Some Results for q-Functions of Many Variables.

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Abstract - We will give some new formulas and expansions for q-Appell-, q-Lauricella- and q-Kampé de Fériet functions. This is an area which has been explored by Jackson, R. P. Agarwal, W. A Al-Salam, Andrews, Kalnins, Miller, Manocha, Jain, Srivastava, and Van der Jeugt. The multiple q-hypergeometric functions are defined by the q-shifted factorial and the tilde operator. By a method invented by the author [20]-[23], which also involves the Ward-Alsalam q-addition and the Jackson-Hahn q-addition, we are able to find q-analogues of corresponding formulas for the multiple hypergeometric case. In the process we give a new definition of q-hypergeometric series, illustrated by some examples, which elucidate the integration property of q-calculus, and helps to try to make a first systematic attempt to find summation- and reduction theorems for multiple basic series. The Jackson Γ_q-function will be frequently used. Two definitions of a generalized q-Kampé de Fériet function, in the spirit of Karlsson and Srivastava [69], which are symmetric in the variables, and which allow q to be a vector are given. Various connections of transformation formulas for multiple q-hypergeometric series with Lie algebras and finite groups known from the literature are cited.

1. Historical introduction.

After Gauss' introduction of the hypergeometric series in 1812, various versions of the multiple hypergeometric series (MHS) were developed by people whose names are now associated to the functions they introduced. In 1880, Paul Emile Appell (1855-1930) [7], [8] introduced so-

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me 2-variable hypergeometric series now called Appell functions

(1) $\begin{cases} F_1(a; b, b'; c; x_1, x_2) \equiv \displaystyle\sum_{m_1, m_2 = 0}^{\infty} \dfrac{(a)_{m_1 + m_2}(b)_{m_1}(b')_{m_2}}{m_1! \, m_2! \, (c)_{m_1 + m_2}} x_1^{m_1} x_2^{m_2}, \\[2ex] \max(|x_1|, |x_2|) < 1. \end{cases}$

(2) $\begin{cases} F_2(a; b, b'; c, c'; x_1, x_2) \equiv \displaystyle\sum_{m_1, m_2 = 0}^{\infty} \dfrac{(a)_{m_1 + m_2}(b)_{m_1}(b')_{m_2}}{m_1! \, m_2! \, (c)_{m_1}(c')_{m_2}} x_1^{m_1} x_2^{m_2}, \\[2ex] |x_1| + |x_2| < 1. \end{cases}$

(3) $\begin{cases} F_3(a, a'; b, b'; c; x_1, x_2) \equiv \displaystyle\sum_{m_1, m_2 = 0}^{\infty} \dfrac{(a)_{m_1}(a')_{m_2}(b)_{m_1}(b')_{m_2}}{m_1! \, m_2! \, (c)_{m_1 + m_2}} x_1^{m_1} x_2^{m_2}, \\[2ex] \max(|x_1|, |x_2|) < 1. \end{cases}$

(4) $\begin{cases} F_4(a; b; c, c'; x_1, x_2) \equiv \displaystyle\sum_{m_1, m_2 = 0}^{\infty} \dfrac{(a)_{m_1 + m_2}(b)_{m_1 + m_2}}{m_1! \, m_2! \, (c)_{m_1}(c')_{m_2}} x_1^{m_1} x_2^{m_2}, \\[2ex] |x_1| + |x_2| < 1. \end{cases}$

REMARK 1. In the whole paper, the symbol $\equiv$ will denote definitions, except when we work with congruences.

The four Lauricella functions of n variables were introduced in [50]. The first systematic treatment of this subject was written in French by J.M. Kampé de Fériet (1893-1982) and Appell [8].

The Kampé de Fériet functions, in their most general form, are defined in [p. 27] [69]. There are many generalizations of these functions in the literature, and we refer to the books Srivastava and Karlsson [69] and Exton [25].

Much of the research in MHS today is concerned with reduction formulas which reduce the degree of the function in some sense. As in the one-variable case, there are also confluent forms of MHS, which are obtained by using limits.

We will now describe the method invented by the author [20]-[23], which also involves the Ward-Alsalam q-addition and the Jackson-Hahn q-addition. This method is a mixture of Heine 1846 [32] and Gasper-Rahman [26]. The advantages of this method have been summarized in [23, p. 495].

DEFINITION 1.   The power function is defined by $q^a \equiv e^{a \log(q)}$. We always use the principal branch of the logarithm.

The variables $a, b, c, a_1, a_2, \ldots, b_1, b_2, \ldots \in \mathbb{C}$ denote parameters in hypergeometric series or *q*-hypergeometric series. The variables i, j, k, l, m, n, p, r will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit. The *q*-analogues of a complex number a and of the factorial function are defined by:

$$(5) \qquad \{a\}_q \equiv \frac{1 - q^a}{1 - q}, \qquad q \in \mathbb{C} \backslash \{1\},$$

$$(6) \qquad \{n\}_q! \equiv \prod_{k=1}^{n} \{k\}_q, \qquad \{0\}_q! \equiv 1, \qquad q \in \mathbb{C},$$

DEFINITION 2.   In 1908 Jackson [39] reintroduced the Euler-Heine-Jackson *q*-difference operator

$$(7) \qquad (D_q \varphi)(x) \equiv \begin{cases} \dfrac{\varphi(x) - \varphi(qx)}{(1 - q)x}, & \text{if } q \in \mathbb{C} \backslash \{1\}, \quad x \neq 0 ; \\[2mm] \dfrac{d\varphi}{dx}(x) & \text{if } q = 1 ; \\[2mm] \dfrac{d\varphi}{dx}(0) & \text{if } x = 0 . \end{cases}$$

If we want to indicate the variable which the *q*-difference operator is applied to, we write $(D_{q,x} \varphi)(x, y)$ for the operator.

REMARK 2.   The definition (7) is more lucid than the one previously given, which was without the condition for $x = 0$. It leads to new so-called *q*-constants, or solutions to $(D_q \varphi)(x) = 0$.

DEFINITION 3.   Let [16] ε_j denote the operator which maps $f(x_j)$ to $f(q_j x_j)$. In our case all $q_j = q$.

The following operator [41] will be useful in chapter 5.

$$(8) \qquad \theta_j \equiv x_j D_{q, x_j}.$$

DEFINITION 4. Let the q-shifted factorial (compare [p. 38] [28]) be defined by

$$(9) \qquad \langle a\,;\,q\rangle_n \equiv \begin{cases} 1\,, & n = 0\,; \\ \prod_{m=0}^{n-1}(1 - q^{a+m}) & n = 1,\,2,\,\dots, \end{cases}$$

The Watson notation [26] will also be used

$$(10) \qquad (a\,;\,q)_n \equiv \begin{cases} 1\,, & n = 0\,; \\ \prod_{m=0}^{n-1}(1 - aq^m), & n = 1,\,2,\,\dots. \end{cases}$$

Furthermore,

$$(11) \qquad (a\,;\,q)_\infty \equiv \prod_{m=0}^{\infty}(1 - aq^m), \quad 0 < |q| < 1\,.$$

$$(12) \qquad (a\,;\,q)_\alpha \equiv \frac{(a\,;\,q)_\infty}{(aq^\alpha\,;\,q)_\infty}\,, \quad a \neq q^{-m-\alpha},\ m = 0,\,1,\,\dots.$$

$$(13) \qquad \langle a\,;\,q\rangle_\alpha \equiv \frac{\langle a\,;\,q\rangle_\infty}{\langle a + \alpha\,;\,q\rangle_\infty}\,, \quad a \neq -m - \alpha,\ m = 0,\,1,\,\dots.$$

DEFINITION 5. In the following, $\dfrac{\mathbb{C}}{\mathbb{Z}}$ will denote the space of complex numbers mod $\dfrac{2\pi i}{\log q}$. This is isomorphic to the cylinder $\mathbb{R} \times e^{2\pi i\theta}$, $\theta \in \mathbb{R}$. The operator

$$\sim\,:\,\frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$(14) \qquad a \mapsto a + \frac{\pi i}{\log q}\,.$$

Furthermore we define

$$(15) \qquad \widetilde{\langle a\,;\,q\rangle}_n \equiv \langle \tilde{a}\,;\,q\rangle_n.$$

By (14) it follows that

$$(16) \qquad \langle \widetilde{a; q} \rangle_n = \prod_{m=0}^{n-1} (1 + q^{a+m}),$$

so that this time the tilde denotes an involution which changes the minus sign to a plus sign in all the n factors of $\langle a; q \rangle_n$.

The following simple rules follow from (14). Clearly the first two equations are applicable to q-exponents. Compare [72, p. 110].

$$(17) \qquad \tilde{a} \pm b \equiv \widetilde{a \pm b} \bmod \frac{2\pi i}{\log q},$$

$$(18) \qquad \tilde{a} \pm \tilde{b} \equiv a \pm b \bmod \frac{2\pi i}{\log q},$$

$$(19) \qquad q^{\tilde{a}} = -q^a,$$

where the second equation is a consequence of the fact that we work mod $\dfrac{2\pi i}{\log q}$.

DEFINITION 6. Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notation Let $(a) = (a_1, \ldots, a_A)$ be a vector with A elements. Then

$$(20) \qquad \langle (a); q \rangle_n \equiv \langle a_1, \ldots, a_A; q \rangle_n \equiv \prod_{j=1}^{A} \langle a_j; q \rangle_n.$$

The following notation will be convenient.

$$(21) \qquad \mathrm{QE}(x) \equiv q^x.$$

When there are several q:s, we generalize this

$$(22) \qquad \mathrm{QE}(x, q_i) \equiv q_i^x.$$

The q-hypergeometric series was developed by Heine 1846 [32] as a generalization of the hypergeometric series.

DEFINITION 7. Generalizing Heine's series, compare [26, p. 4], we shall define a q-hypergeometric series by

$$(23) \qquad {}_{p+p'}\phi_{r+r'}(\hat{a}_1, \ldots, \hat{a}_p; \hat{b}_1, \ldots, \hat{b}_r | q, z \| s_1, \ldots, s_{p'}; t_1, \ldots, t_{r'}) \equiv$$

$$\equiv {}_{p+p'}\phi_{r+r'}\left[\begin{matrix}\widehat{a}_1, \ldots, \widehat{a}_p, \\ \widehat{b}_1, \ldots, \widehat{b}_r\end{matrix}\bigg|q, z\bigg\|\begin{matrix}s_1, \ldots, s_{p'} \\ t_1, \ldots, t_{r'}\end{matrix}\right] =$$

$$= \sum_{n=0}^{\infty} \frac{\langle \widehat{a}_1; q\rangle_n \ldots \langle \widehat{a}_p; q\rangle_n}{\langle 1; q\rangle_n \langle \widehat{b}_1; q\rangle_n \ldots \langle \widehat{b}_r; q\rangle_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+r+r'-p-p'} \times$$

$$\times z^n \prod_{k=1}^{p'}(s_k; q)_n \prod_{k=1}^{r'}(t_k; q)_n^{-1},$$

where $q \neq 0$ when $p + p' > r + r' + 1$, and

(24)
$$\widehat{a} \equiv \begin{cases} a, & \text{if no tilde is involved} \\ \tilde{a} & \text{otherwise}. \end{cases}$$

REMARK 3.   In a few cases, when $0 < |q| < 1$, the parameter $\widehat{a}$ in (23) will be the real plus infinity.

The factors $\langle \widehat{a}; q\rangle_n$ then correspond to multiplication by 1.

REMARK 4.   The parameters $\widehat{a}_i$ and $\widehat{b}_i$ to the left of $|$ in (23) are to be thought of as exponents; they are periodic mod $\dfrac{2\pi i}{\log q}$.

The general equation (23), with nonzero parameters to the left of $\|$ is used in certain special cases when we need factors $(t; q)_n$ in the q-series. One example is the q-analogue of a bilinear generating formula for Laguerre polynomials [23].

This notation will be further extended as follows.

