REND. SEM. MAT. UNIV. PADOVA, Vol. 112 (2004)

An Interpolation Inequality in Exterior Domains.

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1. Introduction.

This note is concerned with some interpolation inequalities of Gagliardo-Nirenberg type. In [4, 13] it is proved that if $\Omega \subset \mathbb{R}^n$ is a bounded domain having the cone property or if $\Omega = \mathbb{R}^n$, then any function u belonging to $L^q(\Omega)$ with derivatives $D^\alpha u$, α *multi-index*, belonging to $L^p(\Omega)$, $q, p \geq 1$, satisfies the following inequality: for $m = |\alpha|$ and $j = 0, \ldots, m-1$

$$(1.1) \qquad |D^j u|_{L^r(\Omega)} \leq C_1 \, |D^m u|^a_{L^p(\Omega)} \, |u|^{1-a}_{L^q(\Omega)} + C_2 \, |u|_{L^q(\Omega)} \ (^1),$$

where $\dfrac{1}{r} = \dfrac{j}{n} + a\left(\dfrac{1}{p} - \dfrac{m}{n}\right) + (1-a)\dfrac{1}{q}$, with $a \in \left[\dfrac{j}{m}, 1\right]$ either if $p = 1$ or if $p > 1$ and $m - j - \dfrac{n}{p} \notin \mathbb{N} \cup \{0\}$, while $a \in \left[\dfrac{j}{m}, 1\right)$ for $p > 1$ and $m - j - \dfrac{n}{p} \in \mathbb{N} \cup \{0\}$. The constants C_1, C_2 are independent of u. In interpolation inequality (1.1) the constant C_2 can be equal to zero either if $u \in W_0^{m,p}(\Omega)$ or if we assume $\Omega = \mathbb{R}^n$ (see [7, 10, 13, 15]). An extension of inequality (1.1) has been given in [11, 14]. It is known that the above inequality is an *improvement* of the Sobolev embedding theorem. Indeed, if $\dfrac{1}{p} - \dfrac{m}{n} > \dfrac{1}{q}$ and $u \in W^{1,p}(\Omega) \cap L^q(\Omega)$, then inequality (1.1) implies, for the values of the exponent r, a range wider than the Sobolev embedding one. Moreover, inequality (1.1) is interesting in the study of partial diffe-

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(1) Using the notation of [4, 13], for $h \in \mathbb{N}$, by the symbol $|D^h u|_{L^p(\Omega)}$ we mean $\sup\limits_{|\alpha| = h} |D^\alpha u|_{L^p(\Omega)}$.

rential equations and, of course, the inequality is improved proving that $C_2 = 0$.

Assuming that Ω is an exterior domain, we are going to investigate the validity of inequality (1.1) with $C_2 = 0$ without requiring that u has trace equal zero on $\partial\Omega$. For such a problem, partial results are already known. In [5, 6] the above problem was studied in connection with Sobolev's inequality: $|u - u_\circ|_{L^r(\Omega)} \leq C|\nabla u|_{L^p(\Omega)}$, $\frac{1}{r} = \frac{1}{p} - \frac{1}{n}$, $p \in [1, n)$, where $u_\circ$ is a constant depending on u and Ω is an exterior domain. *A priori* function u has trace different from zero on $\partial\Omega$, but $\lim_{|x| \to \infty} \int_{S_n} |u - u_\circ| d\sigma = 0$, where S_n is the surface of the n-dimensional unit sphere. If, in some sense, $u \to 0$ for $|x| \to \infty$, then constant $u_\circ = 0$. Moreover, in [9], in relation with some applications in partial differential equations, inequality (1.1) is studied with respect to the interpolation by derivatives of first order: $|u|_{L^r(\Omega)} \leq C_1 |\nabla u|^a_{L^p(\Omega)} |u|^{1-a}_{L^q(\Omega)}$. However, also in the case of first order derivatives, the result in [9] is partial since $r \in [q, +\infty)$ is required. The aim of this paper is to establish inequality (1.1) with $C_2 = 0$ in a complete form in the case of an exterior domain.

The proof is rather elementary: it makes use of an appropriate construction of cut-off functions and of inequalities of Sobolev and Gagliardo-Nirenberg type. It consists of two main steps: the first goal is to prove inequality (1.1) with $C_2 = 0$ interpolating function u by first and second order derivatives; subsequently, it is given the generalization of the inequality to any order derivatives $j < m$.

A referee of this paper has pointed out that in the case of Lipschitz boundary (special case of our assumption) the result can be obtained employing a modified version of the extension theorem of Burenkov (cf. [2]) and the Gagliardo-Nirenberg results.

2. Notations and statement of the theorem.

Throughout this paper $\Omega \subset \mathbb{R}^n$ will denote an exterior domain, that is an unbounded domain with compact boundary, having the cone property. By d we denote the diameter of $\mathbb{R}^n \setminus \Omega$. $L^p(\Omega)$ $(p \geq 1)$ denotes the Banach space of all Lebesgue measurable functions u endowed with the usual L^p norm; likewise $L^\infty(\Omega)$ denotes the Banach space of Lebesgue measurable functions such that $|u|_\infty = \text{ess sup}_\Omega |u(x)| < +\infty$. We shall denote by $|\cdot|_p$ the norm on Ω, while if $D \neq \Omega$ is a subdomain of $\mathbb{R}^n$ we shall write $|\cdot|_{L^p(D)}$. We let $W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), 0 \leqslant |\alpha| \leqslant m\}$, with the norm $|u|_{m,p} = \left(\sum_{|\alpha|=0}^{m} |D^\alpha u|_p^p \right)^{1/p}$ if $p \geqslant 1$, $|u|_{m,\infty} = \max_{0 \leqslant |\alpha| \leqslant m} |D^\alpha u|_\infty$ if $p = +\infty$, where $D^\alpha u$ denotes weak derivatives of $u(x)$ of order $|\alpha|$. For $m \geqslant 0$, $p \geqslant 1$ we set $\widehat{W}^{m,p}(\Omega) = \{u \in L^1_{\text{loc}}(\Omega) : D^\alpha u \in L^p(\Omega), |\alpha| = m\}$. As described in [1, 3], any domain Ω having the cone property can be expressed as a union of finitely many subdomains, each of which is union of translates of parallelepipeds. Following Miranda, [10, 12], if $\mu \in (0, 1]$, we define $C^{[0,\mu]}(\Omega)$ as the set of locally Hölder continuous functions, that is all functions such that

$$\sup_{x, y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\mu} < \infty,$$

for any x, y belonging to a parallelepiped $\Pi \subseteq \Omega$. We will denote the upper bound, which is independent of Π, by $[u]_\mu^\Omega$. Then, for $p < 0$ we set $k = [-n/p]$, $\mu = -k - n/p$, and define the *Nirenberg seminorm*, [13], setting

$$|u|_p = \begin{cases} \sup_{|\sigma|=k} [D^\sigma u]_\mu^\Omega & \text{if } \mu > 0, \\ \sup_{|\sigma|=k} |D^\sigma u|_\infty & \text{if } \mu = 0. \end{cases}$$

Since we will often make use of it later on, we define an infinitely differentiable function $h(x)$ of the positive variable x satisfying the conditions $h(x) \in [0, 1]$, $h(x) = 1$ for $x \leqslant 1$, $h(x) = 0$ for $x \geqslant 2$. If a is a positive constant and x is a point of $\mathbb{R}^n$, we let $h^a(x) = h(|x|/a)$. If w is a function defined on Ω, we can decompose it in the sum $w = w_1^a + w_2^a$, where $w_1^a = (1 - h^a(x))\, w(x), w_2^a(x) = h^a(x)\, w(x)$. The symbol $S_a(x)$ denotes an open ball of radius a centered at x and $K_a(x)$ the set $\{x \in \mathbb{R}^n : a \leqslant |x| \leqslant 2a\}$.

