have, for $r \to 0$,

$$\begin{aligned} |F(r) - \boldsymbol{q}(\boldsymbol{x}) \cdot \boldsymbol{n}| &= \Big| \int_{\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{n}, r)} (\boldsymbol{q}(\boldsymbol{y}) - q(\boldsymbol{x})) \cdot G_r(\boldsymbol{y}) \, d\mathscr{L}^n(\boldsymbol{y}) \Big| \\ &\le n\boldsymbol{a}_{n-1}^{-1} r^{-n} \int_{\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{n}, r)} |\boldsymbol{q}(\boldsymbol{y}) - \boldsymbol{q}(\boldsymbol{x})| d\mathscr{L}^{n-1}(\boldsymbol{y}) \to 0. \end{aligned}$$ □

PROOF OF THEOREM 4.6. Note that under the hypotheses of Theorem 4.6, the conclusions of Lemma 7.2(i), (ii) are available. We have to prove that the measure ν^M from Lemma 7.2 is H^{n-1} absolutely continuous, which in turn means that σ^M is $\mathscr{H}^{n-1}$ absolutely continuous. Note first that $\operatorname{div} \boldsymbol{q} \llcorner \partial M$ is $\mathscr{H}^{n-1}$ absolutely continuous since $|\operatorname{div} \boldsymbol{q}|_{<n-1}(\partial M) = 0$. By the Radon Nikodym theorem, there exists a $q_2 \in L^1(\partial M, \mathscr{H}^{n-1})$ such that

$$\operatorname{div} \boldsymbol{q} \llcorner \partial M = q_2 H^{n-1} \llcorner \partial M.$$

By (7.6) we have

$$\langle \sigma^M, \varphi \rangle = \lim_{\rho \to 0} \int_{\partial M} \varphi_\rho \operatorname{div} \boldsymbol{q} = \lim_{\rho \to 0} \int_{\partial M} \varphi_\rho q_2 \, d\mathscr{H}^{n-1}$$

where φ_ρ is given by (7.8). Noting that

$$\varphi_\rho(\boldsymbol{x}) \to \frac{1}{2}\varphi(\boldsymbol{x}) \qquad \text{for } \mathscr{H}^{n-1} \text{ a.e.} \qquad \boldsymbol{x} \in \partial M$$

we see that

$$\langle \sigma^M, \varphi \rangle = \frac{1}{2} \int_{\partial M} \varphi q_2 \, d\mathscr{H}^{n-1}.$$

Thus $\sigma^M = \frac{1}{2} q_2 \mathscr{H}^{n-1} \llcorner \partial M$ and (4.6) holds with $q^M := q_0 - \frac{1}{2} q_2$. Equation (4.11) follows from (4.2) by noting that $\operatorname{div} \boldsymbol{q} \llcorner \partial M = J \mathscr{H}^{n-1} \llcorner \partial M$. □

Proof of Theorem 4.4. (ii): If $\boldsymbol{q} \in L^1_{\text{loc}} \mathscr{M}^{\text{div}}(\boldsymbol{R}^n) \cap L^\infty_{\text{loc}}(\boldsymbol{R}^n)$ and $\boldsymbol{q}$ is dominated on ∂M, then by Theorem 3.2(ii), $|\operatorname{div} \boldsymbol{q}|$ is $\mathscr{H}^{n-1}$ absolutely continuous, hence $|\operatorname{div} \boldsymbol{q}|_{<n-1}$ vanishes by Proposition 4.5 and thus (4.10) is automatically satisfied. That the normal trace is an integrable function then follows from Theorem 4.6. Proof of (4.7): Prove first that if $\boldsymbol{q}, M$ satisfy the hypothesis of Theorem 4.6, then for $\mathscr{H}^{n-1}$ a.e. $\boldsymbol{x} \in \partial M$ we have

$$q^M(\boldsymbol{x}) = \lim_{r \to 0} \frac{n}{\boldsymbol{a}_{n-1} r^n} \int_{M \cap \boldsymbol{B}(\boldsymbol{x}, r)} \boldsymbol{q}(\boldsymbol{y}) \cdot \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|} \, d\mathscr{L}^n(\boldsymbol{y}). \tag{8.4}$$