EXAMPLE 1.   Let

(25)
$$E_q(x) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} x^k.$$

We have

(26)
$$D_q E_q(-x^2) = -x \sum_{k=0}^{\infty} \frac{(-x^2)^k (1+q^{k+1})}{\{k\}_q!}.$$

This can't be expressed in usual q-hypergeometric notation in a simple way. That's why we suggest the following extension of the notation, where an extra k appears in the index to indicate the summation variable. This elucidates the integration property of q-calculus, i.e. to get the proper q-analogue, every factor in the hypergeometric formula has to be

integrated in the (multiple) sum. The sum is the natural q-analogue of the integral, as is manifested by the definition of the q-integral

$$(27) \quad D_q E_q(-x^2) = -x\,_1\phi_{0,\,k}\left[\begin{matrix} \infty \\ - \end{matrix}\Big| q,\ -x^2(1-q)\Big\|\begin{matrix}- \\ -\end{matrix}\Big\|\Big|\Big|\begin{matrix}1+q^{k+1}\\ -\end{matrix}\right].$$

In general the new notation should look like, compare (72), (90).

DEFINITION 8.

$$(28) \quad {}_{p+p'+p''}\phi_{r+r'+r'',\,k}\left[\begin{matrix}\widehat{a}_1,\,\ldots,\,\widehat{a}_p \\ \widehat{b}_1,\,\ldots,\,\widehat{b}_r\end{matrix}\Big|q,\,z\Big\|\begin{matrix}s_1,\,\ldots,\,s_{p'}\\ t_1,\,\ldots,\,t_{r'}\end{matrix}\Big\|\Big|\dfrac{\prod\limits_i f_i(k)}{\prod\limits_j g_j(k)}\right] \equiv$$

$$\equiv \sum_{k=0}^{\infty} \frac{\langle \widehat{a}_1;\,q\rangle_k \ldots \langle \widehat{a}_p;\,q\rangle_k}{\langle 1;\,q\rangle_k \langle \widehat{b}_1;\,q\rangle_k \ldots \langle \widehat{b}_r;\,q\rangle_k}\Big[(-1)^k q^{\binom{k}{2}}\Big]^{1+r+r'-p-p'} \times$$

$$\times z^k \prod_{l=1}^{p'}(s_l;\,q)_k \prod_{l=1}^{r'}(t_l;\,q)_k^{-1}\frac{\prod\limits_i f_i(k)}{\prod\limits_j g_j(k)}\,,$$

where $q \neq 0$ when $p+p'+p'' > r+r'+r''+1$. We assume that the $f_i(k)$ and $g_j(k)$ contain p'' and r'' factors of the form $\langle \widehat{a(k)};\,q\rangle_k$ or $(s(k);\,q)_k$ respectively.

In the same way as the Γ function plays a basic role in complex analysis, the Γ_q function plays a fundamental role for q-calculus.

DEFINITION 9. Let the Γ_q-function be defined by [71], [36], [47]

$$(29) \quad \Gamma_q(x) \equiv \begin{cases} \dfrac{\langle 1;\,q\rangle_\infty}{\langle x;\,q\rangle_\infty}(1-q)^{1-x}, & \text{if } 0 < |q| < 1 \\[3mm] \dfrac{\langle 1;\,q^{-1}\rangle_\infty}{\langle x;\,q^{-1}\rangle_\infty}(q-1)^{1-x}q^{\binom{x}{2}}, & \text{if } 1 < q\,. \end{cases}$$

The following notation for quotients of Γ_q functions will sometimes be used.

DEFINITION 10.

$$(30) \quad \Gamma_q\left[\begin{matrix}a_1,\,\ldots,\,a_p\\ b_1,\,\ldots,\,b_r\end{matrix}\right] \equiv \frac{\Gamma_q(a_1)\ldots\Gamma_q(a_p)}{\Gamma_q(b_1)\ldots\Gamma_q(b_r)}\,.$$

We will give a longer exposition of the Γ_q function in [24]. Basically

it has similar properties as the Γ function except for the simple poles at $x = -n \pm \dfrac{2k\pi i}{\log q}$, $n, k \in \mathbb{N}$.

As was pointed out by Jackson [41, p. 78] the q-analogue of hypergeometric formulas with sums in the function arguments must contain a so-called q-addition. Jackson meant the Jackson-Hahn q-addition, compare [31, p. 362], which is given by

$$(31) \quad [x \pm y]_q^n \equiv \sum_{k=0}^{n} (\pm)^k \binom{n}{k}_q q^{\binom{k}{2}} y^k x^{n-k} = x^n \left(\mp \frac{y}{x} \, ; q \right)_n, \quad n = 0, 1, 2, \dots .$$

The Ward-AlSalam q-addition was invented by Ward 1936 [76, p. 256] and Al-Salam 1959 [4, p. 240]

$$(32) \qquad (a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}, \quad n = 0, 1, 2, \dots .$$

Unlike the Ward-AlSalam q-addition, the Jackson-Hahn q-addition is neither commutative nor associative, but on the other hand, it can be written as a finite product.

We will find the Ward-AlSalam q-addition more convenient for our purposes.

The paper [11] was a breakthrough in the studies of expansions for MHS. By using an inverse pair of symbolic operators, Burchnall & Chaundy were able to prove a large number of expansion formulas for Appell functions. These results were later generalized by Verma [75] and Srivastava [64] to Kampé de Fériet functions.

In 1942 Jackson [41] made the first attempt to find a q-analogue of [11]. Jackson [41, p. 70] and Agarwal [2, p. 187] defined four so-called normal q-Appell functions.

DEFINITION 11.

$$(33) \qquad \Phi_1(a; b, b'; c \,|\, q; x_1, x_2) \equiv$$

$$\equiv \sum_{m_1, m_2 = 0}^{\infty} \frac{\langle a; q \rangle_{m_1 + m_2} \langle b; q \rangle_{m_1} \langle b'; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1 + m_2}} x_1^{m_1} x_2^{m_2}.$$

$$(34) \qquad \Phi_2(a; b, b'; c, c' \,|\, q; x_1, x_2) \equiv$$

$$\equiv \sum_{m_1, m_2 = 0}^{\infty} \frac{\langle a; q \rangle_{m_1 + m_2} \langle b; q \rangle_{m_1} \langle b'; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1} \langle c'; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}.$$

(35) $\Phi_3(a, a'; b, b'; c \,|\, q; x_1, x_2) \equiv$

$$\equiv \sum_{m_1, m_2 = 0}^{\infty} \frac{\langle a; q \rangle_{m_1} \langle a'; q \rangle_{m_2} \langle b; q \rangle_{m_1} \langle b'; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1 + m_2}} x_1^{m_1} x_2^{m_2}.$$

(36) $\Phi_4(a; b; c, c' \,|\, q; x_1, x_2) \equiv$

$$\equiv \sum_{m_1, m_2 = 0}^{\infty} \frac{\langle a; q \rangle_{m_1 + m_2} \langle b; q \rangle_{m_1 + m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1} \langle c'; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2},$$

with the same convergence regions as before, and with even sharper convergence when $|q| < 1$ [41, p. 70].

For the four so-called abnormal q-Appell functions, $x_2^{m_2} q^{k \binom{m_2}{2}}$ replaces $x_2^{m_2}$ in the general term. We will see that normal functions (in a broader sense) correspond to the Ward-AlSalam q-addition, and abnormal functions, with $k = 1$, (in a broader sense) correspond to the Jackson-Hahn q-addition.

q-Contiguity relations for $_2\phi_1$ were found by Heine [33], and for q-Appell functions by Agarwal [3]. The first Saalschützian theorems for double series were published by Carlitz 1963 [13] and 1967 [14]. q-Analogues of [13] were found by W. A. Al-Salam [5]. W. N. Bailey (1893-1961) had been greatly influenced by Ramanujan as an undergraduate at Cambridge 1914 and wrote the first systematic treatment of hypergeometric series [10]. L. J. Slater attended Bailey's lectures on q-hypergeometric series in 1947-50 at London University and wrote many important papers on this subject; among her pupils were Howard Exton. R. P. Agarwal [2] visited Bailey, in 1953, and made the aforementioned contributions to the subject.

In 1966 Slater concluded [61, p. 234] that there seemed to be no systematic attempt to find summation theorems for basic Appell series, but Andrews [6] managed to prove some summation- and transformation-formulas for basic Appell series. Some of these do not have a hypergeometric counterpart. This is typical for q-calculus, we almost always get more equations than we had before.

In a series of papers [52], [53] Miller has proved that by using the maximal Lie algebra generated by the first order differential recurrence relations satisfied by a MHS, we obtain addition theorems, transformations and generating functions. Then in 1980 Kalnins, Manocha and Miller [45] developed a theory which enabled the powerful use of Lie algebraic methods for the solutions of partial differential equations by two-variable hypergeometric series. As was shown in [1], this technique can

also be adapted to multiple q-hypergeometric series. This was the Lie algebra approach.

There is also a finite group approach to transformation-formulas for q-MHS, which we will briefly summarize. In the footsteps of Rogers [57], [58], and Whipple [77], Lievens & Van der Jeugt [51] and Van der Jeugt [73] proved some formulas for q-MHS by using the invariance group or symmetry group of the particular double series.

In 1996 Kuznetsov & Sklyanin [49] showed the connection between (multiple) Macdonald polynomials and q-Lauricella functions.

Obviously we have not seen the end of this story yet as the subject gets more and more complex.

Various q-analogues of Kampé de Fériet functions occur in the literature, compare [69, p. 349-350] and [70, p. 50]. We give a definition reminiscent of [26], which allows easy confluence to diminish the dimensions in (37) and (38), and has the advantage of being symmertic in the variables. Furthermore, q is allowed to be a vector and the full machinery of tilde operators and q-additions will be used. The first definition is a q-analogue of [69, (24), p. 38], in the spirit of Srivastava. The second definition is a q-analogue of [69, (24), p. 38] with the restraint [69, (29), p. 38], due to Karlsson. It will be clear from the context which of the definitions we use.

DEFINITION 12.   The notation $\sum\limits_{m}$ denotes a multiple summation with the indices $m_1, \ldots, m_n$ running over all non-negative integer values. In this connection we put $m \equiv \sum\limits_{j=1}^{n} m_j$.

DEFINITION 13.   Let

$$(a), (b), (g_i), (h_i), (a'), (b'), (g_i'), (h_i')$$

have dimensions

$$A, B, G_i, H_i, A', B', G_i', H_i'.$$

Let

$$1 + B + B' + H_i + H_i' - A - A' - G_i - G_i' \geqslant 0, \; i = 1, \ldots, n.$$

Then the generalized q-Kampé de Fériet function is defined by

$$(37) \quad \Phi_{B+B':\, H_1+H_1';\, \ldots;\, H_n+H_n'}^{A+A':\, G_1+G_1';\, \ldots;\, G_n+G_n'} \begin{bmatrix} (\widehat{a}):(\widehat{g}_1);\, \ldots;(\widehat{g}_n); & (a'):(g_1');\, \ldots;(g_n'); \\ & \left| \boldsymbol{q}\, ;\, \boldsymbol{x} \right\| \\ (\widehat{b}):(\widehat{h}_1);\, \ldots;(\widehat{h}_n); & (b'):(h_1');\, \ldots;(h_n'); \end{bmatrix} \equiv$$

$$\equiv \sum_{m} \frac{\langle(\widehat{a}); q_0\rangle_{m_1 + \ldots + m_n}((a'); q_0)_{m_1 + \ldots + m_n} \prod_{j=1}^{n}(\langle(\widehat{g_j}); q_j\rangle_{m_j}((g_j'); q_j)_{m_j} x_j^{m_j})}{\langle(\widehat{b}); q_0\rangle_{m_1 + \ldots + m_n}((b'); q_0)_{m_1 + \ldots + m_n} \prod_{j=1}^{n}(\langle(\widehat{h_j}); q_j\rangle_{m_j}((h_j'); q_j)_{m_j}\langle 1; q_j\rangle_{m_j})} \times$$

$$\times (-1)^{\sum_{j=1}^{n} m_j(1 + H_j + H_j' - G_j - G_j' + B + B' - A - A')} \times$$

$$\mathrm{QE}\left((B + B' - A - A')\binom{m}{2}, q_0\right)\prod_{j=1}^{n}\mathrm{QE}\left((1 + H_j + H_j' - G_j - G_j')\binom{m_j}{2}, q_j\right).$$