The symbol C denotes a numerical constant whose value is unessential to our aims. As in [8], when there is no ambiguity, if $A \geqslant 0$ the quantity $C(1 + AC)$ is majorized by C.

Now we are in a position to state our result.

THEOREM 2.1. *Let $w(x)$ be in $\widehat{W}^{m,p}(\Omega) \cap L^q(\Omega)$, $p \in [1, +\infty]$, $q \geqslant 1$. Then, for $k \in \{0, 1, \dots, m-1\}$, the following inequality holds*

$$|D^k w|_r \leqslant C |D^m w|_p^a \, |w|_q^{1-a}, \tag{2.2}$$

where

$$\frac{1}{r} = \frac{k}{n} + a\left(\frac{1}{p} - \frac{m}{n}\right) + (1-a)\frac{1}{q}, \tag{2.3}$$

with $a \in \left[\frac{k}{m}, 1\right]$ *either if* $p = 1$ *or if* $p > 1$ *and* $m - k - n/p \notin \mathbb{N} \cup \{0\}$, *while* $a \in \left[\frac{k}{m}, 1\right)$ *if* $p > 1$ *and* $m - k - n/p \in \mathbb{N} \cup \{0\}$. *The result also holds if* $q = +\infty$; *however, in the case* $k = 0$ *and* $mp < n$ *the following additional condition is required:* w *tends to zero at infinity or* $w \in L^{q'}(\Omega)$ *for some finite* $q' \geq 1$. *The constant* C *in inequality* (2.2) *is independent of* $w(x)$.

REMARK 2.1. In virtue of Lemma 3.1 below, if we replace w by $w - w_0$, we can omit the additional hypothesis of the case $k = 0$ and $mp < n$.

Observe that, by using Gagliardo-Nirenberg's theorem, $D^k w$ $(k = 0, \dots m-1)$ certainly enjoys the summability properties, as we point out in Lemma 3.7 given in section 3. Hence, the aim of the theorem is to show the validity of the interpolation inequality of the form (2.2).

3. Some preliminary results.

LEMMA 3.1. *Let* $u(x)$ *be in* $\widehat{W}^{1,p}(\Omega)$, $p \in [1, n)$. *Then, there exist constants* u_0 *and* C *such that*

$$|u - u_0|_q \leq C|\nabla u|_p, \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{n},$$

with C *independent of* u.

PROOF. In [5], Chapter II Theorem 5.1, the result has been proved assuming $\partial\Omega$ locally lipschitzian. However, the same proof works also if we assume Ω with the cone property. The case $\Omega \subseteq \mathbb{R}^3$ and $\partial\Omega$ of class C^2 can be found in [6].

THEOREM 3.1. *Let* D *be a bounded domain of* $\mathbb{R}^n$ *having the cone property and* $u(x)$ *be in* $\widehat{W}^{m,p}(D) \cap L^q(D)$, $p, q \in [1, \infty]$. *Then, for* $k \in \{0, \dots, m-1\}$, *the following inequality holds*

$$|D^k u|_{L^r(D)} \leq C_1 |D^m u|^a_{L^p(D)} |u|^{1-a}_{L^q(D)} + C_2 |u|_{L^q(D)}, \tag{3.1}$$

provided that

$$\frac{1}{r} = \frac{k}{n} + a\left(\frac{1}{p} - \frac{m}{n}\right) + (1-a)\frac{1}{q}, \tag{3.2}$$

with $a \in \left[\frac{k}{m}, 1\right]$ *either if* $p = 1$ *or if* $p > 1$ *and* $m - k - n/p \notin \mathbb{N} \cup \{0\}$, *while* $a \in \left[\frac{k}{m}, 1\right)$ *if* $p > 1$ *and* $m - k - n/p \in \mathbb{N} \cup \{0\}$. *The result also holds for* $D = \mathbb{R}^n$, *with* $C_2 = 0$. *In this case, if* $q = \infty$, $k = 0$ *and* $mp < n$ *the following additional condition is required: w tends to zero at infinity or* $w \in L^{q'}(\Omega)$ *for some finite* $q' \geq 1$. *The constants* C_1 *and* C_2 *in inequality* (3.1) *are independent of* $w(x)$.

PROOF. See Gagliardo [4], Nirenberg [13] and Miranda [10] also.

LEMMA 3.2. *Let u be a twice continuously differentiable function on the open interval* $(0, 1)$, *u in* L^q *and* u'' *in* L^p, $q \in [1, \infty]$, $p \geq 1$. *Moreover, suppose* u' *equals zero at least in a point of the interval* $(0, 1)$. *Then, the following inequality holds*

$$|u'|_{q_1} \leq C|u''|_p^{1/2}\,|u|_q^{1/2}, \qquad q_1 = \frac{2pq}{p+q}. \tag{3.3}$$

The constant C in inequality (3.3) *is independent of* $u(x)$.

PROOF. See Miranda [10], Lemma 2.II.

LEMMA 3.3. *Let u be a twice continuously differentiable function on the open interval* $(0, 1)$, *u in* L^q *and* u'' *in* L^p, $q \in [1, \infty]$, $p \geq 1$. *Then, the following inequality holds*

$$|u'|_{q_1} \leq C|u''|_p^{1/2}\,|u|_q^{1/2} + |u|_{\overline{p}}, \quad \forall \overline{p} \in [1, q], \tag{3.4}$$

with $q_1 = \dfrac{2pq}{p+q}$. *The constant C in inequality* (3.4) *is independent of* $u(x)$.

PROOF. We can assume $u' \neq 0$ for all $x \in (0, 1)$, since if there is $\overline{x} \in (0, 1)$ with $u'(\overline{x}) = 0$ then the result follows trivially from Lemma 3.2. Then just to fix the ideas, let us assume that

$$u'(x) \geq u'(c) > 0\,.$$

If there exists a point $x_0 \in (0, 1)$ such that $u(x_0) = 0$, we have

$$|u(x)| \geqslant u'(c)\,|x - x_0|,$$

and then, by a simple computation, we get

$$|u'(c)| \leqslant C(\overline{p})\,|u|_{\overline{p}}, \qquad \forall \overline{p} \in [1, q]. \tag{3.5}$$

Setting

$$\widetilde{u}(x) = u(x) - u'(c)(x - c),$$

we have

$$\widetilde{u}'(c) = 0; \qquad |\widetilde{u}'(x)| \leqslant u'(x); \qquad |\widetilde{u}|_{\overline{p}} \leqslant (1 + C(\overline{p}))\,|u|_{\overline{p}}, \qquad \forall \overline{p} \in [1, q].$$

Then, by Lemma 3.2 we have

$$|\widetilde{u}'|_{q_1} \leqslant C|\widetilde{u}''|_p^{1/2}\,|\widetilde{u}|_q^{1/2} \leqslant C|u''|_p^{1/2}\,|u|_q^{1/2}$$

and, since

$$|\widetilde{u}'|_{q_1} = |u'(x) - u'(c)|_{q_1} \geqslant |u'(x)|_{q_1} - |u'(c)|,$$

from (3.5) we also get

$$|u'|_{q_1} \leqslant C|u''|_p^{1/2}\,|u|_q^{1/2} + C(\overline{p})\,|u|_{\overline{p}}.$$

Finally, let us suppose $u(x) \neq 0$ for all $x \in (0, 1)$ and assume $u(x) > 0$. In this case, setting $u(c) = \min_{x \in [0, 1]} u(x)$, we define the following function

$$\widehat{u}(x) = u(x) - u(c).$$

From the definition it is immediate that $|\widehat{u}|_{\overline{p}} \leqslant C|u|_{\overline{p}}$. Moreover, the function $\widehat{u}$ has the same properties of regularity of u. So, we can reproduce for $\widehat{u}$ the same arguments used in the previous case for u, and we get

$$|u'|_{q_1} = |\widehat{u}'|_{q_1} \leqslant C|\widehat{u}''|_p^{1/2}\,|\widehat{u}|_q^{1/2} + \\ + C|\widehat{u}|_{\overline{p}} \leqslant C|u''|_p^{1/2}\,|u|_q^{1/2} + C|u|_{\overline{p}}, \qquad \forall \overline{p} \in [1, q].$$

The case $u(x) \neq 0$ for all $x \in (0, 1)$ and $u(x) < 0$ can be treated in a similar way.

