REND. SEM. MAT. UNIV. PADOVA, Vol. 112 (2004)

A Note on Extremality and Completeness in Financial Markets with Infinitely many Risky Assets (*).

LUCIANO CAMPI (**)

ABSTRACT - In this paper we study the interplay existing between completeness of financial markets with infinitely many risky assets and extremality of equivalent martingale measures. In particular, we obtain a version of the Douglas-Naimark Theorem for a dual system $\langle X, Y\rangle$ of locally convex topological real vector spaces equipped with the weak topology $\sigma(X, Y)$, and we apply it to the space L^∞ with the topology $\sigma(L^\infty, L^p)$ for $p \geqslant 1$. Thanks to these results, we obtain a condition equivalent to the market completeness and based on the notion of extremality of measures, which allows us to give new and simpler proofs of the second fundamental theorems of asset pricing. Finally, we discuss also the completeness of a slight generalization of the Artzner and Heath example.

1. Introduction.

Artzner and Heath (1995) constructed a market with an infinite number of equivalent martingale probability measures, which is complete under two of such measures, the extremal ones. This market has the essential property that the set of risky assets is infinite, in other words it is a

(*) I wish to thank Marc Yor for his help, support and suggestions, and Massimo Marinacci for his interest in this paper.

(**) Indirizzo dell'A.: Laboratoire de Probabilités et Modèles aléatoires, CNRS-UMR 7599, Université Paris VI & Université Paris VII, 4 Place Jussieu, Tour 56, F-75252 Paris Cedex 05, France & Università degli Studi di Padova, Dipartimento di Matematica Pura e Applicata, via Belzoni 7, 35131 Padova, Italy. E-Mail: campi@ccr.jussieu.fr

large financial market. The existence of such an economy implies that the equivalence between completeness and uniqueness of the equivalent martingale measure is not verified in an infinite assets setting.

Bättig (1999), Jin, Jarrow and Madan (1999) and Jarrow and Madan (1999) adopted a different notion of market completeness in order to extend this equivalence even to a large financial market. They give a definition of completeness which is independent from the notion of no-arbitrage, and show that if the market is complete, then there exists at most one equivalent martingale signed measure and if the market is arbitrage-free, then this signed measure is a true probability. In order to demonstrate them, these authors have to verify the surjectivity of a certain operator and then the injectivity of its adjoint.

Here we will examine the interplay existing between the extremality of martingale probability measures and the various notions of market completeness introduced by Artzner and Heath (1995) and Jin, Jarrow and Madan (1999). For this we will need two versions of the Douglas-Naimark Theorem, which is a functional analysis result connecting the density of the subsets of some space L^p with the extremality on a certain subset of measures of the underlying probability. Now, we quote them without proofs, for which one can consult Douglas (1964) (Theorem 1, p. 243) or Naimark (1947) for the first and Yor (1976) (Proposition 4 of the Appendice, p. 306) for the second.

THEOREM 1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let F be a subspace of $L^1(P)$ such that $1 \in F$. The following three assertions are equivalent*:

1) *F is dense in $(L^1(P), \|\cdot\|_1)$*;

2) *if $g \in L^\infty(P)$ satisfies $\int fgdP = 0$ for each $f \in F$, then $g = 0$ P-a.s.*;

3) *P is an extremal point of the set*

$$\tilde{\Xi}_1(P) = \{Q \in \mathcal{P} : \textit{for each } f \in F,\ f \in L^1(Q) \textit{ and } \mathbb{E}_Q(f) = \mathbb{E}_P(f)\},$$

where $\mathcal{P}$ is the space of all probability measures over $(\Omega, \mathcal{F})$.

We denote $ba(\Omega, \mathcal{F})$, or simply ba, the space of all additive bounded measures on the measurable space $(\Omega, \mathcal{F})$. It is well known that one can identify ba with the topological dual of the space $L^{\infty}(P)$ equipped with the strong topology. Finally, with an obvious notation, one has the decomposition $ba = ba^{+} - ba^{+}$. For further information on ba, one can consult Dunford and Schwartz (1957).

THEOREM 2. *Let* $ba^{+}(P) = \{\nu \in ba^{+};\, \nu \ll P\}$, *and let* F *be a subspace of* $L^{\infty}(P)$ *such that* $1 \in F$. *The following two assertions are equivalent*:

1) F *is dense in* $(L^{\infty}(P), \|\cdot\|_{\infty})$;

2) *every additive measure* $\nu \in ba^{+}(P)$ *is an extremal point of the set*

$$\Xi_{ba}(\nu) = \{\lambda \in ba^{+}(\nu) \colon \textit{for each } f \in F,\ \lambda(f) = \nu(f)\}.$$

We observe that the spaces L^{p} considered in the previous theorems are equipped with their respective strong topologies.

In Section 2 we obtain a version of the Douglas-Naimark Theorem for a dual system $\langle X, Y\rangle$ of ordered locally convex topological real vector spaces, and we apply it to the special case $\langle X, Y\rangle = \langle L^{\infty}, L^{p}\rangle$ for $p \geqslant 1$. In Subsection 2.3 we obtain also a Douglas-Naimark Theorem for L^{∞} with L^{p}-norm topologies for $p \geqslant 1$, which we will use for the discussion of the completeness of the AH-market.