For $\mathscr{H}^{n-1}$ a.e. $\boldsymbol{x} \in \partial M$ we have the following assertions:

$$q^M(\boldsymbol{x}) = \lim_{r \to 0} \frac{1}{\boldsymbol{a}_{n-1} r^{n-1}} \int_{\partial M \cap \boldsymbol{B}(\boldsymbol{x}, r)} q^M \, d\mathscr{H}^{n-1}, \tag{8.5}$$

$$\lim_{r \to 0} \frac{1}{r^{n-1}} |\operatorname{div} \boldsymbol{q}|(M \cap \boldsymbol{B}(\boldsymbol{x}, r)) = 0. \tag{8.6}$$

Here (8.5) follows from the rectifiability of ∂M ([10; Theorem 4.5.6(2)]) and (8.6) follows from Lemma 8.2 applied to $m := n - 1$ and $\eta := |\operatorname{div} \boldsymbol{q}| \llcorner M$ by noting that $\eta(\partial M) = 0$. Let $\boldsymbol{x}$ be a such point and for any $r > 0$ set

$$M_r := M \cap \boldsymbol{B}(\boldsymbol{x}, r), \qquad \varphi_r(\boldsymbol{y}) = \max \{ r - |\boldsymbol{y} - \boldsymbol{x}|, 0 \}$$

$\boldsymbol{y} \in \boldsymbol{R}^n$. We apply the divergence theorem (4.6) and divide by $n / \boldsymbol{a}_{n-1} r^n$ to obtain

$$(8.7)\qquad \frac{n}{\boldsymbol{a}_{n-1}r^n}\int\limits_{\partial M\cap \boldsymbol{B}(\boldsymbol{x},r)}\varphi_r q^M\,d\mathscr{H}^{n-1}=\int\limits_{M_r}\boldsymbol{q}\cdot G_r\,d\mathscr{L}^n+\frac{n}{\boldsymbol{a}_{n-1}r^n}\int\limits_{M_r}\varphi_r\operatorname{div}\boldsymbol{q},$$

where G_r is given by $(8.3)_1$. Noting that $\varphi_r(\boldsymbol{y})=\int_0^r 1_{\boldsymbol{B}(\boldsymbol{x},s)}(\boldsymbol{y})\,d\mathscr{L}^1(s)$ we find that

$$\int\limits_{\partial M\cap \boldsymbol{B}(\boldsymbol{x},r)}\varphi_r q^M(\boldsymbol{y})\,d\mathscr{H}^{n-1}=\int_0^r\int\limits_{\partial M\cap \boldsymbol{B}(\boldsymbol{x},s)}q^M d\mathscr{H}^{n-1}d\mathscr{L}^1(s);$$

combining with l'Hopital's rule and with (8.5), one finds that the left hand side of (8.7) converges to $q^M(\boldsymbol{x})$. Since

$$\Big|\int\limits_{M_r}(r-|\boldsymbol{y}-\boldsymbol{x}|)\operatorname{div}\boldsymbol{q}(\boldsymbol{y})\Big|\le r|\operatorname{div}\boldsymbol{q}|(M_r),$$

the limit of the second term on the right hand side of (8.7) vanishes by (8.6). Thus (8.4). To prove (4.7), note that for $\mathscr{H}^{n-1}$ a.e. $\boldsymbol{x}\in\partial M$ (see [10; §§ 4.5.5-6])

$$(8.8)\qquad D(x,\,M\,\triangle\,\{\boldsymbol{y}\in\boldsymbol{R}^n\colon(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})<0\})=0.$$

Here $\triangle$ denotes the symmetric difference, i.e., $A\triangle B:=(A\backslash B)\cup(B\backslash A)$ for any two sets A, B. Using (8.4), (8.8), and the local boundedness of $\boldsymbol{q}$ near $\boldsymbol{x}$, a straightforward estimate shows that (8.4) remains valid with $\partial M\cap\boldsymbol{B}(\boldsymbol{x},r)$ replaced by $\boldsymbol{B}(\boldsymbol{x},\,\boldsymbol{n}(\boldsymbol{x}),\,r)$. (i): If additionally $\boldsymbol{q}\in L^\infty(\boldsymbol{R}^n)$ then $\boldsymbol{q}$ is dominated on ∂M and the conclusions follow from (i). □