REMARK 3.1. Inequality (3.4) has been proved in [4, 10, 13] for $\bar{p} = q$. Our proof follows the lines of the one given in [10], with suitable changes in order to obtain the inequality for any $\bar{p} \in [1, q]$.

LEMMA 3.4. *Let D be a bounded domain of $\mathbb{R}^n$ having the cone property and $u(x)$ be in $\widehat{W}^{2,p}(D) \cap L^q(D)$, $p \in [1, n/2)$, $q > np/(n-2p)$. Then, the following inequality holds*

$$(3.6) \qquad |\nabla u|_{L^r(D)} \leqslant$$

$$\leqslant C\left(|D^2 u|^a_{L^p(D)} |u|^{1-a}_{L^q(D)} + |u|_{L^{\bar{p}}(D)} + \frac{2a-1}{a} |u|^a_{L^{\bar{p}}(D)} |u|^{1-a}_{L^q(D)} \right), \quad \forall \bar{p} \in [1, q],$$

where

$$\frac{1}{r} = \frac{1}{n} + a\left(\frac{1}{p} - \frac{2}{n}\right) + (1-a)\frac{1}{q},$$

with $a \in \left[\frac{1}{2}, 1\right]$. The constant C in (3.6) *is independent of $w(x)$.*

PROOF. Let us consider the cases $a = 1$ and $a = \frac{1}{2}$. For $a = 1$ inequality (3.6) follows from Theorem 3.1. For $a = \frac{1}{2}$ we can obtain inequality (3.6) using Lemma 3.3 and following the proof given in [4, 13] or [10] ([2]). Finally, employing the convexity theorem for $L^r(D)$ we deduce the inequality in the complete form.

REMARK 3.2. Lemma 3.4 can be proved for arbitrary exponents of summability p and q and for any order derivatives. In this way, since $\bar{p}$ is arbitrary, we obtain a sort of generalization of Gagliardo-Nirenberg's theorem. However, we think that such a result does not give a real improvement to the theory.

LEMMA 3.5. *Let $u \in \widehat{W}^{1,p}(\Omega)$, $p > n$. Then $u \in C^{[0,\mu]}(\Omega)$, $\mu = 1 - \frac{n}{p}$, and*

$$(3.7) \qquad [u]^{\Omega}_{\mu} \leqslant C |\nabla u|_p,$$

with the constant C independent of $u(x)$.

PROOF. See Gagliardo [3], Nirenberg [13].

([2]) In particular we suggest [4], par. 6, pg. 45.

LEMMA 3.6. *Let u be in $C^{[0,\mu]}(\Omega)\cap L^\infty(\Omega)$. Setting $s=-\dfrac{n}{\mu}$, the following inequality holds*

$$|u|_q \leqslant C|u|_s^a\,|u|_\infty^{1-a}, \qquad a=\frac{s}{q}\in[0,1]. \tag{3.8}$$

The constant C in inequality (3.8) *is independent of $u(x)$.*

PROOF. The proof is quite immediate. Indeed, choosing two points x, y in the same parallelepiped Π in Ω, we have

$$\frac{|u(x)-u(y)|}{|x-y|^{\mu_1}}=\left(\frac{|u(x)-u(y)|}{|x-y|^{\mu}}\right)^{\mu_1/\mu}|u(x)-u(y)|^{1-\mu_1/\mu}\leqslant \tag{3.9}$$

$$\leqslant C|u|_s^{\mu_1/\mu}\,|u|_\infty^{1-\mu_1/\mu}, \qquad \text{with } \mu_1=-\frac{n}{q}.$$

Then, considering the upper bound of (3.9), which is independent of Π, we obtain inequality (3.8).

LEMMA 3.7. *Let $u(x)$ be in $\widehat{W}^{m,p}(\Omega)\cap L^q(\Omega)$, $p\in[1,+\infty]$, $q\geqslant 1$. Then, for $k\in\{0,1,\dots,m-1\}$,*

$$D^k u\in L^r(\Omega),$$

where

$$\frac{1}{r}=\frac{k}{n}+a\left(\frac{1}{p}-\frac{m}{n}\right)+(1-a)\frac{1}{q},$$

with $a\in\left[\dfrac{k}{m},1\right]$ either if $p=1$ or if $p>1$ and $m-k-n/p\notin\mathbb{N}\cup\{0\}$, while $a\in\left[\dfrac{k}{m},1\right)$ if $p>1$ and $m-k-n/p\in\mathbb{N}\cup\{0\}$. The result also holds if $q=+\infty$; however, in the case $k=0$ and $mp<n$ the following additional condition is required: w tends to zero at infinity or $w\in L^{q'}(\Omega)$ for some finite $q'\geqslant 1$.

PROOF. Let $R>d$. The result is a consequence of the decomposition $u=u_1^R+u_2^R$. Indeed, u_1^R and u_2^R can be considered as functions defined on $\mathbb{R}^n$ and $\Omega\cap S_{2R}$, respectively. For these two functions the summability of the kth order derivatives is ensured by Theorem 3.1.

4. Proof of the theorem.

LEMMA 4.1. *Let $w(x)$ be in $\widehat{W}^{1,p}(\Omega) \cap L^q(\Omega)$, $p \in [1, +\infty]$, $q \geq 1$. Then, the following inequality holds*

$$|w|_r \leq C |\nabla w|_p^a |w|_q^{1-a}, \tag{4.1}$$

where

$$\frac{1}{r} = a\left(\frac{1}{p} - \frac{1}{n}\right) + (1-a)\frac{1}{q}, \tag{4.2}$$

with $a \in [0, 1]$ if $p \neq n$, $a \in [0, 1)$ if $p = n$. The result also holds if $q = +\infty$; however, in the case $p < n$ the following additional condition is required: w tends to zero at infinity or $w \in L^{q'}(\Omega)$ for some finite $q' \geq 1$. The constant C in (4.1) *is independent of $w(x)$.*

PROOF. For $p \in [1, n)$ inequality (4.1) is an easy consequence of the convexity theorem for L^p spaces and Lemma 3.1. Indeed, if $q \leq np/(n-p)$, we have

$$|w|_r \leq |w|^a_{np/(n-p)} |w|_q^{1-a} \leq C |\nabla w|_p^a |w|_q^{1-a}, \quad \forall r \in \left[q, \frac{np}{n-p}\right].$$

If $q > np/(n-p)$ the same argument works.

For $p \in [n, +\infty]$, either r belongs to $[q, +\infty]$ or r is a negative real number.