DEFINITION 14.   Let

$$(a), (b), (g_i), (h_i), (a'), (b'), (g_i'), (h_i')$$

have dimensions

$$A, B, G, H, A', B', G', H', \forall i.$$

Let

$$1 + B + B' + H + H' - A - A' - G - G' \geqslant 0.$$

Then the generalized q-Kampé de Fériet function is defined by

$$(38) \qquad \Phi_{B+B':H+H'}^{A+A':G+G'}\left[\begin{matrix}(\widehat{a}):(\widehat{g}_1);\ \ldots;(\widehat{g}_n);\\ (\widehat{b}):(\widehat{h}_1);\ \ldots;(\widehat{h}_n);\end{matrix}\Big|\boldsymbol{q}; \boldsymbol{x}\Big\|\begin{matrix}(a'):(g_1');\ \ldots;(g_n');\\ (b'):(h_1');\ \ldots;(h_n');\end{matrix}\right] \equiv$$

$$\equiv \sum_{m} \frac{\langle(\widehat{a}); q_0\rangle_{m_1 + \ldots + m_n}((a'); q_0)_{m_1 + \ldots + m_n} \prod_{j=1}^{n}(\langle(\widehat{g_j}); q_j\rangle_{m_j}((g_j'); q_j)_{m_j} x_j^{m_j})}{\langle(\widehat{b}); q_0\rangle_{m_1 + \ldots + m_n}((b'); q_0)_{m_1 + \ldots + m_n} \prod_{j=1}^{n}(\langle(\widehat{h_j}); q_j\rangle_{m_j}((h_j'); q_j)_{m_j}\langle 1; q_j\rangle_{m_j})} \times$$

$$\times (-1)^{\sum_{j=1}^{n} m_j(1 + H + H' - G - G' + B + B' - A - A')} \times$$

$$\times \mathrm{QE}\left((B + B' - A - A')\binom{m}{2}, q_0\right)\prod_{j=1}^{n}\mathrm{QE}\left((1 + H + H' - G - G')\binom{m_j}{2}, q_j\right).$$

REMARK 5.   In case we want to get rid of the ∞ symbol, we just take away all the -1 and q-powers in the definitions. In the one-variable case, this was the original approach, see Thomae 1871 [72, p. 111], [62]. However in this case we have to watch out for convergence and we lose the confluence property.

2. Transformations for basic double series.

In some of the proofs we will need the following q-summation formulas. A q-analogue of the Dixon-Schafheitlin theorem [19], [59, p. 24 (22)] is a corollary of Jackson's $_8\phi_7$ sum.

(39) $\quad _4\phi_3 \begin{bmatrix} a,\, b,\, c,\, \overset{\frown}{1 + \dfrac{1}{2}a} \\[2mm] 1 + a - b,\, 1 + a - c,\, \overset{\frown}{\dfrac{1}{2}a} \end{bmatrix} q,\, q^{1 + \frac{a}{2} - b - c} \Big] =$

$$= \Gamma_q \begin{bmatrix} 1 + a - b,\, 1 + a - c,\, 1 + \dfrac{a}{2},\, 1 + \dfrac{a}{2} - b - c \\[2mm] 1 + a,\, 1 + a - b - c,\, 1 + \dfrac{a}{2} - b,\, 1 + \dfrac{a}{2} - c \end{bmatrix}.$$

A q-analogue of Kummer's formula [48], the Bailey-Daum summation formula, was proved independently by Bailey 1941 [9] and Daum 1942 [18]. This was the first example of a q-analogue of a summation formula for a hypergeometric series with argument -1.

THEOREM 2.1.

(40) $\quad _2\phi_1(a,\, b;\, 1 + a - b\,|\,q,\, -q^{1-b}) =$

$$= \Gamma_q \begin{bmatrix} 1 + a - b,\, 1 + \dfrac{a}{2} \\[2mm] 1 + a,\, 1 + \dfrac{a}{2} - b \end{bmatrix} \frac{\langle \overset{\frown}{1 + \dfrac{a}{2} - b},\, \tilde{1};\, q \rangle_\infty}{\langle \overset{\frown}{1 + \dfrac{a}{2}},\, \overset{\frown}{1 - b};\, q \rangle_\infty}.$$

where $|q^{a+1-b}| < 1$ *and* $1 + a - b \neq 0,\, -1,\, -2 \ldots$.

PROOF. Put $c = \dfrac{\tilde{a}}{2}$ in the previous formula or use [26, (1.8.1) p. 14]. The idea to use the q-Dixon-Schafheitlin theorem for this proof was mentioned in Daum's thesis [17], chapter 3. ∎

If a is a negative integer, (40) may be reformulated in the form

THEOREM 2.2.

(41) $\quad {}_2\phi_1(-2N, b; 1-2N-b\,|\,q, -q^{1-b}) \equiv$

$$\equiv \sum_{k=0}^{2N} \binom{2N}{k}_q q^{\binom{k}{2}+k(1-2N-b)} \frac{\langle b; q\rangle_k}{\langle 1-2N-b; q\rangle_k} \equiv$$

$$\equiv 2 \sum_{k=0}^{N-1} \binom{2N}{k}_q (-1)^k \frac{\langle b; q\rangle_k}{\langle 2N+b-k; q\rangle_k} + \binom{2N}{N}_q (-1)^N \frac{\langle b; q\rangle_N}{\langle N+b; q\rangle_N} =$$

$$= \frac{\langle \tilde{b}; q\rangle_N}{\langle N+b; q\rangle_N} \left\langle \frac{1}{2}; q^2 \right\rangle_N .$$

(42) $\qquad {}_2\phi_1(-N, b; 1-N-b\,|\,q, -q^{1-b}) = 0, \; N \text{ odd}.$

PROOF. The validity has been confirmed by Axel Riese (RISC) in Linz using Mathematica. ■

REMARK 6. For $b = +\infty$ this is a result of Gauss [27], [29]. Equation (41) is a q-analogue of [55, p. 43].

The following corollary enables us to find a relation for Γ_q-functions with negative integer argument.

COROLLARY 2.3.

(43) $\quad \Gamma_q \begin{bmatrix} 1-2N-b, \, 1-N \\ 1-2N, \, 1-N-b \end{bmatrix} = \frac{\langle \widetilde{1-N}, \tilde{b}; q\rangle_N}{\langle 1\, \widetilde{-N}-b, \, b+N; q\rangle_N} \left\langle \frac{1}{2}; q^2 \right\rangle_N .$

PROOF. Just equate (40) and (41). ■

We continue with a few examples where the Jackson [40], [26, p. 11, 1.5.4] q-analogue of the Euler-Pfaff-Kummer transformation is used.

(44) $\quad {}_2\phi_1(a, b; c\,|\,q, t) = \frac{1}{(t; q)_a} {}_2\phi_2(a, c-b; c\,|\,q, tq^b \| -; tq^a).$

When no convergence region is given for a certain formula, we first assume that the absolute value of the function arguments are small.

The following three expansions of Φ_1 in terms of one-variable ϕ are

q-analogues of the Appell and Kampé de Fériet [8, p. 24 (27), (27′), (27″)] formulas.

$$(45) \quad \Phi_1(\alpha; \beta, \beta'; \gamma | q; x_1, x_2) = \sum_{m_1=0}^{\infty} \frac{\langle \alpha, \beta; q \rangle_{m_1} x_1^{m_1}}{\langle 1, \gamma; q \rangle_{m_1} (x_2; q)_{\alpha+m_1}} \times$$
$$\times \, {}_2\phi_2(\alpha + m_1, \gamma - \beta' + m_1; \gamma + m_1 | q, x_2 q^{\beta'} \| -; x_2 q^{\alpha+m_1}).$$

$$(46) \quad \Phi_1(\alpha; \beta, \beta'; \gamma | q; x_1, x_2) = \sum_{m_1=0}^{\infty} \frac{\langle \alpha, \beta; q \rangle_{m_1} x_1^{m_1}}{\langle 1, \gamma; q \rangle_{m_1} (x_2; q)_{\beta'}} \times$$
$$\times \, {}_2\phi_2(\beta', \gamma - \alpha; \gamma + m_1 | q, x_2 q^{\alpha+m_1} \| -; x_2 q^{\beta'}).$$

$$(47) \quad \Phi_1(\alpha; \beta, \beta'; \gamma | q; x_1, x_2) = \sum_{m_1=0}^{\infty} \frac{\langle \alpha, \beta; q \rangle_{m_1} x_1^{m_1}}{\langle 1, \gamma; q \rangle_{m_1} (x_2; q)_{\alpha+\beta'-\gamma}} \times$$
$$\times \, {}_2\phi_1(\gamma - \alpha, \gamma - \beta' + m_1; \gamma + m_1 | q, x_2 q^{\alpha+\beta'-\gamma}).$$

By Jackson [40] we obtain the following two expansions of ${}_2\phi_1(\,|q, x_1 \oplus_q x_2)$:

$$(48) \quad {}_2\phi_1(\alpha, \beta; \gamma | q, x_1 \oplus_q x_2) = \sum_{m_1=0}^{\infty} \frac{\langle \alpha, \beta; q \rangle_{m_1} x_1^{m_1}}{\langle 1, \gamma; q \rangle_{m_1} (x_2; q)_{\alpha+m_1}} \times$$
$$\times \, {}_2\phi_2(\alpha + m_1, \gamma - \beta; \gamma + m_1 | q, x_2 q^{\beta+m_1} \| -; x_2 q^{\alpha+m_1}).$$

$$(49) \quad {}_2\phi_1(\alpha, \beta; \gamma | q, x_1 \oplus_q x_2) = \sum_{m_1=0}^{\infty} \frac{\langle \alpha, \beta; q \rangle_{m_1} x_1^{m_1}}{\langle 1, \gamma; q \rangle_{m_1} (x_2; q)_{\beta+m_1}} \times$$
$$\times \, {}_2\phi_2(\beta + m_1, \gamma - \alpha; \gamma + m_1 | q, x_2 q^{\alpha+m_1} \| -; x_2 q^{\beta+m_1}).$$

By [34, p. 115] we obtain

$$(50) \quad {}_2\phi_1(\alpha, \beta; \gamma | q, x_1 \oplus_q x_2) = \sum_{m_1=0}^{\infty} \frac{\langle \alpha, \beta; q \rangle_{m_1} x_1^{m_1}}{\langle 1, \gamma; q \rangle_{m_1} (x_2; q)_{\alpha+\beta-\gamma+m_1}} \times$$
$$\times \, {}_2\phi_1(\gamma - \alpha, \gamma - \beta; \gamma + m_1 | q, x_2 q^{\alpha+\beta-\gamma+m_1}).$$

The following three expansions of Φ_2 in terms of one-variable ϕ are q-analogues of [8, p. 24 (30), (30′), (30″)]

$$(51) \quad \Phi_2(\alpha; \beta, \beta'; \gamma, \gamma' | q; x_1, x_2) = \sum_{m_1=0}^{\infty} \frac{\langle \alpha, \beta; q \rangle_{m_1} x_1^{m_1}}{\langle 1, \gamma; q \rangle_{m_1} (x_2; q)_{\alpha+m_1}} \times$$
$$\times \, {}_2\phi_2(\alpha + m_1, \gamma' - \beta'; \gamma' | q, x_2 q^{\beta'} \| -; x_2 q^{\alpha+m_1}).$$

$$(52) \quad \Phi_2(\alpha; \beta, \beta'; \gamma, \gamma' \,|\, q; x_1, x_2) = \sum_{m_1=0}^{\infty} \frac{\langle \alpha, \beta; q \rangle_{m_1} x_1^{m_1}}{\langle 1, \gamma; q \rangle_{m_1} (x_2; q)_{\beta'}} \times$$

$$\times \, {}_2\phi_2(\beta', \gamma' - \alpha - m_1; \gamma' \,|\, q, x_2 q^{\alpha + m_1} \| -; x_2 q^{\beta'}).$$

$$(53) \quad \Phi_2(\alpha; \beta, \beta'; \gamma, \gamma' \,|\, q; x_1, x_2) = \sum_{m_1=0}^{\infty} \frac{\langle \alpha, \beta; q \rangle_{m_1} x_1^{m_1}}{\langle 1, \gamma; q \rangle_{m_1} (x_2; q)_{\alpha + \beta' - \gamma' + m_1}} \times$$

$$\times \, {}_2\phi_1(\gamma' - \alpha - m_1, \gamma' - \beta'; \gamma' \,|\, q, x_2 q^{\alpha + \beta' - \gamma' + m_1}).$$