Proof of Remark 4.7. By Lemma 7.2 and the proof of Theorem 4.6 we have

$$q^{M_i}=q_0^{M_i}-\frac{1}{2}q_2^{M_i},$$

$i=1,\,2,$ where

$$(8.9)\qquad q_0^{M_i}=w-\lim_{\rho\to 0}\boldsymbol{q}_\rho|_{\partial M_i}\cdot n^{M_i}\quad\text{in } L^1(\partial M_i,\,\mathscr{H}^{n-1}),$$

$$(8.10)\qquad \operatorname{div}\boldsymbol{q}\llcorner\partial M_i=q_2^{M_i}\mathscr{H}^{n-1}\llcorner\partial M_i.$$

We have $\boldsymbol{q}_\rho|_{\partial M_1}\cdot\boldsymbol{n}^{M_1}.=\boldsymbol{q}_\rho|_{\partial M_2}\cdot\boldsymbol{n}^{M_2}$ on S and thus testing (8.9) on functions $f\in L^\infty(\partial M_1,\,\mathscr{H}^{n-1})\cap L^\infty(\partial M_2,\,\mathscr{H}^{n-1})$ which vanish outside S we obtain $q_0^{M_1}=q_0^{M_2}$ for $\mathscr{H}^{n-1}$ a.e. point of S. Similarly, $\operatorname{div}\boldsymbol{q}\llcorner\partial M_1(A)=$

$= \operatorname{div} \boldsymbol{q} \llcorner \partial M_2(A)$ for each Borel set $A \subset S$ and thus (8.10) implies $q_2^{M_1} = q_2^{M_2}$ for H^{n-1} a.e. point of S. □

9. Proof of Theorem 5.2.

PROOF OF THEOREM 5.2. Prove first that $\mathscr{S}^*$ contains almost all of $\mathscr{S}$. Let

$$\boldsymbol{q}_\rho(\boldsymbol{x}) = \frac{1}{\boldsymbol{a}_n \rho^n} \int_{\boldsymbol{B}(\boldsymbol{x}, \rho)} \boldsymbol{q} \, d\mathscr{L}^n$$

for any $\boldsymbol{x} \in \boldsymbol{R}^n$ and $\rho > 0$, let $\boldsymbol{q}$ be any Borel representation of the class $\boldsymbol{q} \in L^1_{\text{loc}} \mathscr{M}^{\text{div}}(\boldsymbol{R}^n)$, and let $B_m := \boldsymbol{B}(\boldsymbol{0}, m)$, $m \in \boldsymbol{N}$. We have $\boldsymbol{q}_\rho \to \boldsymbol{q}$ in $L^1(B_1)$ as $\rho \to 0$. Thus there exists a sequence $\rho^{(1)} \equiv \{\rho_k^{(1)}\}_{k=1}^\infty$ converging monotonically to 0 such that $|\boldsymbol{q}_{\rho_k^{(1)}} - \boldsymbol{q}|_{L^1(B_1)} < 2^{-k}$, $k \in \boldsymbol{N}$. Define $h_1 : B_1 \to [0, \infty]$ by

$$h_1(\boldsymbol{x}) = |\boldsymbol{q}(\boldsymbol{x})| + \sum_{k=1}^\infty |\boldsymbol{q}_{\rho_k^{(1)}}(\boldsymbol{x}) - \boldsymbol{q}(\boldsymbol{x})|,$$