Case 1 - Let us consider the case $p \geq n$, $r \in [q, +\infty)$ and $q \geq p$. Let $w = w_1^R + w_2^R$. Since $w_1^R(x)$ is defined on the whole $\mathbb{R}^n$, we can apply Theorem 3.1 and, subsequently, Hölder's inequality. Thus,

$$\begin{aligned} |w_1^R|_r &\leq C_1 |\nabla w_1^R|_p^a |w_1^R|_q^{1-a} \leq C_1 \left(|\nabla w|_p + \frac{1}{R} |w|_{L^p(K_R)} \right)^a |w|_q^{1-a} \leq \\ &\leq C_1 \left(|\nabla w|_p + \frac{1}{R^{1 - n(q-p)/qp}} |w|_q \right)^a |w|_q^{1-a}. \end{aligned} \tag{4.3}$$

Since $w_2^R(x)$ is defined on a bounded domain, using Theorem 3.1, it follows that

$$|w_2^R|_r \leq C_2 |\nabla w_2^R|_p^a |w_2^R|_q^{1-a} + C_3 |w_2^R|_q. \tag{4.4}$$

We can assume $|\nabla w|_p \neq 0$, otherwise $w(x)$ is a constant and the assumption $w(x) \in L^q(\Omega)$ implies $w = 0$. Then the inequality (4.1) becomes trivial. There are two possible cases:

$$\frac{|w|_q}{|\nabla w|_p} \leqslant d^{\,1-n(q-p)/qp} \quad \text{or} \quad \frac{|w|_q}{|\nabla w|_p} > d^{\,1-n(q-p)/qp}.$$

In the first case we fix $R > d$ and increase $|w|_q$ in the right-hand side of (4.3) and (4.4) by $R^{\,1-n(q-p)/qp}|\nabla w|_p$. Therefore,

$$|w|_r \leqslant |w_1^R|_r + |w_2^R|_r \leqslant (C_1 2^a + C_2 + C_3 R^{\,(1-n(q-p)/qp)a})\,|\nabla w|_p^a\,|w|_q^{1-a}.$$

In the latter case we modify the estimate for $w_2^R(x)$ and, subsequently, we choose R in a suitable way. For any $r \geqslant q$, there exists $\overline{p} \in [1, n)$ such that $n\overline{p}/(n-\overline{p}) > r$. In virtue of (4.1) for $\overline{p} \in [1, n)$, we get

$$|w_2^R|_r \leqslant C|\nabla w_2^R|_{\overline{p}}^b\,|w_2^R|_q^{1-b},$$

with C independent of R and $w_2^R(x)$. Since $p > \overline{p}$ and $w_2^R(x)$ is 0 for $|x| \geqslant$ $\geqslant 2R$, applying Hölder's inequality, we deduce

$$\begin{aligned}|w_2^R|_r \leqslant CR^{\,bn(p-\overline{p})/p\overline{p}}\,|\nabla w_2^R|_p^b\,|w_2^R|_q^{1-b} \leqslant \\ \leqslant CR^{\,bn(p-\overline{p})/p\overline{p}}\Big(|\nabla w|_p + \frac{1}{R}|w|_{L^p(K_R)}\Big)^b |w|_q^{1-b} \leqslant \\ \leqslant CR^{\,bn(p-\overline{p})/p\overline{p}}\Big(|\nabla w|_p + \frac{1}{R^{\,1-n(q-p)/qp}}|w|_q\Big)^b |w|_q^{1-b}.\end{aligned}$$

Since $p \geqslant n$ then $1 > n(q-p)/qp$, for any $q \geqslant 1$. Thus choosing $R^{\,1-n(q-p)/qp} = |w|_q/|\nabla w|_p > d^{\,1-n(q-p)/qp}$, the estimate for $|w|_r$ becomes

$$|w|_r \leqslant |w_1^R|_r + |w_2^R|_r \leqslant 2^a C_1\,|\nabla w|_p^a\,|w|_q^{1-a} + 2^b C|\nabla w|_p^{b(1-\beta)}\,|w|_q^{1-b(1-\beta)},$$

$$\text{with } \beta = \frac{qn(p-\overline{p})}{\overline{p}(qp-n(q-p))}.$$

Taking into account the values of b and a in (4.2), a simple computation gives $a = b(1-\beta)$. The proof is then complete in the case $q \geqslant p$ and $p \in [n, +\infty]$.

Case 2 - Let us consider the case $q < p$. First of all we prove that $w(x)$ is in $L^p(\Omega)$. Of course, Poincaré inequality implies $w(x) \in L^p(B)$ for all bounded domain B contained in Ω. Moreover, considering $w_1^{\widetilde{R}}$, $\widetilde{R} > d$, from Theorem 3.1 we have

$$|w_1^{\tilde{R}}|_p \leqslant C|\nabla w_1^{\tilde{R}}|_p^{a_1}\,|w_1^{\tilde{R}}|_q^{1-a_1},$$

which implies $w(x) \in L^p(\Omega - S_{2\tilde{R}})$. So, we have proved that $w(x) \in \in L^p(\Omega)$. Since for $w(x) \in W^{1,p}(\Omega)$ we have already obtained estimate (4.1), then for any $r \geqslant p$

$$|w|_r \leqslant C|\nabla w|_p^{a_2}\,|w|_p^{1-a_2}. \tag{4.5}$$

By applying the convexity theorem for L^p spaces to inequality (4.5), we have

$$|w|_r \leqslant C|\nabla w|_p^{a_2}\,|w|_r^{b(1-a_2)}\,|w|_q^{(1-b)(1-a_2)}, \quad \text{with } b = \frac{r(p-q)}{p(r-q)},$$

and it is easy to deduce

$$|w|_r \leqslant C|\nabla w|_p^{a}\,|w|_q^{1-a}, \tag{4.6}$$

with a given in (4.2). In order to obtain the inequality (4.1) in the case $r \in \in (q, p)$ it is sufficient to apply again the convexity theorem for L^p spaces:

$$|w|_r \leqslant |w|_{r_1}^{b}\,|w|_q^{1-b}, \quad \frac{1}{r} = \frac{b}{r_1} + \frac{1-b}{q}, \quad \forall r_1 \geqslant p\,. \tag{4.7}$$

Since $r_1 \geqslant p$, we majorize the right-hand side of inequality (4.7) by (4.6). Therefore,

$$|w|_r \leqslant C(|\nabla w|_p^{a_3}\,|w|_q^{1-a_3})^b\,|w|_q^{1-b}$$

and a simple computation gives $a_3 b = a$, with a satisfying (4.2).