By [40], we get a q-analogue of [8, p. 25 (31)]

$$\Phi_2(\alpha; \beta, \beta'; \gamma, \gamma' \,|\, q; x_1, x_2) = \frac{1}{(x_2; q)_\alpha} \sum_{m_1=0}^{\infty} \frac{\langle \alpha, \beta; q \rangle_{m_1} x_1^{m_1}}{\langle 1, \gamma; q \rangle_{m_1}} (x_2 q^\alpha; q)_{m_1} \times$$

$$\times \sum_{m_2=0}^{\infty} \frac{\langle \alpha + m_1, \gamma' - \beta'; q \rangle_{m_2} (x_2 q^{\beta'})^{m_2}}{\langle 1, \gamma'; q \rangle_{m_2} (x_2 q^{\alpha + m_1}; q)_{m_2}} (-1)^{m_2} q^{\binom{m_2}{2}} =$$

$$= \frac{1}{(x_2; q)_\alpha} \Phi_{1:1;2}^{1:1;1} \begin{bmatrix} \alpha : \beta; \gamma' - \beta'; \\ - : \gamma; \gamma', \infty; \end{bmatrix} q; x_1, x_2 q^{\beta'} \, \Big\| \begin{matrix} -: -; \\ x_2 q^\alpha : -; \end{matrix} \Big].$$

The following three expansions of Φ_3 in terms of one-variable ϕ are q-analogues of [8, p. 25 (33), (33'), (33'')]

$$(54) \quad \Phi_3(\alpha, a'; \beta, \beta'; \gamma \,|\, q; x_1, x_2) = \sum_{m_1=0}^{\infty} \frac{\langle \alpha, \beta; q \rangle_{m_1} x_1^{m_1}}{\langle 1, \gamma; q \rangle_{m_1} (x_2; q)_{a'}} \times$$

$$\times \, {}_2\phi_2(\alpha', \gamma - \beta' + m_1; \gamma + m_1 \,|\, q, x_2 q^{\beta'} \| -; x_2 q^{a'}).$$

$$(55) \quad \Phi_3(\alpha, a'; \beta, \beta'; \gamma \,|\, q; x_1, x_2) = \sum_{m_1=0}^{\infty} \frac{\langle \alpha, \beta; q \rangle_{m_1} x_1^{m_1}}{\langle 1, \gamma; q \rangle_{m_1} (x_2; q)_{\beta'}} \times$$

$$\times \, {}_2\phi_2(\beta', \gamma - \alpha' + m_1; \gamma + m_1 \,|\, q, x_2 q^{a'} \| -; x_2 q^{\beta'}).$$

$$(56) \quad \Phi_3(\alpha, a'; \beta, \beta'; \gamma \,|\, q; x_1, x_2) = \sum_{m_1=0}^{\infty} \frac{\langle \alpha, \beta; q \rangle_{m_1} x_1^{m_1}}{\langle 1, \gamma; q \rangle_{m_1} (x_2; q)_{a' + \beta' - \gamma - m_1}} \times$$

$$\times \, {}_2\phi_1(\gamma - \alpha' + m_1, \gamma - \beta' + m_1; \gamma + m_1 \,|\, q, x_2 q^{a' + \beta' - \gamma - m_1}).$$

Our next task is to find q-analogues of some general formulas of Carlson [15]. We first recall the following two equivalent forms of the q-Vandermonde theorem. The second one also appeared in [41, (29) p. 74] and [42, (60) p. 56].

LEMMA 2.4.

$$(57) \qquad \sum_{m+n=N} \frac{\langle b; q \rangle_m \langle b'; q \rangle_n}{\langle 1; q \rangle_m \langle 1; q \rangle_n} q^{-nb'} = q^{-Nb'} \frac{\langle b+b'; q \rangle_N}{\langle 1; q \rangle_N}.$$

$$(58) \qquad \sum_{m+n=N} \frac{\langle b; q \rangle_m \langle b'; q \rangle_n}{\langle 1; q \rangle_m \langle 1; q \rangle_n} q^{nb} = \frac{\langle b+b'; q \rangle_N}{\langle 1; q \rangle_N}.$$

The order classification [35] for double hypergeometric series (DHS) was prevalent for a long time. Carlson [15] showed evidence in support of the need for a new classification of DHS.

THEOREM 2.5. *A q-analogue of Carlson's transformation* [15, (8), p. 223].

$$(59) \qquad \sum_{m+n=N} \frac{\langle b;q \rangle_m \langle b';q \rangle_n}{\langle 1;q \rangle_m \langle 1;q \rangle_n} q^{-nb'} (x_1 \oplus_q x_2)^m x_2^n =$$

$$= \langle b+b'; q \rangle_N \sum_{m+n=N} \frac{\langle b; q \rangle_m}{\langle b+b', 1; q \rangle_m \langle 1; q \rangle_n} q^{-nb'} x_1^m x_2^n.$$

PROOF.

$$(60) \qquad LHS = \sum_{n=0}^{N} \frac{\langle b; q \rangle_{N-n} \langle b'; q \rangle_n}{\langle 1; q \rangle_{N-n} \langle 1; q \rangle_n} q^{-nb'} \sum_{m'=0}^{N-n} \binom{N-n}{m'}_q x_1^{m'} x_2^{N-m'} =$$

$$= \sum_{m=0}^{N} \frac{x_1^m x_2^{N-m}}{\langle 1; q \rangle_m} \sum_{n=0}^{N-m} \frac{\langle b; q \rangle_{N-n} \langle b'; q \rangle_n}{\langle 1; q \rangle_n \langle 1; q \rangle_{N-n-m}} q^{-nb'} \overset{\text{by}(57)}{=} =$$

$$= \sum_{m=0}^{N} \frac{x_1^m x_2^{N-m}}{\langle 1; q \rangle_m} q^{-b'(N-m)} \frac{\langle b; q \rangle_m \langle b+b'+m; q \rangle_{N-m}}{\langle 1; q \rangle_{N-m}} =$$

$$= \sum_{m+n=N} \frac{x_1^m x_2^n}{\langle 1; q \rangle_m} q^{-b'n} \frac{\langle b; q \rangle_m \langle b+b'+m; q \rangle_{N-m}}{\langle 1; q \rangle_n} = RHS.$$

REMARK 7.   One obtains three similar results by i) using equation (58) in place of (57), or ii) using $[x_2 + x_1]_q^m$ in place of $(x_1 \oplus_q x_2)^m$ or iii) making both of these changes.

COROLLARY 2.6. *A q-analogue of* [15, (4), p. 222], *which is valid when*

$$|x_1| + |x_2| < 1, \quad a \ne c, \quad c \ne b + b', \quad b \ne 1, \quad b' \ne 1,$$

neither c nor $b + b' = 0$ *or* $-N$.

$$(61) \qquad \sum_{m,n=0}^{\infty} \frac{\langle a; q\rangle_{m+n}\langle b; q\rangle_m\langle b'; q\rangle_n}{\langle c; q\rangle_{m+n}\langle 1; q\rangle_m\langle 1; q\rangle_n} q^{-nb'}(x_1 \oplus_q x_2)^m x_2^n =$$

$$= \sum_{m,n=0}^{\infty} \frac{\langle a, b+b'; q\rangle_{m+n}\langle b; q\rangle_m}{\langle c; q\rangle_{m+n}\langle b+b', 1; q\rangle_m\langle 1; q\rangle_n} q^{-nb'} x_1^m x_2^n.$$

REMARK 8. Three other *q*-analogues of Carlson's transformation may be obtained via the three variants of Theorem 2.5 mentioned above.

3. *q*-Analogues of some of Srivastava's formulas.

In this chapter we will derive some more complex *q*-analogues than before. When certain special cases of these (non-*q*)-formulas have been published, we try to find *q*-analogues of them too. We will prove two general multiple formulas via the *q*-Vandermonde sums, and two general formulas by use of (41). The Bailey-Daum theorem will also be used.

We first derive two *q*-analogues of the general formula [65, (4), p. 295] (62) and (66).

THEOREM 3.1. *If* $\{C_n\}_{n=0}^{\infty}$ *is a sequence of arbitrary complex numbers then*

$$(62) \qquad \sum_{m,n=0}^{\infty} \frac{C_{m+n}x^{m+n}q^{2\binom{n}{2}+\sigma n}}{\langle 1, \nu; q\rangle_m\langle 1, \sigma; q\rangle_n} = \sum_{N=0}^{\infty} \frac{C_N x^N}{\langle 1, \nu, \sigma; q\rangle_N}\langle \sigma - 1 + \nu + N; q\rangle_N.$$

PROOF.

$$(63) \qquad LHS =$$

$$= \sum_{N=0}^{\infty} \frac{C_N x^N}{\langle 1, \nu; q\rangle_N} \sum_{n=0}^{N} \frac{\langle -\nu + 1 - N, -N; q\rangle_n}{\langle 1, \sigma; q\rangle_n} q^{n(-1+\nu+2N+\sigma)} = RHS. \qquad \blacksquare$$

COROLLARY 3.2. *A first q-analogue of [65, (8), p. 296]. If* $|x| < \dfrac{1}{4}$
then

(64)
$$\sum_{m,\,n=0}^{\infty} \frac{\langle \lambda, \mu; q \rangle_{m+n}\, x^{m+n}\, q^{2\binom{n}{2}+\sigma n}}{\langle \nu, 1; q \rangle_m \langle \sigma, 1; q \rangle_n} =$$

$$= \sum_{n=0}^{\infty} \frac{\left\langle \lambda, \mu, \dfrac{\nu+\sigma-1}{2}, \dfrac{\widetilde{\nu+\sigma-1}}{2}, \dfrac{\nu+\sigma}{2}, \dfrac{\widetilde{\nu+\sigma}}{2}; q \right\rangle_n x^n}{\langle \nu, 1, \sigma, \nu+\sigma-1; q \rangle_n}.$$

REMARK 9.   The convergence radius $R(q)$ in Corollary 3.2 is the one
for $q = 1$. We feel that the convergence radius for $|q| < 1$ is in fact big-
ger, and that $R(q) < 1$. This is justified by the following computation.
Denote the terms of the RHS sum in (64) $a_n(x)$. Then the quotient crite-
rion gives

(65) $\lim n \to +\infty \left| \dfrac{a_{n+1}(x)}{a_n(x)} \right| = \lim n \to +\infty \times$

$$\times \left| \frac{\left\langle \lambda+n, \mu+n, \dfrac{\nu+\sigma-1}{2}+n, \dfrac{\widetilde{\nu+\sigma-1}}{2}+n, \dfrac{\nu+\sigma}{2}+n, \dfrac{\widetilde{\nu+\sigma}}{2}+n; q \right\rangle_1}{\langle \nu+n, 1+n, \sigma+n, \nu+\sigma-1+n; q \rangle_1} x \right| = |x|.$$

This implies that the RHS of (64) should converge for $|x| < 1$, when
$|q| < 1$. However, things are not so easy as we have an equality between
a double sum and a simple sum as functions of the two complex variables
x and q. In general, the convergence works best when both $|q|$ and $|x|$
are far away from 1. The same reasoning obtains for Corollary 3.4 and
Corollary 3.6.