$\boldsymbol{x} \in B_1$, and note that h_1 is a Borel function in $L^1(B_1)$. Next, since $\boldsymbol{q}_{\rho_k^{(1)}} \to \boldsymbol{q}$ in $L^1(B_2)$ as $k \to \infty$, there exists a subsequence $\rho^{(2)} \equiv \{\rho_k^{(2)}\}_{k=1}^\infty$ of $\rho^{(1)}$ such that $\|\boldsymbol{q}_{\rho_k^{(2)}} - \boldsymbol{q}\|_{L^1(B_2)} < 2^{-k}$, $k \in \boldsymbol{N}$. Define $h_2 : B_2 \to [0, \infty]$ by

$$h_2(\boldsymbol{x}) = |\boldsymbol{q}(\boldsymbol{x})| + \sum_{k=1}^\infty |\boldsymbol{q}_{\rho_k^{(2)}}(\boldsymbol{x}) - \boldsymbol{q}(\boldsymbol{x})|,$$

$\boldsymbol{x} \in B_2$, and note that h_2 is a Borel function in $L^1(B_2)$ with $h_2 \le h_1$ on B_1. Proceeding inductively, for each $m \in \boldsymbol{N}$ there exists a subsequence $\rho^{(m)} \equiv \{\rho_k^{(m)}\}_{k=1}^\infty$ of $\rho^{(m-1)}$ such that the function

$$h_m(\boldsymbol{x}) := |q(\boldsymbol{x})| + \sum_{k=1}^\infty |q_{\rho_k^{(m)}}(\boldsymbol{x}) - \boldsymbol{q}(\boldsymbol{x})| \tag{9.1}$$

is a Borel function in $L^1(B_m)$ and $h_m \le h_{m-1}$ on B_{m-1}. Define $h : \boldsymbol{R}^n \to [0, \infty]$ by $h(\boldsymbol{x}) = h_m(\boldsymbol{x})$ where m is the unique integer such that $\boldsymbol{x} \in B_m \backslash B_{m-1}$, with the convention $B_0 := \emptyset$. Note that h is a Borel function in $L^1_{\text{loc}}(\boldsymbol{R}^n)$. Set $\eta := |\operatorname{div} \boldsymbol{q}|_{<n-1}$ and prove that $\mathscr{P}_{h\eta} \subset \mathscr{P}^*$. Thus we have to prove that if $M \in \mathscr{P}_{h\eta}$ then $\boldsymbol{q}$ is dominated on ∂M. Let $M \in \mathscr{P}_{h\eta}$. Since M is bounded, we have $M \subset B_m$ for some m; the condition $\int_{\partial M} h \, d\mathscr{H}^{n-1} < \infty$, which follows from $M \in \mathscr{P}_{h\eta}$, and the fact that $h_m \le h$ on B_m, implies that $\int_{\partial M} h_m d\mathscr{H}^{n-1} < \infty$. By (9.1) then

$$|\boldsymbol{q}_{\rho_k}(\boldsymbol{x})| \le h_m(\boldsymbol{x})$$

where $\rho_k := \rho_k^{(m)}$, which implies (4.4) with $g = h|_{\partial M}$. Thus $\boldsymbol{q}$ is dominated on ∂M; hence $\mathscr{P}_{h\eta} \subset \mathscr{P}^*$ and consequently $\mathscr{S}^*$ contains almost all S. Furthermore, Conditions (i)–(iii) of the definition of the Cauchy flux are clearly satisfied on $\mathscr{D}_0 := \mathscr{S}_{h\eta}$ where h, η are as above, with h as above in (ii) and $\eta := |\operatorname{div} \boldsymbol{q}|$ in (iii). Finally, if $M \in \mathscr{P}_{h\eta}$ then the normal trace q^M of $\boldsymbol{q}$ on ∂M is given by $q^M(\boldsymbol{x}) = \boldsymbol{q}(\boldsymbol{x}) \cdot \boldsymbol{n}^M(\boldsymbol{x})$ for $\mathscr{H}^{n-1}$ a.e. point x of ∂M. Thus

$$F^*(S) \int_S \boldsymbol{q} \cdot \boldsymbol{n}^S d\mathscr{H}^{n-1}$$

for any $S \in \mathscr{S}_{h\eta}$. A comparison with (5.2) shows that $F(S) = F^*(S)$ for almost every $S \in \mathscr{S}$.

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