Now, let us prove the inequality for $r = +\infty$. For this value of r, inequality (4.1) takes the form

$$|w|_\infty \leqslant C|\nabla w|_p^{a}\,|w|_q^{1-a}, \tag{4.8}$$

where

$$a\left(\frac{1}{p} - \frac{1}{n}\right) + (1-a)\frac{1}{q} = 0\,. \tag{4.9}$$

The proof can be obtained following some ideas given in [1, 13]. Assume

$R_1 > d$. Let x be in $S^c_{R_1} = S^c(0, R_1)$, $\mathcal{C}_x$ a finite cone ([3]) of height ℓ contained in $S^c_{R_1}$ $\forall \ell > 0$. Introducing polar coordinates (r, ϑ) in $\mathbb{R}^n$ with origin at x such that $0 < r < \ell$ and $\vartheta \in \mathcal{A}$, we have

$$w(x) = w(0, \vartheta) = w(r, \vartheta) - \int_0^r \frac{\partial}{\partial t} w(t, \vartheta)\, dt\,, \qquad \forall r \in (0, \ell), \quad \forall \vartheta \in \mathcal{A}\,.$$

If $r^{n-1}\omega(\vartheta)\,dr d\vartheta$ denotes the volume element, multiplying by $r^{n-1}\omega(\vartheta)$, integrating r from 0 to ℓ and ϑ over $\mathcal{A}$ and using Hölder's inequality, we obtain

$$|w(x)|\, vol(\mathcal{C}_x) \leq$$

$$\leq \int_{\mathcal{C}_x} |w(y)|\, dy + \frac{\ell^n}{n} \int_{\mathcal{C}_x} \frac{|\nabla w(y)|}{|x-y|^{n-1}}\, dy \leq |w|_{L_q(\mathcal{C}_x)} (vol(\mathcal{C}_x))^{(q-1)/q} +$$

$$+ \frac{\ell^n}{n} |\nabla w|_{L^p(\mathcal{C}_x)} \left| \int_{\mathcal{C}_x} |x-y|^{(1-n)p/(p-1)}\, dy \right|^{(p-1)/p} \leq$$

$$\leq |w|_{L_q(\mathcal{C}_x)} \ell^{n(q-1)/q} + C \frac{\ell^{n+1-n/p}}{n} |\nabla w|_{L^p(\mathcal{C}_x)}.$$

Hence,

$$|w(x)| \leq C |w|_{L_q(\mathcal{C}_x)} \ell^{-n/q} + \frac{\ell^{1-n/p}}{n} |\nabla w|_{L^p(\mathcal{C}_x)}. \tag{4.10}$$

Actually, we can choose ℓ such that

$$\ell^{1-n(q-p)/qp} = \frac{|w|_q}{|\nabla w|_p}\,.$$

Thus, inequality (4.10) becomes

$$|w(x)| \leq C |w|_q^{1-np/(qp-nq+np)} |\nabla w|_p^{np/(qp-nq+np)} +$$

$$+ C |w|_q^{q(p-n)/(qp-nq+np)} |\nabla w|_p^{1+q(n-p)/(qp-nq+np)} = C |\nabla w|_p^a |w|_q^{1-a},$$

([3]) As in [1], given a point $x \in \mathbb{R}^n$, an open ball B_1 with center x, an open ball B_2 not containing x, we call a finite cone in $\mathbb{R}^n$ with vertex at x the set $\mathcal{C}_x = B_1 \cap \cap \{x + \lambda(y - x) : y \in B_2, \lambda > 0\}$.

with $a = np/(qp + np - nq)$, which implies

$$|w|_{L^\infty(S^c_{R_1})} \leqslant C|\nabla w|^a_p |w|^{1-a}_q. \tag{4.11}$$

Let us prove inequality (4.11) in $\Omega \cap S_{R_1}$. Define the function $w_3(x) = h^{R_1}(x)w(x)$. Since $w_3(x)$ is defined on the bounded domain $\Omega_3 = \Omega \cap S_{2R_1}$, we can use Theorem 3.1 and obtain

$$|w_3|_{L^\infty(\Omega_3)} \leqslant C|\nabla w_3|^a_{L^p(\Omega_3)} |w_3|^{1-a}_{L^q(\Omega_3)} + C|w_3|_{L^q(\Omega_3)}. \tag{4.12}$$

First, let us estimate the last term on the right-hand side of relation (4.13). If $q \leqslant p$, applying Poincarè inequality and, subsequently, Hölder's inequality, we have

$$|w_3|_{L^q(\Omega_3)} = |w_3|^a_{L^q(\Omega_3)} |w_3|^{1-a}_{L^q(\Omega_3)} \leqslant$$

$$\leqslant C|\nabla w_3|^a_{L^q(\Omega_3)} |w_3|^{1-a}_{L^q(\Omega_3)} \leqslant C|\nabla w_3|^a_{L^p(\Omega_3)} |w_3|^{1-a}_{L^q(\Omega_3)}.$$

If $n < p < q$, using Poincarè inequality, we majorize $|w_3|_{L^q(\Omega_3)}$ as follows

$$|w_3|_{L^q(\Omega_3)} =$$

$$= |w_3|^t_{L^q(\Omega_3)} |w_3|^{1-t}_{L^q(\Omega_3)} \leqslant |w_3|^t_{L^q(\Omega_3)} |w_3|^{p(1-t)/q}_{L^p(\Omega_3)} \sup_{\Omega_3} |w_3|^{(q-p)(1-t)/q} \leqslant$$

$$\leqslant |w_3|^t_{L^q(\Omega_3)} |\nabla w_3|^{p(1-t)/q}_{L^p(\Omega_3)} |w_3|^{(q-p)(1-t)/q}_{L^\infty(\Omega_3)}, \quad \forall t \in (0, 1).$$

By Cauchy inequality and for $t = p(1-a)/(p - a(p-q))$, we have

$$|w_3|_{L^q(\Omega_3)} \leqslant$$

$$\leqslant C|\nabla w_3|^a_{L^p(\Omega_3)} |w_3|^{1-a}_{L^q(\Omega_3)} + \eta |w_3|_{L^\infty(\Omega_3)}, \quad n < p < q, \ \forall \eta \in (0, 1).$$

So, in both the cases, inequality (4.12) becomes

$$|w_3|_{L^\infty(\Omega_3)} \leqslant C|\nabla w_3|^a_{L^p(\Omega_3)} |w_3|^{1-a}_{L^q(\Omega_3)}.$$

From this inequality, using estimate (4.11) for $R_1 \leqslant |x| \leqslant 2R_1$, we obtain:

$$|w_3|_{L^\infty(\Omega_3)} \leqslant C(|\nabla w|_p + |w|_{L^\infty(K_{R_1})})^a |w_3|^{1-a}_{L^q(\Omega_3)} \leqslant$$

$$\leqslant C|\nabla w|^a_p |w|^{1-a}_q + C(|\nabla w|^a_p |w|^{1-a}_q)^a |w_3|^{1-a}_{L^q(\Omega_3)},$$

hence

$$|w_3|_{L^\infty(\Omega_3)} \leqslant C|\nabla w|_p^a |w|_q^{1-a} + C|\nabla w|_p^{a^2} |w|_q^{a-a^2} mis(\Omega_3)^{(1-a)/q} |w_3|^{1-a}_{L^\infty(\Omega_3)}.$$

Using Cauchy inequality we get

$$|w_3|_{L^\infty(\Omega_3)} \leqslant C|\nabla w|_p^a |w|_q^{1-a}. \tag{4.13}$$

Inequality (4.11) and (4.13) imply inequality (4.1) for $r = +\infty$.

Finally, let us prove the inequality for $p > n$, $r < 0$. By Lemma 3.5, function w belongs to $C^{[0,\alpha]}(\Omega)$, $\alpha = (p-n)/p$. Moreover, we have just proved that w belongs to $L^\infty(\Omega)$. Hence, we can apply the interpolation lemma 3.6 and we get

$$|w|_r \leqslant |w|^e_{\frac{np}{n-p}} |w|_\infty^{1-e}, \qquad \frac{1}{r} = e\left(\frac{1}{p} - \frac{1}{n}\right).$$

Then, majorizing the right-hand side of the previous inequality using Lemma 3.5 and inequality (4.8), we obtain our estimate for $r < 0$.