In Section 3, we apply these results to mathematical finance. In particular, in subsections 3.2 and 3.3 we give new proofs of the versions of the Second Fundamental Theorems of Asset Pricing (abbr. SFTAP) obtained by Jarrow, Jin and Madan (1999) and Bättig (1999), based on the notion of extremality of measures thanks to the results established in Section 2. The advantage of this approach is that it permits to work directly on the equivalent martingale measures set of the market, using only some elementary geometrical argument. In Subsection 3.4 we discuss the completeness of the Artzner and Heath market with respect to several topologies and we obtain a more general construction of it.

2. A weak version of the Douglas-Naimark Theorem.

2.1. *Weak Douglas-Naimark Theorem for a dual system*

We recall some basic facts about duality for a locally convex topological real vector space (abbr. LCS). Let X, Y be a pair of real vector spaces, and let f be a bilinear form on $X \times Y$, satisfying the separation axioms:

$$f(x_0, y) = 0 \text{ for each } y \in Y \text{ implies } x_0 = 0,$$

$$f(x, y_0) = 0 \text{ for each } x \in X \text{ implies } y_0 = 0.$$

The triple (X, Y, f) is called a dual system or duality (over $\mathbb{R}$). To distinguish f from other bilinear forms on $X \times Y$, f is called the canonical bilinear form of the duality, and is usually denoted by $(x, y) \mapsto \langle x, y \rangle$. The triple (X, Y, f) is more conveniently denoted by $\langle X, Y \rangle$.

If $\langle X, Y \rangle$ is a duality, the mapping $x \mapsto \langle x, y \rangle$ is, for each $y \in Y$, a linear form f_y on X. Since $y \mapsto f_y$ is linear and, by virtue of the second axiom of separation, biunivocal, it is an isomorphism of Y into the algebraic dual X^* of X; thus Y can be identified with a subspace of X^*. Note that under this identification, the canonical bilinear form of $\langle X, Y \rangle$ is induced by the canonical bilinear form of $\langle X, X^* \rangle$.

We recall that the weak topology $\sigma(X, Y)$ is the coarsest topology on X for which the linear forms f_y, $y \in Y$, are continuous; by the first axiom of separation, X is a LCS under $\sigma(X, Y)$.

Let $\langle X, Y \rangle$ be a duality between LCS's and let $K \subset X$ be a cone, which introduce in X a natural order $\leqslant$, which we call K-order, i.e. $x \leqslant x'$ if $x' - x \in K$. Now, we set

$$H_K = \{y \in Y : \langle x, y \rangle \geqslant 0 \text{ for each } x \in K\}$$

and we observe that it is a cone contained in Y. If there is no ambiguity about the cone K we will consider, we will simply write H instead of H_K.

We assume that Y is a vector lattice. We recall that a vector lattice is an ordered vector space Y over $\mathbb{R}$ such that for each pair $(y_1, y_2) \in Y$, $\sup(y_1, y_2)$ and $\inf(y_1, y_2)$ exist. Thus, we can define the positive and the negative part of each $y \in Y$ by

$$y^+ = \sup(0, y)$$

$$y^- = \sup(0, -y)$$

and its absolute value $|y| = \sup(y, -y)$ which satisfies $|y| = y^+ + y^-$. Finally, we have $y = y^+ - y^-$.

We need some additional notation. If $y \in Y$ and $F \subset X$, we set

$$\Xi_{y, F} = \{z \in H_K : \langle x, z \rangle = \langle x, y \rangle \text{ for every } x \in F\}.$$

If there is no confusion about the subset F, we will simply write Ξ_y. Finally, we set $K_0 = K\backslash\{0\}$ and $H_0 = H\backslash\{0\}$.

For more information on topological vector spaces, see e.g. Schaefer (1966) or Narici and Berenstein (1985).

THEOREM 3. *Let F be a subspace of X.The following assertions are equivalent*:

1) *F is dense in $(X, \sigma(X, Y))$*;
2) *every $y \in H_0$ is extremal in Ξ_y.*

PROOF. Firstly, we show that 1) implies 2). It is known (e.g. exercise 9.108(a) in Narici and Berenstein (1985), p. 222) that F is dense in $(X, \sigma(X, Y))$ if and only if, for every $y \in Y$, $\langle x, y\rangle = 0$ for each $x \in F$ implies $y = 0$. Now, we proceed by contradiction and we assume that there exists $y \in H_0$ not extremal in Ξ_y, i.e. we can write $y = \alpha y_1 + (1-\alpha)\, y_2$ where $\alpha \in (0, 1)$ and $y_i \in \Xi_y$ for $i = 1, 2$. Then, we have

$$\langle x, y_1\rangle = \langle x, y_2\rangle = \langle x, y\rangle,$$

which implies

$$\langle x, y_1 - y_2\rangle = 0\,.$$

Then $y_1 = y_2 = y$.

In order to prove the other direction of the equivalence, we note that it is sufficient to show, for every $y \in Y$, that if $\langle x, y\rangle = 0$ for each $x \in F$, then $y = 0$. We assume that there exist $y_0 \in Y$ and $x_0 \in X\backslash F$ such that $\langle x, y_0\rangle = 0$ for every $x \in F$ and $\langle x_0, y_0\rangle \neq 0$. Since Y is a vector lattice, we can write $y_0 = y_0^+ - y_0^-$ and $|y_0| = y_0^+ + y_0^- \in H_0$, where $y_0^+, y_0^- \in H_0$. Now, we observe that

$$|y_0| = \frac{1}{2}(2y_0^+ + 2y_0^-)$$

and, since $2y_0^+, 2y_0^- \in \Xi_{|y_0|}$, we have, by the extremality hypothesis, that $|y_0| = 2y_0^+ = 2y_0^-$, which implies $y_0 = 0$. ■

We note that we have used the assumption that Y is a lattice only in the second part of the proof. Then, even if Y is not a lattice, 1) still implies 2).