THEOREM 3.3.   *If* $\{C_n\}_{n=0}^{\infty}$ *is a sequence of arbitrary complex num-*
bers then

(66)
$$\sum_{m,\,n=0}^{\infty} \frac{C_{m+n}\, x^{m+n}\, q^{2\binom{n}{2}+n(-2(m+n)-\nu+2)}}{\langle 1, \nu; q \rangle_m \langle 1, \sigma; q \rangle_n} =$$

$$= \sum_{N=0}^{\infty} \frac{C_N\, x^N}{\langle 1, \nu, \sigma; q \rangle_N} \langle \sigma-1+\nu+N; q \rangle_N\, q^{N(1-\nu-N)}.$$

COROLLARY 3.4. *A second q-analogue of* [65, (8), p. 296].

$$(67) \quad \sum_{m,n=0}^{\infty} \frac{\langle \lambda, \mu; q \rangle_{m+n} \, x^{m+n} \, q^{2\binom{n}{2} + n(-2(m+n) - \nu + 2)}}{\langle \nu, 1; q \rangle_m \langle \sigma, 1; q \rangle_n} =$$

$$= \sum_{n=0}^{\infty} \frac{\langle \lambda, \mu, \dfrac{\nu + \sigma - 1}{2}, \dfrac{\widetilde{\nu + \sigma - 1}}{2}, \dfrac{\nu + \sigma}{2}, \dfrac{\widetilde{\nu + \sigma}}{2}; q \rangle_n \, x^n \, q^{n(1 - \nu - n)}}{\langle \nu, 1, \sigma, \nu + \sigma - 1; q \rangle_n}, \quad |x| < \frac{1}{4}.$$

THEOREM 3.5. *A q-analogue of* [65, (5), p. 295].
If $\{C_n\}_{n=0}^{\infty}$ is a sequence of arbitrary complex numbers then

$$(68) \quad \sum_{m,n=0}^{\infty} \frac{(-1)^n \, C_{m+n} \, x^{m+n} \, q^{-mn}}{\langle 1, \sigma; q \rangle_m \langle 1, \sigma; q \rangle_n} =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{C_{2n} \, x^{2n}}{\langle \sigma; q \rangle_{2n}} \frac{\langle -\sigma \, \widetilde{+1-} \, 2n; q \rangle_n \, q^{\binom{n}{2} + n\sigma}}{\langle 1, \tilde{1}, \sigma; q \rangle_n}.$$

PROOF.

$$(69) \quad LHS = \sum_{N=0}^{\infty} \frac{C_N \, x^N}{\langle 1, \sigma; q \rangle_N} \sum_{n=0}^{N} (-1)^n \frac{\langle -\sigma + 1 - N, -N; q \rangle_n}{\langle 1, \sigma; q \rangle_n} \, q^{n(N + \sigma)} =$$

$$= \sum_{N=0, N \text{ even}}^{\infty} \frac{C_N \, x^N}{\langle 1, \sigma; q \rangle_N} \frac{\langle -\sigma \, \widetilde{+1-} \, N; q \rangle_{N/2}}{\langle -\sigma + 1 - \dfrac{N}{2}; q \rangle_{N/2}} \left\langle \frac{1}{2}; q^2 \right\rangle_{N/2} =$$

$$= \sum_{n=0}^{\infty} \frac{C_{2n} \, x^{2n}}{\langle 1, \sigma; q \rangle_{2n}} \frac{\langle -\sigma \, \widetilde{+1-} \, 2n; q \rangle_n}{\langle -\sigma + 1 - n; q \rangle_n} \left\langle \frac{1}{2}; q^2 \right\rangle_n = RHS. \quad \blacksquare$$

COROLLARY 3.6. *A q-analogue of* [65, (9), p. 296]. *If $|x| < \dfrac{1}{4}$*
then

$$(70) \quad \sum_{m,n=0}^{\infty} (-1)^n \frac{\langle \lambda, \mu; q \rangle_{m+n} \, x^{m+n} \, q^{-mn}}{\langle \sigma, 1; q \rangle_m \langle \sigma, 1; q \rangle_n} =$$

$$= \sum_{n=0}^{\infty} \frac{\left\langle \dfrac{\lambda}{2}, \dfrac{\lambda+1}{2}, \dfrac{\mu}{2}, \dfrac{\mu+1}{2}, \dfrac{\widetilde{\lambda}}{2}, \dfrac{\widetilde{\lambda+1}}{2}, \dfrac{\widetilde{\mu}}{2}, \dfrac{\widetilde{\mu+1}}{2}; q \right\rangle_n x^{2n}}{\left\langle \dfrac{\sigma}{2}, \dfrac{\sigma+1}{2}, \dfrac{\widetilde{\sigma}}{2}, \dfrac{\widetilde{\sigma+1}}{2}, \sigma, 1; q \right\rangle_n} \times$$

$$\times \frac{\left\langle -\sigma \ \widetilde{+1-} \ 2n; q \right\rangle_n}{\left\langle \tilde{1}; q \right\rangle_n} (-1)^n q^{\binom{n}{2}+n\sigma}.$$

By (62) we get

COROLLARY 3.7. *A first q-analogue of a reduction formula for the Humbert function* [65, (12), p. 296].

$$(71) \qquad \Phi_{1:1;1}^{1:2;0} \begin{bmatrix} \mu : \infty, \ \infty; \ - ; \\ \infty : \nu; \ \sigma; \end{bmatrix} q; x, xq^{\sigma} \end{bmatrix} =$$

$$= {}_5\phi_4 \begin{bmatrix} \mu, \dfrac{\nu+\sigma-1}{2}, \dfrac{\nu+\sigma}{2}, \dfrac{\widetilde{\nu+\sigma-1}}{2}, \dfrac{\widetilde{\nu+\sigma}}{2} \ \Big| q, x \\ \nu, \sigma, \nu+\sigma-1, \ \infty \end{bmatrix}.$$

By (68) we get

COROLLARY 3.8. *A q-analogue of a reduction formula for the Humbert function* [65, (13), p. 296].

$$(72) \qquad \sum_{m,n=0}^{\infty} \frac{(-1)^n \langle \mu; q \rangle_n x^{m+n} q^{-mn}}{\langle 1, \nu; q \rangle_m \langle 1, \nu; q \rangle_n} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \langle -\nu \ \widetilde{+1-} \ 2n; q \rangle_n x^{2n} q^{\binom{n}{2}+n\nu}}{\langle 1, \tilde{1}; q \rangle_n} \frac{\left\langle \dfrac{\mu}{2}, \dfrac{\mu+1}{2}, \dfrac{\widetilde{\mu}}{2}, \dfrac{\widetilde{\mu+1}}{2}; q \right\rangle_n}{\left\langle \dfrac{\nu}{2}, \dfrac{\nu+1}{2}, \dfrac{\widetilde{\nu}}{2}, \dfrac{\widetilde{\nu+1}}{2}; q \right\rangle_n} \equiv$$

$$\equiv {}_6\phi_{6,k}\left[\begin{array}{c} \dfrac{\mu}{2},\dfrac{\mu+1}{2},\dfrac{\tilde{\mu}}{2},\widetilde{\dfrac{\mu+1}{2}},\infty \\[2mm] \dfrac{\nu}{2},\dfrac{\nu+1}{2},\dfrac{\tilde{\nu}}{2},\widetilde{\dfrac{\nu+1}{2}},\nu,\tilde{1} \end{array}\Big|q,\,-x^2q^\nu\Big\|-\|\Big|\right.\genfrac{\langle}{}{0pt}{}{-\nu\ \widetilde{+1-}\ 2k;\,q\rangle_k}{-}\,.$$

COROLLARY 3.9. *A q-analogue of* [65, (14), p. 297], *which leads to a relation for the product of the two Jackson q-Bessel functions* [26], [37], [38].

(73) $\qquad {}_2\phi_1(\infty,\,\infty\,;\,\nu\,|\,q,\,x)\,{}_0\phi_1(-\,;\,\sigma\,|\,q,\,xq^\sigma) =$

$$= {}_4\phi_3\left[\begin{array}{c} \dfrac{\nu+\sigma-1}{2},\dfrac{\nu+\sigma}{2},\widetilde{\dfrac{\nu+\sigma-1}{2}},\widetilde{\dfrac{\nu+\sigma}{2}} \\[2mm] \nu,\sigma,\nu+\sigma-1 \end{array}\Big|q,\,x\right].$$

THEOREM 3.10. *The first q-analogue of* [65, (16), p. 297].

(74) $\qquad \displaystyle\sum_{m,\,n=0}^{\infty}\frac{C_{m+n}x^{m+n}\langle\nu;q\rangle_m\langle\sigma;q\rangle_n q^{-n\sigma}}{\langle1;q\rangle_m\langle1;q\rangle_n}=\sum_{n=0}^{\infty}\frac{C_n x^n\langle\nu+\sigma;q\rangle_n}{\langle1;q\rangle_n}q^{-n\sigma}\,.$

PROOF.

(75) $\qquad LHS=\displaystyle\sum_{N=0}^{\infty}\sum_{n=0}^{N}\frac{C_N x^N\langle-N,\sigma;q\rangle_n\langle\nu;q\rangle_N q^{n(-\nu+1-\sigma)}}{\langle1;q\rangle_N\langle1,-\nu+1-N;q\rangle_n}=$

$$=\sum_{N=0}^{\infty}\frac{C_N x^N\langle\nu,-\nu+1-\sigma-N;q\rangle_N}{\langle1,-\nu+1-N;q\rangle_N}=RHS\,.\qquad\blacksquare$$

There is also a second q-analogue of [65, (16), p. 297].

THEOREM 3.11.

(76) $\qquad \displaystyle\sum_{m,\,n=0}^{\infty}\frac{C_{m+n}x^{m+n}\langle\nu;q\rangle_m\langle\sigma;q\rangle_n q^{n\nu}}{\langle1;q\rangle_m\langle1;q\rangle_n}=\sum_{n=0}^{\infty}\frac{C_n x^n\langle\nu+\sigma;q\rangle_n}{\langle1;q\rangle_n}\,.$

By putting

(77) $\qquad C_n=\dfrac{\langle a;q\rangle_n}{\langle c;q\rangle_n}\,,\quad \nu=b,\quad \sigma=b\,',\quad x=q^{c-a-b}$

in the two equations above, we get the following two special cases, which are q-analogues of [8, (24), p. 22].

COROLLARY 3.12.

(78) $\Phi_1(a; b, b'; c|q; q^{c-a-b}, q^{c-a-b-b'}) =$

$$= \Phi_1(a; b, b'; c|q; q^{c-a-b-b'}, q^{c-a-b'}) = \Gamma_q \begin{bmatrix} c, c-a-b-b' \\ c-a, c-b-b' \end{bmatrix},$$

where $|q^{c-a-b-b'}| < 1$.

For $n = 2$, the above theorem implies the following reduction theorem for a q-Lauricella function from [21, p. 5 (27)].

THEOREM 3.13.

(79) $\Phi_D^{(n)}(a, b_1, \ldots, b_n; c|q; x, xq^{-b_2}, xq^{-b_2-b_3}, \ldots, xq^{-b_2-\cdots-b_n}) =$

$$= {}_2\phi_1(a, b_1 + \ldots + b_n; c|q, xq^{-b_2-\cdots-b_n}),$$

where

(80) $\Phi_D^{(n)}(a, b_1, \ldots, b_n; c|q; x_1, \ldots, x_n) \equiv$

$$\equiv \sum_m \frac{\langle a; q\rangle_{m_1+\ldots+m_n} \prod_{j=1}^{n} \langle b_j; q\rangle_{m_j} x_j^{m_j}}{\langle c; q\rangle_{m_1+\ldots+m_n} \prod_{j=1}^{n} \langle 1; q\rangle_{m_j}}, \ \max(|x|, \ldots, |x_n|) < 1.$$

THEOREM 3.14.   *A q-analogue of [65, (17), p. 297].*

(81) $\displaystyle\sum_{m,n=0}^{\infty} \frac{(-1)^n C_{m+n} x^{m+n} \langle v; q\rangle_m \langle v; q\rangle_n}{\langle 1; q\rangle_m \langle 1; q\rangle_n} = \sum_{n=0}^{\infty} \frac{C_{2n} x^{2n} \langle v, \tilde{v}; q\rangle_n}{\langle 1; q^2\rangle_n}.$

PROOF.   See the proof of (88).   ∎

By applying the Bailey-Daum theorem, we can prove the following

THEOREM 3.15.   *A q-analogue of* [63, p. 104]. *If a* ≠ *negative integer then*

(82) $\Phi_1(a; b, b'; 1 + a + b - b' \,|\, q; q^{1-b'}, -q^{1-b'}) =$

$$= \Gamma_q \left[\begin{matrix} 1 + a + b - b', 1 - b', 1 + \dfrac{a}{2} \\ 1 + a, 1 + b - b', 1 + \dfrac{a}{2} - b' \end{matrix} \right] \frac{\left\langle 1 + \overbrace{\dfrac{a}{2} - b'}, \tilde{1}; q \right\rangle_\infty}{\left\langle \overbrace{1 + \dfrac{a}{2}}, \overbrace{1 - b'}; q \right\rangle_\infty},$$

where $|q^{1-b'}| < 1$.