LEMMA 4.2. *Let $w(x)$ be in $\widehat{W}^{2,p}(\Omega) \cap L^q(\Omega)$, $p \in [1, n/2)$, $q > np/(n-2p)$. Then, for $k \in \{0, 1\}$, the following inequality holds*

$$|D^k w|_r \leqslant C|D^2 w|_p^a |w|_q^{1-a}, \tag{4.14}$$

where

$$\frac{1}{r} = \frac{k}{n} + a\left(\frac{1}{p} - \frac{2}{n}\right) + (1-a)\frac{1}{q}, \tag{4.15}$$

with $a \in \left[\frac{k}{2}, 1\right]$. The result also holds if $q = +\infty$; however in the case $k = 0$ the following additional condition is required: w tends to zero at infinity or $w \in L^{q'}(\Omega)$ for some finite $q' \geqslant 1$. The constant C in inequality (4.14) *is independent of $w(x)$.*

PROOF. First we show inequality (4.14) in the case $a = 1$. Since $D^2 w \in L^p(\Omega)$, $p < n/2$, by Lemma 3.1 there exist a constant vector a and a constant C such that $|\nabla w - a|_{\frac{np}{n-p}} \leqslant C|D^2 w|_p$. Setting $\widehat{w} = w - a \cdot x$, we have $\nabla \widehat{w} \in L^{\frac{np}{n-p}}(\Omega)$ and then we can apply Lemma 3.1 to the function $\widehat{w}$. Therefore, there exist two constants w_0 and C, C independent of $\widehat{w}$, such that $|\widehat{w} - w_0|_{\frac{np}{n-2p}} \leqslant C|\nabla \widehat{w}|_{\frac{np}{n-p}}$, so that the function $w - a\cdot x - w_0$ belongs to $L^{\frac{np}{n-2p}}(\Omega)$. However, in some sense $w - a\cdot x - w_0 \to 0$ for $|x| \to \infty$, hence $|a\cdot x + w_0| \to 0$ for $|x| \to \infty$, which implies $a = 0$, $w_0 =$

$= 0$. Therefore, the previous inequalities become

(4.16) $$|w|_{\frac{np}{n-2p}} \leqslant C|\nabla w|_{\frac{np}{n-p}} \leqslant C|D^2 w|_p.$$

Now, let us consider the case $k = 1$, $a \in \left[\frac{1}{2}, 1\right]$. Fix $R_0 > d$. Since $w_1^{R_0}$ is defined on $\mathbb{R}^n$, we apply Theorem 3.1 and obtain

$$|\nabla w_1^{R_0}|_r \leqslant C|D^2 w_1^{R_0}|_p^a |w_1^{R_0}|_q^{1-a} \leqslant$$
$$\leqslant C(|D^2 w|_p + 2|\nabla h^{R_0} \nabla w|_p + |wD^2 h^{R_0}|_p)^a |w|_q^{1-a} \leqslant$$
$$\leqslant C(|D^2 w|_p + |\nabla w|_{L^p(K_{R_0})} + |w|_{L^p(K_{R_0})})^a |w|_q^{1-a}.$$

In virtue of the Ehrling-Gagliardo-Nirenberg theorem [1], it follows that

$$|\nabla w_1^{R_0}|_r \leqslant C(|D^2 w|_p + |w|_{L^p(K_{R_0})})^a |w|_q^{1-a}.$$

Applying Hölder's inequality we get

$$|\nabla w_1^{R_0}|_r \leqslant C(|D^2 w|_p + |w|_{\frac{np}{n-2p}})^a |w|_q^{1-a},$$

and then, using (4.16), inequality (4.17) becomes

$$|\nabla w_1^{R_0}|_r \leqslant C|D^2 w|_p^a |w|_q^{1-a}.$$

Now, let us estimate $w_2^{R_0}$. Since this function is defined on the bounded domain $\Omega_0 = \Omega \cap S_{2R_0}$, we apply Lemma 3.4 and consider $a = \frac{1}{2}$. Thus, for some $\overline{p} \in [1, q]$ and $\overline{r} = 2pq/(p+q)$ we get

$$|\nabla w_2^{R_0}|_{L^{\overline{r}}(\Omega_0)} \leqslant C|D^2 w_2^{R_0}|_{L^p(\Omega_0)}^{1/2} |w_2^{R_0}|_{L^q(\Omega_0)}^{1/2} + C_1 |w_2^{R_0}|_{L^{\overline{p}}(\Omega_0)} \leqslant$$
$$\leqslant C(|D^2 w|_p + |\nabla w|_{L^p(K_{R_0})} + |w|_{L^p(K_{R_0})})^{1/2} |w|_q^{1/2} + C_1 |w|_{\overline{p}}.$$

Then, using the Ehrling-Gagliardo-Nirenberg theorem and, subsequently, Hölder's inequality we obtain

$$|\nabla w_2^{R_0}|_{L^{\overline{r}}(\Omega_0)} \leqslant C(|D^2 w|_p + |w|_{L^p(K_{R_0})})^{1/2} |w|_q^{1/2} + C_1 |w|_{\overline{p}} \leqslant$$
$$\leqslant C(|D^2 w|_p + |w|_{\frac{np}{n-2p}})^{1/2} |w|_q^{1/2} + C_1 |w|_{\overline{p}}.$$

We apply the convexity theorem to the term $|w|_{\overline{p}}$ choosing $\overline{p} = n\overline{r}/(n-\overline{r})$

and we obtain

$$(4.18)\qquad |\nabla w_2^{R_0}|_{L^{\bar r}(\Omega_0)} \leqslant C(|D^2 w|_p + |w|_{\frac{np}{n-2p}})^{1/2}|w|_q^{1/2} + |w|^{1/2}_{\frac{np}{n-2p}}|w|_q^{1/2}.$$

Therefore, from (4.16) inequality (4.18) becomes

$$|\nabla w_2^{R_0}|_{L^{\bar r}(\Omega_0)} \leqslant C|D^2 w|_p^{1/2}|w|_q^{1/2}.$$

Moreover, by Lemma 3.4 with $a=1$ we deduce

$$|\nabla w_2^{R_0}|_{L^{\bar{\bar r}}(\Omega_0)} \leqslant C(|D^2 w_2^{R_0}|_p + |w_2^{R_0}|_{\bar p}) \leqslant$$

$$\leqslant C(|D^2 w|_p + |\nabla w|_{L^p(K_{R_0})} + |w|_{L^p(K_{R_0})} + |w_2^{R_0}|_{\bar p}), \qquad \bar{\bar r} = \frac{np}{n-p}.$$

Then, applying the Ehrling-Gagliardo-Nirenberg theorem and, subsequently, Hölder's inequality we obtain

$$|\nabla w_2^{R_0}|_{L^{\bar{\bar r}}(\Omega_0)} \leqslant$$

$$\leqslant C(|D^2 w|_p + |w|_{L^p(K_{R_0})}) + C|w|_{\bar p} \leqslant C(|D^2 w|_p + |w|_{\frac{np}{n-2p}}) + C|w|_{\bar p}.$$

Choosing $\bar p = np/(n-2p)$ and using (4.16) we obtain

$$|\nabla w_2^{R_0}|_{L^{\bar{\bar r}}(\Omega_0)} \leqslant C(|D^2 w|_p + |w|_{\frac{np}{n-2p}}) \leqslant C|D^2 w|_p.$$

Hence, using the convexity theorem for L^p spaces we finally deduce

$$|\nabla w_2^{R_0}|_{L^r(\Omega_0)} \leqslant C|D^2 w|_p^a |w|_q^{1-a},$$

for any r satisfying (4.15). Since $|\nabla w|_r \leqslant |\nabla w_1^{R_0}|_r + |\nabla w_2^{R_0}|_r$, summing $|\nabla w_1^{R_0}|_r$ and $|\nabla w_2^{R_0}|_r$ we obtain inequality (4.14) for $k=1$.