2.2. *The space L^∞ equipped with weak topologies.*

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, where μ is a positive finite measure.

Now, we want to apply Theorem 3 to the special case $X = L^\infty(\mu)$ equipped with a family of weak topologies. In this case, the order we consider is the usual one, i.e. for each $f, g \in L^\infty(\mu)$, $f \geqslant g$ if $f(\omega) \geqslant g(\omega)$ for every $x \in \Omega$. In other words, we choose $K = L^\infty_+(\mu)$.

COROLLARY 4. *Let F be a subspace of $L^\infty(\mu)$, where μ is a non null finite positive measure and let $p \geqslant 1$. The following assertions are equivalent*:

1) *F is dense in $(L^\infty(\mu), \sigma(L^\infty, L^p))$;*

2) *every $g \in L^p(\mu)$, such that $g \geqslant 0$ and $\mu(\{g > 0\}) > 0$, is extremal in*

$$\Xi_p(g) = \left\{ h \in L^p(\mu) : h \geqslant 0 \ \ and \ \ \int fh d\mu = \int fg d\mu \ for \ each \ f \in F \right\};$$

3) *every non null finite positive measure $\nu \ll \mu$, such that $\frac{d\nu}{d\mu} \in$ $\in L^p(\mu)$, is extremal in*

$$\Xi_p(\nu) = \left\{ \varrho \in \mathcal{M}_p(\nu) : \int f d\varrho = \int f d\nu \ \text{ for each } f \in F \right\}$$

where $\mathcal{M}_p(\nu)$ is the space of finite positive measures ϱ absolutely continuous with respect to ν and such that $\frac{d\varrho}{d\nu} \in L^p(\nu)$.

PROOF. It is an immediate application of Theorem 3. ■

COROLLARY 5. *Under the same assumptions of Corollary 4, if μ is a probability measure and $1 \in F$, the following two assertions are equivalent*:

1) *F is dense in $(L^\infty(\mu), \sigma(L^\infty, L^p))$;*

2) *every probability measure $\nu \ll \mu$, such that $\frac{d\nu}{d\mu} \in L^p(\mu)$, is extremal in*

$$\tilde{\Xi}_p(\nu) = \left\{ \varrho \in \mathcal{P}_p(\nu) : \int f d\varrho = \int f d\nu \ \text{ for each } f \in F \right\}$$

where

$$\mathcal{P}_p(\nu) = \{ \varrho \in \mathcal{M}_p(\nu) : \varrho(1) = 1 \}.$$

PROOF. If F is dense in $(L^{\infty}(\mu), \sigma(L^{\infty}, L^{p}))$, then, by Corollary 4, every probability measure $\nu \ll \mu$ such that $\frac{d\nu}{d\mu} \in L^{p}(\mu)$ is extremal in $\Xi_{p}(\nu) \supset \tilde{\Xi}_{p}(\nu)$ and it follows that ν is extremal in $\tilde{\Xi}_{p}(\nu)$. Then, 1) implies 2).

Now, we assume that there exists a positive finite measure $\nu \ll \mu$, such that $\frac{d\nu}{d\mu} \in L^{p}(\mu)$, which satisfies $\nu = \alpha \varrho_{1} + (1-\alpha) \varrho_{2}$ with $\alpha \in (0, 1)$ and $\varrho_{i} \in \Xi_{p}(\nu)$ for $i = 1, 2$. By setting $\bar{\nu} = \frac{\nu}{\nu(1)}$, we have

$$\bar{\nu} = \alpha \frac{\varrho_{1}}{\nu(1)} + (1-\alpha) \frac{\varrho_{2}}{\nu(1)},$$

and, since $1 \in F$, $\varrho_{1}(1) = \varrho_{2}(1) = \nu(1)$ and so $\bar{\varrho}_{i} = \frac{\varrho_{i}}{\nu(1)} \in \tilde{\Xi}_{p}(\bar{\nu})$ for $i = 1, 2$. ■

2.3. *The space L^{∞} equipped with L^{p}-norm topologies.*

In this subsection, we obtain a Douglas-Naimark Theorem for $L^{\infty}(\mu)$ equipped with L^{p}-norm topologies, i.e. the topologies induced by the norms $\|\cdot\|_{p}$ for $p \geqslant 1$. In the proof we will essentially use the same argument as in the proof of Theorem 3.

THEOREM 6. *Let F be a subspace of $L^{\infty}(\mu)$ such that $1 \in F$ and let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The following assertions are equivalent*

1) *F is dense in $(L^{\infty}(\mu), \|\cdot\|_{p})$;*

2) *for each $g \in L^{q}$ such that $g \geqslant 0$ and $\int g d\mu = 1$, the probability $\nu = g \cdot \mu$ is extremal in $\tilde{\Xi}_{q}(\nu)$.*

PROOF. We first show that 1) implies 2). The Hölder inequality shows that if F is dense in $L^{\infty}(\mu)$ for the $L^{p}(\mu)$-topology, then it is dense even for the $\sigma(L^{\infty}, L^{q})$-topology. So Corollary 5 applies and the thesis follows.