PROOF.

(83) $LHS = \displaystyle\sum_{n=0}^{\infty} \frac{\langle a, b'; q \rangle_n}{\langle 1, 1 + a + b - b'; q \rangle_n} (-1)^n q^{n(1-b')} =$

$= \displaystyle\sum_{m=0}^{\infty} \frac{\langle a + n, b; q \rangle_m}{\langle 1, 1 + a + b - b' + n; q \rangle_m} q^{m(1-b')} =$

$= \displaystyle\sum_{n=0}^{\infty} \frac{\langle a, b'; q \rangle_n}{\langle 1; q \rangle_n} (-1)^n q^{n(1-b')} \Gamma_q \left[\begin{matrix} 1 + a + b - b', 1 - b' \\ 1 + b - b', 1 + a - b' + n \end{matrix} \right] (1-q)^{-n} =$

$= \displaystyle\sum_{n=0}^{\infty} \frac{\langle a, b'; q \rangle_n}{\langle 1, 1 + a - b'; q \rangle_n} (-1)^n q^{n(1-b')} \Gamma_q \left[\begin{matrix} 1 + a + b - b', 1 - b' \\ 1 + b - b', 1 + a - b' \end{matrix} \right] =$

$= RHS.$ ∎

When *a* is a negative integer, the above theorem may be reformulated as

THEOREM 3.16.

(84) $\Phi_1(-2N; b, b'; 1 + a + b - b' \,|\, q; q^{1-b'}, -q^{1-b'}) =$

$$= \Gamma_q \left[\begin{matrix} 1 - 2N + b - b', 1 - b' \\ 1 - 2N - b', 1 + b - b' \end{matrix} \right] \frac{\langle \widetilde{b'}; q \rangle_N}{\langle N + b'; q \rangle_N} \left\langle \frac{1}{2}; q^2 \right\rangle_N.$$

(85) $\Phi_1(-N; b, b'; 1 + a + b - b' \,|\, q; q^{1-b'}, -q^{1-b'}) = 0,$ *N* odd.

The following theorem is a *q*-analogue of Srivastavas generalization

of Carlson's identity from [67, (3), p. 139]. Notice the difference in character to the equations in chapter two.

THEOREM 3.17. *If* $\{C_n\}_{n=0}^{\infty}$ *is a bounded sequence of complex numbers,*

$$x_1 \neq 0, \quad v \neq 1, \quad \sigma \neq 1, \quad v + \sigma \neq 0, -1, -2\ldots$$

and the two double series are absolutely convergent, then

(86)
$$\sum_{m,\,n=0}^{\infty} \frac{C_{m+n} x_2^{m+n} \left(-\frac{x_1}{x_2} q^{-\sigma}; q \right)_m \langle v; q \rangle_m \langle \sigma; q \rangle_n}{\langle 1; q \rangle_m \langle 1; q \rangle_n} \times$$

$$\times \mathrm{QE} \left(-\binom{m}{2} + \binom{n}{2} + \sigma m + n(1 - m - n) \right) =$$

$$= \sum_{m,\,n=0}^{\infty} \frac{C_{m+n} x_1^{m} x_2^{n} \langle v; q \rangle_m \langle v + \sigma + m; q \rangle_n}{\langle 1; q \rangle_m \langle 1; q \rangle_n} \mathrm{QE} \left(-\binom{n}{2} - mn \right).$$

PROOF. We will use the *q*-Euler-Pfaff equation (44).

(87) $RHS =$

$$= \sum_{N=0}^{\infty} \sum_{n=0}^{N} \frac{(-1)^n C_N x_1^N \left(\frac{x_2}{x_1} \right)^n \langle v; q \rangle_N \langle -N, 1 - v - \sigma - N; q \rangle_n}{\langle 1; q \rangle_N \langle 1, 1 - v - N; q \rangle_n} q^{n(1+\sigma)} =$$

$$= \sum_{N=0}^{\infty} \frac{C_N x_1^N \langle v; q \rangle_N}{\langle 1; q \rangle_N \left(-\frac{x_2}{x_1} q^{1+\sigma}; q \right)_{-N}} {}_2\phi_2 \left[\begin{matrix} \sigma, -N \\ 1 - v - N \end{matrix} \Big| q, -\frac{x_2}{x_1} q^{2-v-N} \Big\| -\frac{-}{x_1} q^{1+\sigma - N} \right] =$$

$$= \sum_{N=0}^{\infty} \frac{C_N x_1^N \langle v; q \rangle_N}{\langle 1; q \rangle_N} \sum_{n=0}^{N} \frac{\langle -N, \sigma; q \rangle_n}{\langle 1, 1 - v - N; q \rangle_n} \frac{\left(\frac{x_2}{x_1} q^{2-v-N} \right)^n q^{\binom{n}{2}}}{\left(-\frac{x_2}{x_1} q^{1+\sigma}; q \right)_{n-N}} =$$

$$= \sum_{N=0}^{\infty} \frac{C_N x_1^N \langle v; q \rangle_N}{\langle 1; q \rangle_N} \sum_{n=0}^{N} \frac{\langle -N, \sigma; q \rangle_n}{\langle 1, 1 - v - N; q \rangle_n} \left(\frac{y}{x_1} q^{2-v-N} \right)^n \times$$

$$\times \left(-\frac{x_1}{x_2} q^{-\sigma}; q \right)_{N-n} \mathrm{QE} \left(-\binom{N-n}{2} + \binom{n}{2} + \sigma(N-n) \right) \left(\frac{x_2}{x_1} \right)^{N-n} =$$

$$= \sum_{N=0}^{\infty} \sum_{n=0}^{N} \frac{C_N x_1^N \langle \sigma; q \rangle_n \langle v; q \rangle_{N-n}}{\langle 1; q \rangle_n \langle 1; q \rangle_{N-n}} \left(-\frac{x_1}{x_2} q^{-\sigma}; q \right)_{N-n} \times$$

$$\times \mathrm{QE} \left(-\binom{N-n}{2} + \binom{n}{2} + \sigma(N-n) + n(1-N) \right) \left(\frac{x_2}{x_1} \right)^N = LHS. \quad \blacksquare$$

4. q-Analogues of reducibility theorems of Karlsson.

Karlsson [46] has derived two interesting reduction formulas of general character for triple sums, which can be used to deduce many reduction formulas for hypergeometric functions in three variables [69, p. 311]. The formulas (39) and (41) are used in the proofs. Clearly many more reducibility formulas of this type can be deduced. We illustrate with a q-analogue of one interesting special case known from the literature.

THEOREM 4.1. *A q-analogue of [46, p. 200].*
If $\{C_{m,n}\}_{m,n=0}^{\infty}$ is a sequence of arbitrary complex numbers then

$$(88) \qquad \sum_{m,n,p=0}^{\infty} \frac{C_{m,n+p} \langle b; q \rangle_n \langle b; q \rangle_p x_1^m x_2^n (-x_2)^p}{\langle 1; q \rangle_m \langle 1; q \rangle_n \langle 1; q \rangle_p} =$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{C_{m,2k} \langle b, \tilde{b}; q \rangle_k x_1^m x_2^{2k}}{\langle 1; q \rangle_m \langle 1, \tilde{1}; q \rangle_k}.$$

PROOF.

$$(89) \quad LHS = \sum_{m,t=0}^{\infty} \sum_{p=0}^{t} \frac{C_{m,t} \langle b; q \rangle_t \langle b, -t; q \rangle_p x_1^m x_2^t (-1)^p q^{p(1-b)}}{\langle 1; q \rangle_m \langle 1; q \rangle_t \langle 1, 1-t-b; q \rangle_p} =$$

$$= \sum_{m=0}^{\infty} \sum_{t=0, \, t \text{ even}}^{\infty} \frac{C_{m,t} \langle b; q \rangle_t x_1^m x_2^t}{\langle 1; q \rangle_m \langle 1; q \rangle_t} \frac{\langle \tilde{b}; q \rangle_{t/2} \left\langle \frac{1}{2}; q^2 \right\rangle_{t/2}}{\left\langle b + \frac{t}{2}; q \right\rangle_{t/2}} = RHS. \quad \blacksquare$$

REMARK 10. We obtain (81) by putting $x_1 = 0$ in the theorem above.

COROLLARY 4.2. *A q-analogue of* [46, 2.5, p. 201].

$$(90) \qquad \Phi_D^{(3)}(a+1, 1+a-c, b, b; c\,|\,q;\, -q^{c-a}, x,\, -x) = \Gamma_q \begin{bmatrix} c,\, \dfrac{a}{2} \\[2mm] a,\, c-\dfrac{a}{2} \end{bmatrix} \times$$

$$\times\, _3\phi_{2,\,k} \begin{bmatrix} b, \tilde{b}, \dfrac{a}{2} \\[2mm] \tilde{1},\, c-\dfrac{a}{2} \end{bmatrix} q,\, x^2 \| - \||\, \frac{\left\langle \overbrace{-\dfrac{a}{2}+c+k}; q \right\rangle_{a+1-c}}{\left\langle \overbrace{c-a}\, ;\, q \right\rangle_{a+1-c}(1+q^{\frac{a}{2}+k})} \Bigg].$$

PROOF. Put

$$C_{m,\,n} = \frac{\langle a+1;\, q \rangle_{m+n} \langle 1+a-c;\, q \rangle_m}{\langle c;\, q \rangle_{m+n}}\,;\; x_1 = -q^{c-a} \text{ in } (88).$$

$$(91) \qquad RHS =$$

$$= \sum_{m,\,k=0}^{\infty} \frac{\langle a;\, q \rangle_{2k} \langle 1+a-c,\, a+2k;\, q \rangle_m}{\langle c;\, q \rangle_{2k} \langle 1,\, c+2k;\, q \rangle_m} \frac{\langle b, \tilde{b};\, q \rangle_k}{\langle 1, \tilde{1};\, q \rangle_k} x^{2k}(-1)^m q^{m(c-a)} =$$

$$= \sum_{k=0}^{\infty} \frac{\langle a;\, q \rangle_{2k}}{\langle c;\, q \rangle_{2k}} \frac{\langle b, \tilde{b};\, q \rangle_k}{\langle 1, \tilde{1};\, q \rangle_k} x^{2k} \Gamma_q \begin{bmatrix} c+2k,\, \dfrac{a}{2}+1+k \\[2mm] 1+a+2k,\, c-\dfrac{a}{2}+k \end{bmatrix} \times$$

$$\times\, \frac{\left\langle \overbrace{-\dfrac{a}{2}+c+k}, \tilde{1};\, q \right\rangle_{\infty}}{\left\langle \dfrac{a}{2} \overbrace{+1+}k,\, \overbrace{c-a}\, ;\, q \right\rangle_{\infty}} = \sum_{k=0}^{\infty} \frac{\left\langle \dfrac{a}{2},\, b, \tilde{b};\, q \right\rangle_k}{\left\langle c-\dfrac{a}{2},\, \tilde{1},\, 1;\, q \right\rangle_k} x^{2k} \times$$

$$\times \Gamma_q \begin{bmatrix} c,\, \dfrac{a}{2} \\[2mm] a,\, c-\dfrac{a}{2} \end{bmatrix} \frac{\left\langle \overbrace{-\dfrac{a}{2}+c+k};\, q \right\rangle_{a+1-c}}{\left\langle \overbrace{c-a}\, ;\, q \right\rangle_{a+1-c}(1+q^{\frac{a}{2}+k})} = LHS. \qquad \blacksquare$$

THEOREM 4.3. *A q-analogue of* [46, 4.1 p. 202].