Inequality (4.14) with $k=0$ and $a \in [0, 1]$ can be obtained using Lemma 4.1 and the inequality just proved for $k=1$.

PROOF OF THEOREM 2.1. Lemma 4.1 and Lemma 4.2 are cases of the theorem, therefore the following proof, for $k=1$ and $m=2$, does not involve the exponents $p \in [1, n/2)$ and $q > np/(n-2p)$. First, we consider the case $k=1$, $m=2$, subsequently we shall consider the general case. For $k=1$, $m=2$, inequality (2.2) becomes

$$(4.19)\qquad |\nabla w|_r \leqslant C|D^2 w|_p^a |w|_q^{1-a}.$$

First, we prove the case $r \in [1, +\infty)$. Since w_1^R is defined on the whole

$\mathbb{R}^n$, we apply Theorem 3.1, thus

$$|\nabla w_1^R|_r \leqslant C|D^2 w_1^R|_p^a \, |w_1^R|_q^{1-a} \leqslant$$
$$\leqslant C(|D^2 w|_p + 2|\nabla h^R \nabla w|_p + |w D^2 h^R|_p)^a \, |w|_q^{1-a} \leqslant$$
$$\leqslant C(|D^2 w|_p + \frac{1}{R}|\nabla w|_{L^p(K_R)} + \frac{1}{R^2}|w|_{L^p(K_R)})^a \, |w|_q^{1-a}.$$

In virtue of the Ehrling-Gagliardo-Nirenberg theorem and taking into account the geometry of the set K_R, it follows that

$$R^{-1}|\nabla w|_{L^p(K_R)} \leqslant C\left(|D^2 w|_{L^p(K_R)} + \frac{1}{R^2}|w|_{L^p(K_R)}\right),$$

hence we get

$$\text{(4.20)} \qquad |\nabla w_1^R|_r \leqslant C(|D^2 w|_p^a \, |w|_q^{1-a} + \frac{1}{R^{2a}}|w|_{L^p(K_R)}^a \, |w|_q^{1-a}) =$$
$$= I_1(p, q) + I_2(p, q, R).$$

We notice that from Lemma 3.7 it follows $\nabla w \in L^r(\Omega)$. We can assume $|D^2 w|_p \neq 0$, otherwise ∇w is a constant c. If $c = 0$ the inequality is trivial; if $c \neq 0$, then ∇w is in $L^\infty(\Omega)$ and we get a contradiction, because $w \notin L^q(\Omega)$, $q \in [1, +\infty]$.

There are two possible cases:

$$\text{(4.21)} \qquad \frac{|w|_q}{|D^2 w|_p} \leqslant d^{2-n(q-p)/qp} \quad \text{or} \quad \frac{|w|_q}{|D^2 w|_p} > d^{2-n(q-p)/qp}.$$

In the first case we fix $R = R_2 > d$ and majorize $|w|_q$ by $R_2^{2-n(q-p)/pq}|D^2 w|_p$. Let us estimate $I_2(p, q, R_2)$ in inequality (4.20). If $p \leqslant q$ we apply Hölder's inequality; otherwise, i.e. if $p > q$, we apply Theorem 3.1. Hence, we get:

$$\text{(4.22)} \quad p \leqslant q, \quad I_2(p, q, R_2) \leqslant C R_2^{-(2-n(q-p)/qp)a} \, |w|_q^a \, |w|_q^{1-a},$$

$$\text{(4.23)} \quad p > q, \quad I_2(p, q, R_2) \leqslant \frac{C}{R_2^{2a}}(|D^2 w|_p^c \, |w|_q^{1-c} + |w|_q)^a \, |w|_q^{1-a}.$$

From $(4.21)_1$ inequalities (4.22) and (4.23) imply

$$I_2(p, q, R_2) \leqslant C(R_2)\,|D^2 w|_p^a \, |w|_q^{1-a}.$$

For $w_2^{R_2}$ we employ Gagliardo-Nirenberg's theorem. Therefore, from (3.1) and using the Ehrling-Gagliardo-Nirenberg theorem, we get

$$|\nabla w_2^{R_2}|_r \leqslant C|D^2 w_2^{R_2}|_p^a |w_2^{R_2}|_q^{1-a} + C_1 |w_2^{R_2}|_q \leqslant$$

$$\leqslant C(|D^2 w|_p + |\nabla w|_{L^p(K_{R_2})} + |w|_{L^p(K_{R_2})})^a |w|_q^{1-a} +$$

$$+ C_1 |w|_q \leqslant C(|D^2 w|_p + |w|_{L^p(K_{R_2})})^a |w|_q^{1-a} + C_1 |w|_q .$$

Since $(4.21)_1$ ensures that $|w|_q^a \leqslant C(R_2)|D^2 w|_p^a$, we have

$$|\nabla w_2^{R_2}|_r \leqslant C(R_2)(|D^2 w|_p^a |w|_q^{1-a} + |w|_{L^p(K_{R_2})}^a |w|_q^{1-a}).$$

Taking into account the arguments already used for $|\nabla w_1^R|_r$, we get

$$|\nabla w_2^{R_2}|_r \leqslant C|D^2 w|_p^a |w|_q^{1-a}.$$

Since $|\nabla w|_r \leqslant |\nabla w_1^{R_2}|_r + |\nabla w_2^{R_2}|_r$, summing $|\nabla w_1^{R_2}|_r$ and $|\nabla w_2^{R_2}|_r$ we obtain inequality (4.19) under condition $(4.21)_1$.

To complete the proof of (4.19), we only need to prove it under condition $(4.21)_2$. So, assume now that $(4.21)_2$ holds. In this case we can choose R such that

$$R^{2-n(q-p)/qp} = \frac{|w|_q}{|D^2 w|_p}. \tag{4.24}$$

We now majorize $I_2(p, q, R)$ in (4.20). When $p \leqslant q$, we obtain for $I_2(p, q, R)$ the same estimate as in (4.22):

$$p \leqslant q, \quad I_2(p, q, R) \leqslant CR^{-(2-n(q-p)/qp)a} |w|_q^a |w|_q^{1-a},$$

that, via (4.24), leads to

$$I_2(p, q, R) \leqslant C|D^2 w|_p^a |w|_q^{1-a}.$$

If $q < p < n$, applying Lemma 4.1 to $|w|_{L^p(|x| \geqslant R)} \geqslant |w|_{L^p(K_R)}$, we have

$$I_2(p, q, R) \leqslant \frac{C}{R^{2a}}(|\nabla w|^d_{L^{\frac{np}{n-p}}(|x| \geqslant R)} |w|_q^{1-d})^a |w|_q^{1-a}. \tag{4.25}$$

Applying Sobolev's inequality of Lemma 3.1 to (4.25) and using (4.24), we have

$$I_2(p,q,R) \leqslant \frac{C}{R^{2a}} |D^2 w|_p^{da} |w|_q^{(1-d)a} |w|_q^{1-a} \leqslant$$

$$\leqslant \frac{C}{R^{2a}} |D^2 w|_p^{da} |D^2 w|_p^{(1-d)a} R^{(2-n(q-p)/qp)(1-d)a} |w|_q^{1-a} = C|D^2 w|_p^a |w|_q^{1-a},$$

where we have taken into account that $d = n(p-q)/(2pq + np - nq)$ (because of (4.2)), hence $(2 - n(q-p)/qp)(1-d) = 2$. If $p \geqslant n$ and $p > q$, applying Lemma 4.1 with $s > r$ and $s > \frac{np}{n+p}$, the estimate for $I_2(p,q,R)$ becomes

$$I_2(p,q,R) \leqslant \frac{C}{R^{2a}} (|\nabla w|_s^h |w|_q^{1-h})^a |w|_q^{1-a}.$$