In order to show that 2) implies 1) it is sufficient to show that if $h \in L^{q}(\mu)$ verify $\int fh d\mu = 0$ for each $f \in F$, then $h = 0$ μ-a.s.. We assume, without loss of generality, that $\nu = |h| \cdot \mu$ is a probability. As usually, we denote by h^{+} and h^{-} the positive and negative parts, respectively, of h. Hence, we have

$$\int fh^{+} d\mu = \int fh^{-} d\mu$$

for every $f \in F$. Hence

$$\nu = |h| \cdot \mu = \frac{1}{2}\left[(2h)^+ \cdot \mu + (2h)^- \cdot \mu\right]$$

is a middle-sum of two points of the set $\tilde{\Xi}_q(\nu)$. But, by assumption, ν is an extremal point of $\tilde{\Xi}_q(\nu)$ and then $|h| = 2h^+ = 2h^-$ μ-a.s., which implies $h = 0$ μ-a.s.. ■

An immediate consequence of Corollary 5 and Theorem 6 is the following

COROLLARY 7. *Let $F \subset L^\infty(\mu)$ be a subspace such that $1 \in F$ and let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. F is dense in $(L^\infty(\mu), \sigma(L^\infty, L^p))$ if and only if it is dense in $(L^\infty(\mu), \|\cdot\|_q)$* (1).

REMARK 8. *For the case $p = 1$, being $L^\infty(\mu)$ dense in $(L^1(\mu), \|\cdot\|_1)$, we have the same equivalence as in Theorem* 1: *let F be a linear subspace of $L^\infty(\mu)$ containing* 1, *then F is dense in $(L^\infty(\mu), \|\cdot\|_1)$ if and only if μ is an extremal point of the set $\Xi_1(\mu)$. Indeed, if F is a subspace of $L^\infty(\mu)$ containing* 1 *and dense in $(L^\infty(\mu), \|\cdot\|_1)$, then it is also dense in $(L^1(\mu), \|\cdot\|_1)$ and so item* 3) *of Theorem* 1 *holds. On the other hand, if the latter holds then F is dense in $(L^1(\mu), \|\cdot\|_1)$ and obviously in $(L^\infty(\mu), \|\cdot\|_1)$ too.*

3. Applications to finance.

3.1. *The model.*

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We consider a financial market where the set of trading dates is given by $\mathcal{T} \subseteq [0, 1]$, with $\mathcal{T} = \{0, 1\}$ or $\mathcal{T} = [0, 1]$, and we denote $\mathcal{S}$ the set of discounted price processes of this economy, i.e. $\mathcal{S}$ is a family of stochastic processes indexed by $\mathcal{T}$ and adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathcal{T}}$, where $\mathcal{F}_0$ is the trivial σ-field and $\mathcal{F}_1 = \mathcal{F}$. For simplicity, we assume that, for each $S = (S_t)_{t \in \mathcal{T}} \in \mathcal{S}$, $S_0 = 1$. We note that the set $\mathcal{S}$ may be infinite. In the continuous-time case, we will always

(1) As pointed out by an anonymous referee, the previous equivalence can be directly proved by using Hölder inequality and the duality (L^p, L^q) without assuming $1 \in F$.

suppose that $\mathbb{F}$ satisfies the usual conditions and each price process $S \in \mathcal{S}$ is càdlàg.

Following Jin, Jarrow and Madan (1999) and Bättig and Jarrow (1999), we identify the set of contingent claims with the space of all essentially bounded random variables $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$ equipped with some topology τ. We call P the true probability of the market.

Finally, throughout the sequel, $\mathbb{R}$ will be the set of real numbers and, if $\mathcal{A}$ is an arbitrary subset of L^∞, $v.\,s.(\mathcal{A})$ will denote the vector space generated by $\mathcal{A}$.

Now, we give two notions of market completeness for the discrete and the continuous-time cases.

DEFINITION 9 (*discrete-time case*). *Let $\mathcal{T} = \{0, 1\}$. The market $\mathcal{S}$ is said to be τ-complete if the set*

$$\mathcal{Y}_d = v.s.\,((\mathcal{S}_1 \cup \mathbb{R}) \cap L^\infty)$$

where $\mathcal{S}_1 = \{S_1; S \in \mathcal{S}\}$, is total in L^∞ for the topology τ.

DEFINITION 10 (*continuous-time case*). *Let $\mathcal{T} = [0, 1]$. The market $\mathcal{S}$ is said to be τ-complete if the set*

$$\mathcal{Y}_c = v.s.\,((\mathfrak{I} \cup \mathbb{R}) \cap L^\infty)$$

is dense in L^∞ for the topology τ, where

$$\mathfrak{I} = \{Y(S_\tau - S_\sigma) : \sigma \leqslant \tau \ \mathbb{F}\text{-stopping times},\ Y \in L^\infty(\mathcal{F}_\sigma, P),\ S \in \mathcal{S}\}.$$

We observe that the spaces $\mathcal{Y}_c$ and $\mathcal{Y}_d$ are not empty, both containing 0. The space L^∞ will be equipped with the strong topology, i.e. the topology induced by the supremum norm $\|\cdot\|_\infty$, and the weak topologies $\sigma(L^\infty, L^p)$ for $p \geqslant 1$. If the market is τ-complete with $\tau = \|\cdot\|_\infty$ or $\tau = \sigma(L^\infty, L^1)$, we will say that it is strongly complete or, respectively, weakly* complete.

A detailed discussion of the economic interpretation of the topology $\sigma(L^\infty, L^1)$ can be found in Bättig and Jarrow (1999).

If we apply Corollary 4 to these notions of market completeness, we obtain immediately the following equivalence.

THEOREM 11. *Let $p \geqslant 1$. The following two assertions are equivalent*:

1) *The market is $\sigma(L^\infty, L^p)$-complete*;

2) *every probability measure* $Q \ll P$ *such that* $\frac{dQ}{dP} \in L^p(P)$ *is extremal in* $\widetilde{\Xi}_p(Q)$.