If $\{C_{m,\,n}\}_{m,\,n=0}^{\infty}$ is a sequence of arbitrary complex numbers then

$$(92) \quad \sum_{m,n,p=0}^{\infty} \frac{C_{m,n+p}\langle a,b;q\rangle_n \left\langle a,b,1-\widetilde{\frac{n+p}{2}}\;;q\right\rangle_p x_1^m x_2^n (-x_2)^p q^{\frac{p}{2}(n-1)}}{\langle 1;q\rangle_m \langle 1;q\rangle_n \left\langle 1,\,-\widetilde{\frac{n+p}{2}}\;;q\right\rangle_p} =$$

$$= \sum_{m,k=0}^{\infty} \frac{C_{m,2k}\langle a,\,b,\,\widetilde{1-k},\,\tilde{b};\,q\rangle_k \langle a+b;\,q\rangle_{2k} x_1^m\,x_2^{2k}\,q^{-kb}}{\langle 1;\,q\rangle_m \langle 1,\,a+b,\,\widetilde{1-b-}\,k,\,\tilde{1};\,q\rangle_k}\;.$$

PROOF. Compare [56, p. 56].

$$(93) \quad \sum_{m,\,n=0}^{\infty} \sum_{p=0}^{n} \times$$

$$\times \frac{C_{m,n}\langle a,b;q\rangle_n \left\langle a,b,-n,\,\widetilde{1-\frac{n}{2}}\;;q\right\rangle_p x_1^m x_2^n q^{\binom{p}{2}+p(2-n-a-b)+\frac{p}{2}(n-p-1)}}{\langle 1;q\rangle_m \langle 1;q\rangle_n \left\langle 1,1-n-b,1-n-a,\,-\widetilde{\frac{n}{2}}\;;q\right\rangle_p} \overset{\text{by}(39)}{=}$$

$$= \sum_{m,n=0}^{\infty} \frac{C_{m,n}\langle a,b;q\rangle_n x_1^m x_2^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n}\, \Gamma_q \begin{bmatrix} 1-b-n,1-a-n,1-\dfrac{n}{2},1-a-b-\dfrac{n}{2} \\[2mm] 1-n,1-a-b-n,1-b-\dfrac{n}{2},1-a-\dfrac{n}{2} \end{bmatrix} =$$

$$= \sum_{m=0}^{\infty}\ \sum_{n=0,\,n\text{ even}}^{\infty} \frac{C_{m,n}\langle a,b;q\rangle_n x_1^m x_2^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n}\, \Gamma_q \begin{bmatrix} 1-a-n,\ 1-a-b-\dfrac{n}{2} \\[2mm] 1-a-b-n,\ 1-a-\dfrac{n}{2} \end{bmatrix} \times$$

$$\times \ \frac{\left\langle \widetilde{1-\frac{n}{2}}\,,\tilde{b};\,q\right\rangle_{n/2}}{\left\langle 1\ \widetilde{-\frac{n}{2}-}\ b,\,b+\frac{n}{2}\,;\,q\right\rangle_{n/2}}\left\langle \frac{1}{2}\,;\,q^2\right\rangle_{n/2} = \sum_{m,k=0}^{\infty}\frac{C_{m,\,2k}\langle a,\,b;\,q\rangle_{2k}\,x_1^m\,x_2^{2k}}{\langle 1;\,q\rangle_m\langle 1;\,q\rangle_{2k}}\ \times$$

$$\times \ \frac{\langle 1-a-b-2k,\ \widetilde{1-k}\,,\tilde{b};\,q\rangle_k}{\langle 1-a-2k,\,1\ \widetilde{-k-}\ b,\,b+k;\,q\rangle_k}\left\langle \frac{1}{2}\,;\,q^2\right\rangle_k = RHS. \qquad \blacksquare$$

5. q-Analogues of Burchnall-Chaundy expansions.

This chapter is mainly based on the following inverse pair of symbolic operators defined in [11]. We will get an improved version of [41] in the process, compare [2]. The operators θ_1 and θ_2 from (8), chapter 1 will be extensively used.

DEFINITION 15.

$$(94) \qquad \nabla_q(h) \equiv \Gamma_q\begin{bmatrix} h,\, h+\theta_1+\theta_2 \\ h+\theta_1,\, h+\theta_2 \end{bmatrix}, \quad \Delta_q(h) \equiv \Gamma_q\begin{bmatrix} h+\theta_1,\, h+\theta_2 \\ h+\theta_1+\theta_2,\, h \end{bmatrix}.$$

Then

$$(95) \qquad \nabla_q(h)\langle h;\,q\rangle_m\langle h;\,q\rangle_n x_1^m x_2^n = \langle h;\,q\rangle_{m+n}x_1^m x_2^n.$$

Recall that

$$(96) \qquad {}_2\phi_1(a,\,b;\,c\,|\,q,\,x_1)\,{}_2\phi_1(a',\,b';\,c'\,|\,q,\,x_2) =$$

$$= \sum_{m,\,n=0}^{\infty}\frac{\langle a,\,b;\,q\rangle_m\langle a',\,b';\,q\rangle_n}{\langle 1,\,c;\,q\rangle_m\langle 1,\,c';\,q\rangle_n}\,x_1^m x_2^n.$$

$$(97) \qquad {}_2\phi_1(a,\,b;\,c\,|\,q,\,x_1\oplus_q x_2) = \sum_{m,\,n=0}^{\infty}\frac{\langle a;\,q\rangle_{m+n}\langle b;\,q\rangle_{m+n}}{\langle 1;\,q\rangle_m\langle 1;\,q\rangle_n\langle c;\,q\rangle_{m+n}}\,x_1^m x_2^n.$$

We can write symbolically (the last three equations first appeared in [41,

(14) p. 72])

(98) $\Phi_2(a; b, b'; c, c' | q; x_1, x_2) =$
$$= \nabla_q(a)_2\phi_1(a, b; c | q, x_1)_2\phi_1(a, b'; c' | q, x_2).$$

(99) $\Phi_3(a, a'; b, b'; c | q; x_1, x_2) =$
$$= \Delta_q(c)\,_2\phi_1(a, b; c | q, x_1)\,_2\phi_1(a', b'; c | q, x_2).$$

(100) $\Phi_1(a; b, b'; c | q; x_1, x_2) =$
$$= \nabla_q(a)\,\Delta_q(c)_2\phi_1(a, b; c | q, x_1)_2\phi_1(a, b'; c | q, x_2).$$

(101) $\Phi_1(a; b, b'; c | q; x_1, x_2) = \nabla_q(a)\,\Phi_3(a, a; b, b'; c | q; x_1, x_2).$

(102) $\Phi_1(a; b, b'; c | q; x_1, x_2) = \Delta_q(c)\,\Phi_2(a; b, b'; c, c | q; x_1, x_2).$

(103) $\Phi_4(a, b; c, c' | q; x_1, x_2) = \nabla_q(b)\,\Phi_2(a; b, b; c, c' | q; x_1, x_2).$

(104) $_2\phi_1(a, b; c | q, x_1 \oplus_q x_2) = \nabla_q(b)\,\Phi_1(a; b, b; c | q; x_1, x_2).$

(105) $_2\phi_1(a, b; c | q, x_1 \oplus_q x_2) = \Delta_q(c)\,\Phi_4(a, b; c, c | q; x_1, x_2).$

(106) $_2\phi_1(a, b; c | q, x_1 \oplus_q x_2) =$
$$= \nabla_q(b)\,\Delta_q(c)\,\Phi_2(a; b, b; c, c | q; x_1, x_2).$$

(107) $_2\phi_1(a, b; c | q, [x_1 + x_2]_q) =$
$$= \nabla_q(b)\,\Phi^{1:1;1}_{1:0;1}\left[\begin{matrix} a : b; b; \\ c : - ; \infty; \end{matrix} | q; x_1, x_2 \right].$$

(108) $_2\phi_1(a, b; c | q, [x_1 + x_2]_q) =$
$$= \Delta_q(c)\,\Phi^{2:2;1}_{2:1;1}\left[\begin{matrix} a, b: \infty, \infty; \infty; \\ \infty, \infty: c; c; \end{matrix} | q; x_1, x_2 \right].$$

(109) $_2\phi_1(a, b; c | q, [x_1 + x_2]_q) =$
$$= \nabla_q(b)\,\Delta_q(c)\,\Phi^{1:2;1}_{1:1;1}\left[\begin{matrix} a : b, \infty; b; \\ \infty: c; c; \end{matrix} | q; x_1, x_2 \right].$$

For a simple q-hypergeometric series we obtain

(110) $\dfrac{1}{\langle \theta_1 + c; q \rangle_r}\,_2\phi_1(a, b; c | q, x_1) = \dfrac{1}{\langle c; q \rangle_r}\,_2\phi_1(a, b; c + r | q, x_1),$

and analogous formulae for double series.

The following formulas for θ_1 and θ_2 obtain.

(111) $\quad \dfrac{1}{\langle -a - \theta_1 - \theta_2 + 1; q\rangle_r} \Phi_1(a; b, b'; c|q; x_1, x_2) =$

$\qquad = \dfrac{(-1)^r \langle a; q\rangle_r}{\langle a - r; q\rangle_{2r}} \Phi_1(a - r; b, b'; c|q; x_1 q^r, x_2 q^r) q^{-\binom{r}{2} + r(a-1)}.$

(112) $\quad \langle -\theta_1; q\rangle_r\, {}_2\phi_1(a, b; c|q, x_1) =$

$\qquad = \dfrac{(-1)^r \langle a, b; q\rangle_r x_1^r}{\langle c; q\rangle_r}\, {}_2\phi_1(a + r, b + r; c + r|q, x_1 q^{-r}) q^{\binom{r}{2} - r^2}.$

(113) $\quad \langle -\theta_1, -\theta_2; q\rangle_r \Phi_1(a; b, b'; c|q; x_1, x_2) = \dfrac{\langle a; q\rangle_{2r}\langle b, b'; q\rangle_r}{\langle c; q\rangle_{2r}} x_1^r x_2^r \times$

$\qquad \times \Phi_1(a + 2r; b + r, b' + r; c + 2r|q; x_1 q^{-r}, x_2 q^{-r}) q^{r^2 - r}.$

Combining (111) and (113) we obtain

(114) $\quad \dfrac{\langle -\theta_1, -\theta_2; q\rangle_r}{\langle -a - \theta_1 - \theta_2 + 1; q\rangle_r} \Phi_1(a; b, b'; c|q; x_1, x_2) =$

$= \dfrac{(-1)^r \langle a, b, b'; q\rangle_r}{\langle c; q\rangle_{2r}} x_1^r x_2^r \Phi_1(a + r; b + r, b' + r; c + 2r|q; x_1, x_2) q^{\binom{r}{2} + r(a-1)}.$

We obtain the following set of lemmata

(115) $\qquad\qquad \nabla_q(h) = \displaystyle\sum_{r=0}^{\infty} \dfrac{\langle -\theta_1, -\theta_2; q\rangle_r}{\langle 1, h; q\rangle_r} q^{rh} \varepsilon_1^r \varepsilon_2^r.$

(116) $\qquad\qquad \Delta_q(h) = \displaystyle\sum_{r=0}^{\infty} \dfrac{\langle -\theta_1, -\theta_2; q\rangle_r}{\langle 1, 1 - h - \theta_1 - \theta_2; q\rangle_r} q^r,$

(117) $\quad \Delta_q(h) = \displaystyle\sum_{r=0}^{\infty} \dfrac{(-1)^r \langle -\theta_1, -\theta_2; q\rangle_r \langle h; q\rangle_{2r}}{\langle 1, h + r - 1, h + \theta_1, h + \theta_2; q\rangle_r} q^{\binom{r}{2} + rh} \varepsilon_1^r \varepsilon_2^r,$

and

(118) $\quad \nabla_q(h)\Delta_q(k) = \displaystyle\sum_{r=0}^{\infty} \dfrac{\langle -\theta_1, -\theta_2, k - h; q\rangle_r \langle k; q\rangle_{2r}}{\langle 1, h, k + r - 1, k + \theta_1, k + \theta_2; q\rangle_r} q^{rh} \varepsilon_1^r \varepsilon_2^r,$

(119) $\qquad \nabla_q(h)\Delta_q(k) = \displaystyle\sum_{r=0}^{\infty} \dfrac{\langle h - k, -\theta_1, -\theta_2; q\rangle_r}{\langle 1, h, 1 - k - \theta_1 - \theta_2; q\rangle_r} q^r,$

where the last formula is the q-Pfaff-Saalschütz theorem.