PROOF. Choose $F = \mathcal{Y}_d$ for the discrete time case and $F = \mathcal{Y}_c$ for the continuous time one. ■

3.2. *The second fundamental theorem of asset pricing: the discrete-time case.*

In this subsection we will treat the case $\mathcal{T} = \{0, 1\}$. Finally, we denote by $\mathcal{M}$ the set of all martingale probability measures for $\mathcal{S}$ and we set

$$\mathcal{M}^a = \{Q \in \mathcal{M} : Q \ll P\}$$

and

$$\mathcal{M}^e = \{Q \in \mathcal{M} : Q \sim P\}.$$

In this case a process $S \in \mathcal{S}$ is a Q-martingale if $S_1 \in L^1(Q)$ and $\mathbb{E}_Q(S_1) = 1$.

Thanks to Theorem 11, we can re-demonstrate, using only some geometrical argument based on the notion of extremality, two results which have been initially obtained by Jarrow, Jin and Madan (1999).

THEOREM 12. *Let* $p \geq 1$ *and let the market be* $\sigma(L^\infty, L^p)$*-complete. Then, there exists at most one* $Q \in \mathcal{M}^e$ *such that* $\frac{dQ}{dP} \in L^p(P)$.

PROOF. We assume that there exist two equivalent martingale probability measures Q_1 and Q_2 for $\mathcal{S}$. Since $\mathcal{M}^e$ is a convex set, for each $\alpha \in [0, 1]$, $Q_\alpha = \alpha Q_1 + (1 - \alpha) Q_2$ is an equivalent martingale probability measure for $\mathcal{S}$. But, since the market is $\sigma(L^\infty, L^p)$-complete, by Theorem 11, every Q_α must be extremal in $\widetilde{\Xi}_p(Q_\alpha) = \mathcal{M} \supseteq [Q_1, Q_2]$, which is a contradiction if we choose $\alpha \in (0, 1)$. ■

Let ν be a finite signed measure over the measurable space $(\Omega, \mathcal{F})$. We will say that $S = (1, S_1) \in \mathcal{S}$ is a ν-martingale if S_1 is $|\nu|$-integrable and $\nu(S_1) = \int S_1 \, d\nu = 1$.

We denote by $\mathcal{M}^s$ the space of all finite signed measures ν which are absolutely continuous with respect to P, such that $\nu(\Omega) = 1$ and each $S \in \mathcal{S}$ is a ν-martingale.

THEOREM 13. *Let $\mathcal{M}^s$ be nonempty. The following two assertions are equivalent*:

1) *the market is weakly*-complete*;
2) $\mathcal{M}^s$ *is a singleton.*

PROOF. Firstly, we show that 2) implies 1). We fix $\nu \in \mathcal{M}^s$, which exists by assumption, and assume that the market is not weakly*-complete, i.e. by Theorem 11 there exists a probability $Q \ll P$ such that

$$Q = \alpha Q_1 + (1-\alpha)\, Q_2$$

where $\alpha \in (0, 1)$ and $Q_i \in \widetilde{\Xi}_1(Q)$ for each $i = 1, 2$. Now, we set

$$\nu_i = Q_i - Q + \nu$$

for $i = 1, 2$. Then, since $\nu_i(S_1) = \mathbb{E}_{Q_i}(S_1) - \mathbb{E}_Q(S_1) + \nu(S_1) = 1$, for each $i = 1, 2$, ν_i is martingale signed measure for $\mathcal{S}$. Furthermore, since $Q_1 \leqslant \dfrac{1}{\alpha} Q$ and $Q_2 \leqslant \dfrac{1}{1-\alpha} Q$, we have $Q_i \ll Q \ll P$ for every $i = 1, 2 \ldots$ Then, since $|\nu_i| \leqslant Q_i + Q + |\nu|$, we have $|\nu_i| \ll P$ for each $i = 1, 2$. This shows that ν is not unique in $\mathcal{M}^s$ and so 2) implies 1).

To show that 1) implies 2), proceed by contradiction and suppose that $\mathcal{M}^s \supseteq \{\nu_1, \nu_2\}$, with $\nu_1 \neq \nu_2$. Observe now that, by the definition of $\mathcal{M}^s$, $\nu_1(S_1) = \nu_2(S_1) = 1$ and $\nu_1(\Omega) = \nu_2(\Omega) = 1$, that is

$$\nu_1^+(S_1) - \nu_1^-(S_1) = \nu_2^+(S_1) - \nu_2^-(S_1) = 1 \tag{1}$$

and

$$\nu_1^+(\Omega) - \nu_1^-(\Omega) = \nu_2^+(\Omega) - \nu_2^-(\Omega) = 1\,, \tag{2}$$

where ν_i^+ and ν_i^- $(i = 1, 2)$ are, respectively, the positive and the negative part of ν_i in its Hahn-Jordan decomposition. This implies

$$\nu_1^+(S_1) + \nu_2^-(S_1) = \nu_2^+(S_1) + \nu_1^-(S_1) \tag{3}$$

and

$$\nu_1^+(\Omega) + \nu_2^-(\Omega) = \nu_2^+(\Omega) + \nu_1^-(\Omega) := k > 0\,. \tag{4}$$

Thus, define the two probability measures Q_1 and Q_2 as follows:

$$Q_1 = \frac{\nu_1^+ + \nu_2^-}{k}\,, \qquad Q_2 = \frac{\nu_2^+ + \nu_1^-}{k}\,. \tag{5}$$

Observe that $Q_1 = Q_2$ on $\mathcal{Y}_d$ and define $Q := \alpha Q_1 + (1-\alpha) Q_2$ for some real $\alpha \in (0, 1)$. It is straightforward to verify that $Q \ll P$ (since $|\nu_i| \ll \ll P$, for $i = 1, 2$) and that Q_1 and Q_2 are absolutely continuous to Q, which implies that $Q_1, Q_2 \in \tilde{\Xi}_1(Q)$. We have so built a probability measure Q absolutely continuous to P that is not extremal in $\tilde{\Xi}_1(Q)$. Finally, Theorem 11 applies and gives that $1) \Rightarrow 2)$ ■

We recall that a necessary and sufficient condition for the existence of an equivalent martingale probability measure (resp. finite signed measure) for $\mathcal{S}$ is the absence of free lunch with free disposal (resp. free lunch). For the precise definition of these two conditions, see Jin, Jarrow and Madan (1999). Here, we note only that, under the absence of free lunch, there could exist arbitrage opportunities.

3.3. *The second fundamental theorem of asset pricing: the continuous-time case.*

Here we pass to the continuous-time case, i.e. we take $\mathcal{T} = [0, 1]$, for which our main reference is Bättig (1999). We suppose that the filtration $\mathbb{F}$ satisfies the usual conditions and that each price process $S \in \mathcal{S}$ is càdlàg (right continuous with left limit).

We denote $\mathcal{M}_{\mathrm{loc}}$ the set of all local martingale probability measures for $\mathcal{S}$ and we set

$$\mathcal{M}_{\mathrm{loc}}^a = \{Q \in \mathcal{M}_{\mathrm{loc}}: Q \ll P\}$$

and

$$\mathcal{M}_{\mathrm{loc}}^e = \{Q \in \mathcal{M}_{\mathrm{loc}}: Q \sim P\}.$$

If we use exactly the same argument as in the proof of Theorem 12, we obtain its analogue in the continuous-time case. In order to avoid repetitions, we omit its proof.

THEOREM 14. *Let $p \geq 1$ and let the market be $\sigma(L^\infty, L^p)$-complete. Then, there exists at most one $Q \in \mathcal{M}_{\mathrm{loc}}^e$ such that $\frac{dQ}{dP} \in L^p(P)$.*

Now, let ν be a signed finite measure over $(\Omega, \mathcal{F})$ such that $\nu(\Omega) = 1$. We will say that $S \in \mathcal{S}$ is a ν-local martingale if $\nu(f) = 0$ for all $f \in \mathcal{I}$ and ν-integrable. We let $\mathcal{M}_{\mathrm{loc}}^s$ denote the space of all finite signed measures ν which are absolutely continuous to P and such that $\nu(\Omega) = 1$ and each $S \in \mathcal{S}$ is a ν-local martingale.

REMARK 15. *For a complete treatment of martingales under a finite signed measure but with a definition slightly different from ours, one can consult Beghdadi-Sakrani* (2003); *for a striking extension to signed measures of Lévy's martingale characterization of Brownian Motion, see Ruiz de Chavez* (1984).

THEOREM 16. *Let $\mathcal{M}^s_{\text{loc}}$ be nonempty. The following two assertions are equivalent*:

1) *the market is weakly*-complete*;
2) *$\mathcal{M}^s_{\text{loc}}$ is a singleton.*

PROOF. One may proceed exactly as in the proof of Theorem 13. ■

3.4. *The Artzner-Heath example.*

In this subsection, we study the τ-completeness of an Artzner-Heath market (abbr. AH-market), which is a slight generalization of the pathological economy constructed by Artzner and Heath (1995). Now we give its precise definition. We use the same notation as in the previous section.

DEFINITION 17. *We say that a financial market $\mathcal{S}$ is of the AH-type or that it is an AH-market if, P_0 and P_1 being two different equivalent probability measures,*

$$\mathcal{M} = [P_0, P_1] = \{P_\alpha = \alpha P_0 + (1-\alpha) P_1;\ \alpha \in [0, 1]\}.$$

In this market we can choose P_0 as the true probability measure. So, applying the different versions of Douglas-Naimark Theorem, one has the following result.

PROPOSITION 18. *An AH-market satisfies the following three properties*:

1) *it is $\|\cdot\|_1$-complete under P_α if and only if $\alpha \in \{0, 1\}$*;
2) *it is not strongly complete under P_α for each $\alpha \in [0, 1]$*;
3) *it is not weakly* complete under P_α for each $\alpha \in [0, 1]$.*

PROOF. The first and the third property are simple consequences of, respectively, Theorem 1 and Theorem 11. In order to prove the second property, we assume that there exists $\alpha \in [0, 1]$ such that the market is complete w.r.t. P_α. By Theorem 2, this is equivalent to the extremality of every $\nu \in ba^+(P_\alpha)$ in $\Xi_{ba}(\nu)$. But, if we choose $\nu = P_\beta$ for $\beta \in (0, 1)$, P_β has to be extremal in $\Xi_{ba}(P_\beta) \supset [P_0, P_1]$, which is obviously absurd. ■

REMARK 19. *The previous proposition is a generalization of Proposition* 4.1 *of Artzner and Heath* (1995) *and of the content of Section* 6 *of Jarrow, Jin and Madan* (1999).

Now, we give a little more general construction of an AH-market than the original one contained in Artzner and Heath (1995).