By the same method which was used in [41], we obtain the following q-analogues of the corrected version of Verma [75, (10)-(14)]. The formulas also closely resemble q-analogues of [64, (11), (12), (8), (13), (7)].

When there will be no misunderstanding, we will write G and H for G_1; G_2 and H_1; H_2.

THEOREM 5.1.

$$(120) \quad \Phi_{B:H_1;H_2}^{A:G_1;G_2}\left[\begin{matrix}(a):(g_1);(g_2);\\(b):(h_1);(h_2);\end{matrix}\Big|q; x_1, x_2\right] =$$

$$= \sum_{r=0}^{\infty} \frac{\langle(g_1),(g_2), \alpha; q\rangle_r\langle(a); q\rangle_{2r}}{\langle 1,(h_1),(h_2); q\rangle_r\langle(b), \alpha; q\rangle_{2r}} q^{2\binom{r}{2}+r\alpha} x_1^r x_2^r \, \Phi_{B+1:H_1+1;H_2+1}^{A+1:G_1+1;G_2+1} \times$$

$$\times \left[\begin{matrix}(a+2r), \infty : (g_1+r), \alpha+r; (g_2+r), \alpha+r;\\(b+2r), \alpha+2r : (h_1+r), \infty ; (h_2+r), \infty ;\end{matrix}\Big|q; x_1, x_2\right].$$

$$(121) \quad = \sum_{r=0}^{\infty} \frac{(-1)^r\langle(g_1),(g_2); q\rangle_r\langle(a); q\rangle_{2r}}{\langle 1,(h_1),(h_2), \alpha; q\rangle_r\langle(b); q\rangle_{2r}} q^{\binom{r}{2}+r\alpha} x_1^r x_2^r \times$$

$$\times \Phi_{B+1:H+1}^{A+1:G+1}\left[\begin{matrix}(a+2r), \alpha+r : (g_1+r), \infty ; (g_2+r), \infty ;\\(b+2r), \infty : (h_1+r), \alpha+r; (h_2+r), \alpha+r;\end{matrix}\Big|q; x_1, x_2\right].$$

$$(122) \quad = \sum_{r=0}^{\infty} \frac{\langle(g_1),(g_2), \gamma-\alpha, \alpha; q\rangle_r\langle(a); q\rangle_{2r}}{\langle 1,(h_1),(h_2), \gamma+r-1; q\rangle_r\langle(b), \alpha; q\rangle_{2r}} q^{\binom{r}{2}+r\alpha} x_1^r x_2^r \times$$

$$\times \Phi_{B+1:H+1}^{A+1:G+1}\left[\begin{matrix}(a+2r), \gamma+2r : (g_1+r), \alpha+r; (g_2+r), \alpha+r;\\(b+2r), \alpha+2r : (h_1+r), \gamma+2r; (h_2+r), \gamma+2r;\end{matrix}\Big|q; x_1, x_2\right].$$

$$(123) \quad = \sum_{r=0}^{\infty} \frac{(-1)^r\langle(g_1),(g_2); q\rangle_r\langle(a); q\rangle_{2r}}{\langle 1,(h_1),(h_2), \alpha+r-1; q\rangle_r\langle(b); q\rangle_{2r}} q^{3\binom{r}{2}+r\alpha} x_1^r x_2^r \times$$

$$\times \Phi_{B+1:H+1}^{A+1:G+1}\left[\begin{matrix}(a+2r), \alpha+2r : (g_1+r), \infty ; (g_2+r), \infty ;\\(b+2r), \infty : (h_1+r), \alpha+2r; (h_2+r), \alpha+2r;\end{matrix}\Big|q; x_1, x_2\right].$$

$$(124) \quad = \sum_{r=0}^{\infty} \frac{(-1)^r\langle(g_1),(g_2), \alpha-\gamma, \alpha; q\rangle_r\langle(a); q\rangle_{2r}}{\langle 1,(h_1),(h_2), \gamma; q\rangle_r\langle(b), \alpha; q\rangle_{2r}} q^{\binom{r}{2}+r\gamma} x_1^r x_2^r \times$$

$$\times \Phi_{B+1:H+1}^{A+1: G+1}\left[\begin{matrix}(a+2r), \gamma+r : (g_1+r), \alpha+r; (g_2+r), \alpha+r;\\(b+2r), \alpha+2r : (h_1+r), \gamma+r; (h_2+r), \gamma+r;\end{matrix}\Big|q; x_1, x_2\right].$$

These formulas hold when $A+G_1 \leqq B+H_1+1$, $A+G_2 \leqq B+H_2+1$, and $|x_1|$, $|x_2|$ are chosen in such a way that both sides converge [64, p. 49].

PROOF. We will use the following abbreviation.

$$\Theta \equiv (-1)^{m(1+H_1-G_1+B-A)+n(1+H_2-G_2+B-A)} \times$$

$$\times QE\left((B-A)\binom{m+n}{2} + (1+H_1-G_1)\binom{m}{2} + (1+H_2-G_2)\binom{n}{2}\right).$$

First we prove (122).

$$(125) \qquad LHS = \sum_{r=0}^{\infty} \frac{\langle -\theta_1, -\theta_2, \gamma-\alpha; q\rangle_r \langle \gamma; q\rangle_{2r}}{\langle 1, \alpha, \gamma+r-1, \gamma+\theta_1, \gamma+\theta_2; q\rangle_r} q^{r\alpha} \varepsilon_1^r \varepsilon_2^r \times$$

$$\times \Phi_{B+1:H+1}^{A+1:G+1}\begin{bmatrix}(a), \gamma:(g_1), \alpha;(g_2), \alpha;\\(b), \alpha:(h_1), \gamma;(h_2), \gamma;\end{bmatrix}q; x_1, x_2\end{bmatrix} =$$

$$= \sum_{r,m,n=0}^{\infty} \frac{q^{2\binom{r}{2}+r\alpha}\langle \gamma-\alpha; q\rangle_r \langle \gamma; q\rangle_{2r}}{\langle \alpha, \gamma+r-1, \gamma, \gamma; q\rangle_r} \times$$

$$\times \Phi_{B+1:H+1}^{A+1:G+1}\begin{bmatrix}(a), \gamma:(g_1), \alpha;(g_2), \alpha;\\(b), \alpha:(h_1), \gamma+r;(h_2), \gamma+r;\end{bmatrix}q; x_1, x_2\end{bmatrix} = RHS.$$

Equations (120) and (123) follow from (122) by letting $\gamma \to +\infty$ and $\alpha \to +\infty$ respectively. Equation (124) is proved in a similar way.

$$LHS = \sum_{r=0}^{\infty} \frac{\langle -\theta_1, -\theta_2, \alpha-\gamma; q\rangle_r}{\langle 1, \alpha, 1-\gamma-\theta_1-\theta_2; q\rangle_r} q^r \Phi_{B+1:H+1}^{A+1:G+1} \times$$

$$\times \begin{bmatrix}(a), \gamma:(g_1), \alpha;(g_2), \alpha;\\(b), \alpha:(h_1), \gamma;(h_2), \gamma;\end{bmatrix}q; x_1, x_2\end{bmatrix} =$$

$$= \sum_{r,m,n=0}^{\infty} \frac{(-1)^r q^{\binom{r}{2}+r\gamma}\langle \alpha-\gamma; q\rangle_r \langle \gamma; q\rangle_{m+n-r}}{\langle 1, \alpha; q\rangle_r} \Theta \times$$

$$\times \frac{\langle (a); q\rangle_{m+n}\langle (g_1), \alpha; q\rangle_m \langle (g_2), \alpha; q\rangle_n x_1^m x_2^n}{\langle (b), \alpha; q\rangle_{m+n}\langle (h_1), \gamma; q\rangle_m \langle (h_2), \gamma; q\rangle_n \langle 1; q\rangle_{m-r}\langle 1; q\rangle_{n-r}} =$$

$$= \sum_{r,m,n=0}^{\infty} \frac{(-1)^r q^{\binom{r}{2}+r\gamma} x_1^r x_2^r \langle \alpha-\gamma, (g_1), \alpha, (g_2); q\rangle_r \langle (a); q\rangle_{2r}}{\langle 1, (h_1), \gamma, (h_2); q\rangle_r \langle (b), \alpha; q\rangle_{2r}} \Theta \times$$

$$\times \frac{x_1^m x_2^n \langle (a+2r), \gamma+r; q\rangle_{m+n}\langle (g_1+r), \alpha+r; q\rangle_m \langle (g_2+r), \alpha+r; q\rangle_n}{\langle (b+2r), \alpha+2r; q\rangle_{m+n}\langle (h_1+r), \gamma+r; q\rangle_m \langle (h_2+r), \gamma+r; q\rangle_n} = RHS.$$

Equation (121) follows from (124) by letting $\alpha \to +\infty$. ∎

REMARK 11. As pointed out in [75], specialization of parameters in (120) yields the following results from [41]: 33, 36, 39, a corrected version of 42, a corrected version of 43, 45, and 48. Similarly, (121) gives 34, 40, a corrected version of 44 and 46 from the same paper, while (122) gives [2, 6.8 p. 193], and equation 49 from [41]. Moreover, (123) gives [2, p. 194], as well as equations 41 and 47 from [41], and (124) gives 38 and 50 of [41]. Compare [64, p. 51].

6. Conclusion.

There is a typical trend in some of the new results. In the Watson- and Jackson formulas for $_8\phi_7$ series, we append an extra tilde in numerator and denominator to get the q-analogue. This is also the case in the LHS triple sum of (92). However, in the RHS double sum of (92) we append two extra tildes in numerator and denominator to get the q-analogue. In (64) and (73) the hypergeometric function argument $4x$ is replaced by x times two tildes in the numerator.

The following decomposition of the q-hypergeometric series into even and odd parts is a q-analogue of [68, p. 200-201], and an example of a q-analogue of a hypergeometric function with argument $\dfrac{1}{4}x$.

THEOREM 6.1. [22, (366), p. 71]

$$(126) \qquad {}_r\phi_s((a);(b)\,|\,q,\,z) =$$

$$= {}_{4r}\phi_{4s+3}\left[\begin{array}{c} \dfrac{(a)}{2},\ \dfrac{\widetilde{(a)}}{2},\ \dfrac{(a+1)}{2},\ \dfrac{\widetilde{(a+1)}}{2} \\[2mm] \dfrac{(b)}{2},\ \dfrac{\widetilde{(b)}}{2},\ \dfrac{(b+1)}{2},\ \dfrac{\widetilde{(b+1)}}{2},\ \dfrac{1}{2},\ \dfrac{\tilde{1}}{2},\ \tilde{1} \end{array}\ \bigg|\,q,\,z^2 q^{1+s-r}\right] +$$

$$+(-1)^{1+s-r}\ \dfrac{z}{1-q}\ \dfrac{\displaystyle\prod_{j=1}^{r}(1-q^{a_j})}{\displaystyle\prod_{j=1}^{s}(1-q^{b_j})}\ \times$$

$$\times {}_{4r}\phi_{4s+3}\left[\begin{array}{c} \dfrac{(a+1)}{2},\ \dfrac{\widetilde{(a+1)}}{2},\ \dfrac{(a+2)}{2},\ \dfrac{\widetilde{(a+2)}}{2} \\[2mm] \dfrac{(b+1)}{2},\ \dfrac{\widetilde{(b+1)}}{2},\ \dfrac{(b+2)}{2},\ \dfrac{\widetilde{(b+2)}}{2},\ \dfrac{3}{2},\ \dfrac{\tilde{3}}{2},\ \tilde{1} \end{array}\ \bigg|\,q,\,z^2 q^{3(1+s-r)}\right].$$

So we have 2 cases where the tilde operator is connected to q-analogues of hypergeometric function arguments expressed as $x \times 2^{\pm n}$. The extended q-hypergeometric series (28) follows a similar pattern.

Hopefully the q-Kampé de Fériet functions defined here will enhance the understanding of this interdisciplinary subject. Further developments along these lines will be given in future papers.

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