Firstly, we set $(\Omega, \mathcal{F}) = (\mathbb{Z}^*, \mathcal{P}(\mathbb{Z}^*))$, where $\mathbb{Z}^*$ is the set of integers different from zero, and $\mathcal{S} = \{S^n : n \in \mathbb{Z}\}$. Now, we assume that every random variable S_1^n has a two-points support, i.e.

$$\begin{aligned} \operatorname{supp} S_1^n &= \{n, n+1\} \quad \text{for } n > 0 \\ S_1^n(k) &= S_1^{-n}(-k) \quad \text{for } n < 0 \\ S_1^0 &= \frac{1}{K(p_1+q_1)}(\mathbf{1}_{\{1\}} + \mathbf{1}_{\{-1\}}). \end{aligned} \tag{6}$$

REMARK 20. *The hypothesis on the support of the price processes is not restrictive at all. Actually, thanks to Lemme A of Dellacherie* (1968), *we know that the extremality of a probability P in the set of martingale probabilities for a process $S = (1, S_1) \in \mathcal{S}$, implies $S \equiv 1$ or that the support of the law of S_1 is a two-points set. In financial terms the result of Dellacherie means that, in a two-period setting, the only market model which is both arbitrage-free and complete is the binomial one.*

Then, we fix two different equivalent probabilities P_0 and P_1 over $(\Omega, \mathcal{F})$ and we denote, for every $n \in \mathbb{Z}^*$, $p_n^0 = P_0(\{n\})$ and $p_n^1 = P_1(\{n\})$.

Every process $S \in \mathcal{S}$ has to be a martingale under P_0 and P_1. Then, we have

$$Q(\{n\})\, S_1^n(n) + Q(\{n+1\}) S_1^n(n+1) = 1 \quad \text{for every } n > 0, \tag{7}$$

for $Q \in \{P_0, P_1\}$, i.e. for every $n > 0$

$$\begin{aligned} p_n^0 S_1^n(n) + p_{n+1}^0 S_1^n(n+1) &= 1, \\ p_n^1 S_1^n(n) + p_{n+1}^1 S_1^n(n+1) &= 1. \end{aligned} \tag{8}$$

We solve (8) with respect to S_1^n and we found for each $n > 0$

$$S_1^n = \frac{(p_{n+1}^0 - p_{n+1}^1)\,\mathbf{1}_{\{n\}} - (p_n^0 - p_n^1)\,\mathbf{1}_{\{n+1\}}}{p_n^1 p_{n+1}^0 - p_n^0 p_{n+1}^1}\,. \tag{9}$$

Hence, we have constructed a class $\mathcal{S}$ of processes, which are martingales under both P_0 and P_1, i.e.

$$[P_0, P_1] \subset \mathcal{M}\,.$$

Now, we fix S_1^n and interpret (8) as an equation with respect to the vector Q and we found that its solutions are of the form $P_\lambda = \lambda P_0 + (1-\lambda)\,P_1$ for $\lambda \in \mathbb{R}$. This implies, for this kind of market,

$$\mathcal{M} \subset \{P_\lambda;\ \lambda \in \mathbb{R}\}\,.$$

Now, we look for some conditions on P_0 and P_1 such that P_λ is not a probability when $\lambda \notin [0, 1]$.

LEMMA 21. *Let P_0 and P_1 be two different equivalent probabilities over an arbitrary measurable space $(\Omega, \mathcal{F})$. The following two properties are equivalent*:

1) $P_\lambda = \lambda P_0 + (1-\lambda)\,P_1$ *is a probability if and only if* $\lambda \in \in [0, 1]$;

2) $\dfrac{dP_0}{dP_1}$ *and* $\dfrac{dP_1}{dP_0}$ *are not bounded.*

PROOF. Firstly, we assume that the Radon-Nikodym derivative $\dfrac{dP_1}{dP_0}$ is bounded, i.e. there exists a constant $M > 1$ such that $\dfrac{dP_1}{dP_0} \leqslant M$ almost surely. Let $f\colon \Omega \to \mathbb{R}$ be a measurable, positive and bounded function. If $\lambda > 1$, we have

$$\begin{aligned}\int f dP_\lambda &= \int f \left(\lambda + (1-\lambda)\,\frac{dP_1}{dP_0}\right) dP_0 \\ &\geqslant \int f(\lambda + (1-\lambda)\,M)\,dP_0.\end{aligned}$$

Then, if one chooses $\lambda > 1$ such that $\lambda + (1-\lambda)\,M \geqslant 0$, i.e. $\lambda \leqslant \dfrac{M}{M-1}$, P_λ is a probability measure. If we assume that the other Radon-Nikodym derivative is bounded, then we found that there exists $\lambda < 0$ such that P_λ is a probability.

Now, let $\frac{dP_1}{dP_0}$ be unbounded, i.e. for every $M > 0$

$$P_\alpha\left(\frac{dP_1}{dP_0} \geqslant M\right) > 0 \quad \text{for } \alpha = 0, 1.$$

Let $f = \mathbf{1}_{\left\{\frac{dP_1}{dP_0} \geqslant M\right\}}$ and $\lambda > 1$. Then, we have

$$\begin{aligned}\int f dP_\lambda &= \int_{\{\frac{dP_1}{dP_0} \geqslant M\}} \left(\lambda + (1-\lambda)\frac{dP_1}{dP_0}\right) dP_0 \\ &\leqslant (\lambda + (1-\lambda) M)\, P_0\left(\frac{dP_1}{dP_0} \geqslant M\right) \\ &< 0\end{aligned}$$

for M sufficiently large. For the case $\lambda < 0$, we proceed exactly in the same way, using the fact that $\frac{dP_1}{dP_0}$ is supposed unbounded. ■

Finally, thanks to Lemma 20, we have the following result, which is a generalization of the construction contained in Section 3 of Artzner and Heath (1995).

PROPOSITION 22. *Let P_0 and P_1 two different equivalent probability measures on $(\mathbb{Z}^*, \mathcal{P}(\mathbb{Z}^*))$ which satisfy condition 2 of Lemma 21. Then the class $\mathcal{S}$ defined by (6) and (9) is an AH-market, i.e.*

$$\mathcal{M} = [P_0, P_1].$$

EXAMPLE 23 (*Artzner and Heath* (1995)). *Let $0 < p < q < 1$ be two real numbers. We set, for every $n > 0$,*

$$P_0(\{n\}) = Kp^n \mathbf{1}_{\{n>0\}} + Kq^{-n}\mathbf{1}_{\{n<0\}}$$

$$P_1(\{n\}) = P_0(\{-n\}) \quad \textit{for every } n \in \mathbb{Z}^*,$$

where K is a renormalizing constant. In this case, it is obvious that

$$\lim_{n\to+\infty} \frac{dP_1}{dP_0}(n) = \lim_{n\to+\infty}\left(\frac{q}{p}\right)^n = +\infty$$

and

$$\lim_{n \to -\infty} \frac{dP_0}{dP_1}(n) = \lim_{n \to -\infty} \left(\frac{p}{q} \right)^n = +\infty .$$

So the previous proposition applies, and we find that for

$$S_1^n = \frac{(q^{n+1} - p^{n+1})\, \mathbf{1}_{\{n\}} + (q^n - p^n)\, \mathbf{1}_{\{n+1\}}}{Kp^n q^n (q-p)} \tag{10}$$

the set of all equivalent martingale probabilities for S is equal to the segment $[P_0, P_1]$.

4. Conclusions.

In this paper, we have established in a very easy way a weak version of the Douglas-Naimark theorem, which relates the density (with respect to the weak topology) of a subspace of a vector topological locally convex space with the extremality of a certain family of linear functionals. Then, in Subsection 2, we have applied this result to the space $L^\infty(\mu)$ equipped with the topologies $\sigma(L^\infty, L^p)$ for $p \geqslant 1$, where μ is a probability measure, and in Subsection 2.3 we have shown an analogue result for the spaces $(L^\infty(\mu), \|\cdot\|_p)$, $p \geqslant 1$. Finally, thanks to these results, we have obtained, in Section 3, a condition equivalent to the market completeness and based on the notion of extremality of measures, which has permitted us to give new elementary proofs of the second fundamental theorem of asset pricing and to discuss the completeness of a more general construction of the Artzner-Heath example.

REFERENCES

[1] P. ARTZNER - D. HEATH, *Approximate Completeness with Multiple Martingale Measures*, Math. Finance, **5** (1995), pp. 1-11.

[2] R. BÄTTIG, *Completeness of securities market models-an operator point of view*, Ann. Appl. Prob., **9** (1999), pp. 529-566.

[3] R. BÄTTIG - R.A. JARROW, *The Second Fundamental Theorem of Asset Pricing: A New Approach*, The Review of Financial Studies, **12** (1999), pp. 1219-1235.

[4] S. BEGHDADI-SAKRANI, *Calcul stochastique pour des mesures signées.* In: *Séminaire de Probabilités XXXVI*, Lecture Notes in Math., **1801** (2003), pp. 366-382, Springer, Berlin, 2003.

[5] C. DELLACHERIE, *Une représentation intégrale des surmartingales à temps discret*, Publ. Inst. Statist. Univ. Paris, **17** (2) (1968), pp. 1-17.
[6] R. G. DOUGLAS, *On extremal measures and subspace density*, Michigan Math. J., **11** (1964), pp. 644-652.
[7] R. G. DOUGLAS, *On Extremal Measure and Subspace Density II*, Proceedings of the American Math. Society, **17** (6) (1966), pp. 1363-1365.
[8] N. DUNFORD - J. T. SCHWARTZ, *Linear Operators. Part I: General Theory*, John Wiley and Sons, New York Chichester Brisbane Toronto Singapore, 1957.
[9] R. A. JARROW - X. JIN - D. P. MADAN, *The Second Fundamental Theorem of Asset Pricing*, Math. Finance, **9** (1999), pp. 255-273.
[10] R. A. JARROW - D. P. MADAN, *Hedging contingent claims on semimartingales*, Finance Stochast., **3** (1999), pp. 111-134.
[11] M. A. NAIMARK, *Extremal spectral functions of a symmetric operator*, Bull. Acad. Sci. URSS Sér. Math., **11** (1947), pp. 327-344.
[12] L. NARICI - E. BECKENSTEIN, *Topological vector spaces*, Dekker, New York and Basel, 1985.
[13] J. RUIZ DE CHAVEZ, *Le Théorème de Paul Lévy pour des mesures signées. In:* J. Azema - M. Yor (eds.), *Séminaire de Probabilités XVIII*, Lect. Notes Math., **1059** (1984), pp. 245-255, Springer, Berlin Heidelberg New York.
[14] H. H. SCHAEFER, *Topological vector spaces*, MacMillan, London, 1966.
[15] M. YOR, *Sous-espaces denses dans L^1 ou H^1 et representation des martingales.* In: C. Dellacherie - P. A. Meyer - M. Weil (eds.), *Séminaire de Probabilités XII*, Lect. Notes Math., **649** (1976), pp. 265-309, Springer, Berlin Heidelberg New York.

Manoscritto pervenuto in redazione l'11 marzo 2004.