Subsequently, applying Lemma 4.1 to the gradient on the right-hand side of this last inequality and then Cauchy inequality (we recall that $|\nabla w|_r$ is finite), we deduce

$$I_2(p,q,R) \leqslant \frac{C}{R^{2a}} (|D^2 w|_p^{eh} |\nabla w|_r^{(1-e)h} |w|_q^{1-h})^a |w|_q^{1-a} \leqslant$$

$$\leqslant CR^{-\alpha} |D^2 w|_p^{\beta} |w|_q^{\gamma} + \eta |\nabla w|_r, \quad \forall \eta > 0,$$

with

$$\alpha = \frac{2a}{1 + ha(e-1)}, \quad \beta = \frac{eha}{1 + ha(e-1)}, \quad \gamma = \frac{(1-h)a}{1 + ha(e-1)},$$

with $e = np(s-r)/s(rp + np - nr)$ and $h = sn(p-q)/p(ns + sq - nq)$, both obtained from (4.2). Using (4.24) and setting

$$2 - \frac{n(q-p)}{qp} = z, \tag{4.26}$$

we have

$$I_2(p,q,R) \leqslant C|D^2 w|_p^{\beta + \alpha/z} |w|_q^{\gamma - \alpha/z} + \eta |\nabla w|_r,$$

where $\beta + \alpha/z = a$, $\gamma - \alpha/z = 1 - a$. Therefore, the estimates for

$I_2(p, q, R)$ are the following ones:

$$(4.27)\quad \begin{cases} p \leqslant q, & I_2(p, q, R) \leqslant C|D^2 w|_p^a |w|_q^{1-a}, \\ p > q, & I_2(p, q, R) \leqslant \begin{cases} C|D^2 w|_p^a |w|_q^{1-a} & \text{if } p < n, \\ C|D^2 w|_p^a |w|_q^{1-a} + \eta |\nabla w|_r & \text{if } p \geqslant n. \end{cases} \end{cases}$$

We now estimate $|\nabla w_2^R|_r$. In virtue of the remark made at the beginning of the proof, we have only to consider the cases $p \in \left[1, \frac{n}{2}\right)$ and $q \leqslant \frac{np}{n-2p}$ or $p \geqslant \frac{n}{2}$. Since $p \in \left[1, \frac{n}{2}\right)$ and $q \leqslant \frac{np}{n-2p}$ or $p \in \left[\frac{n}{2}, n\right)$ imply $r \leqslant np/(n-p)$, we can apply Hölder's inequality with exponents $\frac{(n-p)r}{np}$, $\frac{np+rp-nr}{np}$, after which we can use Sobolev's inequality of Lemma 3.1 and we get

$$|\nabla w_2^R|_r \leqslant CR^{(np+rp-nr)/rp} |\nabla w_2^R|_{\frac{np}{n-p}} \leqslant CR^{(np+rp-nr)/rp} |D^2 w_2^R|_p.$$

Otherwise, namely if $p \geqslant n$, we apply Hölder's inequality with exponents p/s, $(s-p)/s$, $s > r$ and then we use Lemma 4.1. Hence, the following estimate holds:

$$(4.28)\quad |\nabla w_2^R|_r \leqslant CR^{n(s-r)/rs} |\nabla w_2^R|_s \leqslant CR^{n(s-r)/rs} |D^2 w_2^R|_p^e |\nabla w_2^R|_r^{1-e}.$$

Taking into account that (4.2) gives $e = np(s-r)/s(rp+np-nr)$, from (4.28) we have

$$|\nabla w_2^R|_r \leqslant CR^{n(s-r)/rse} |D^2 w_2^R|_p =$$

$$= CR^{(np+rp-nr)/rp} |D^2 w_2^R|_p, \quad p \geqslant n,\ s > r.$$

So, in both cases, using the Ehrling-Gagliardo-Nirenberg theorem and taking into account the geometry of the set K_R, we get

$$(4.29)\quad |\nabla w_2^R|_r \leqslant CR^{(np+rp-nr)/rp} |D^2 w_2^R|_p \leqslant$$

$$\leqslant CR^{(np+rp-nr)/rp} \left(|D^2 w|_p + \frac{1}{R} |\nabla w|_{L^p(K_R)} + \frac{1}{R^2} |w|_{L^p(K_R)} \right) \leqslant$$

$$\leqslant CR^{(np+rp-nr)/rp} \left(|D^2 w|_p + \frac{1}{R^2} |w|_{L^p(K_R)} \right) = F_1(p, q) + F_2(p, q).$$

From (4.24) and (4.26), we obtain

$$F_1(p, q) = C|D^2 w|_p^{1-(np+rp-nr)/rpz} |w|_q^{(np+rp-nr)/rpz} = C|D^2 w|_p^a |w|_q^{1-a}.$$

We now estimate $F_2(p, q)$ in (4.29). If $p \leqslant q$, using Hölder's inequality and (4.24), we have

$$F_2(p, q) \leqslant CR^{-z+(np+rp-nr)/rp} |w|_q \leqslant C|D^2 w|_p^a |w|_q^{1-a}.$$

If $p > q$ and $p < n$, we apply Lemma 4.1, with a suitable exponent of summability for ∇w, in order to apply Lemma 3.1. Hence, we get

$$F_2(p, q) \leqslant CR^{(np-rp-nr)/rp} |w|_p \leqslant CR^{(np-rp-nr)/rp} |\nabla w|^d_{\frac{np}{n-p}} |w|_q^{1-d} \leqslant$$

$$\leqslant CR^{(np-rp-nr)/rp} |D^2 w|_p^d |w|_q^{1-d},$$

where $d = n(p-q)/(2pq + np - nq)$ is given from (4.2). Then, using position (4.24) we obtain

$$F_2(p, q) \leqslant C|D^2 w|_p^a |w|_q^{1-a}.$$

If $p > q$ and $p \geqslant n$, we use Lemma 4.1, obtaining for $s > n$

$$F_2(p, q) \leqslant CR^{(n-r)/r} |w|_\infty \leqslant \tag{4.30}$$

$$\leqslant CR^{(n-r)/r} |\nabla w|_s^t |w|_q^{1-t} \leqslant CR^{(n-r)/r} |D^2 w|_p^{vt} |\nabla w|_r^{(1-v)t} |w|_q^{(1-t)}.$$

Let

$$\alpha_1 = \frac{(n-r)/r}{1-(1-v)t}, \quad \beta_1 = \frac{vt}{1-(1-v)t}, \quad \gamma_1 = \frac{1-t}{1-(1-v)t},$$

where $t = sn/(ns + sq - nq)$ because of (4.9) and $v = np(s-r)/s(np + rp - nr)$ because of (4.2). Applying Cauchy inequality and using (4.24), inequality (4.30) becomes

$$F_2(p, q) \leqslant CR^{\alpha_1} |D^2 w|_p^{\beta_1} |w|_q^{\gamma_1} + \eta |\nabla w|_r =$$

$$= C|D^2 w|_p^{\beta_1 - \alpha_1/z} |w|_q^{\gamma_1 + \alpha_1/z} + \eta |\nabla w|_r, \quad \forall \eta > 0,$$

where $\beta_1 - \alpha_1/z = a$, $\gamma_1 + \alpha_1/z = 1 - a$. Therefore, the estimates for

$F_2(p, q)$ are the following

$$(4.31)\qquad \begin{cases} p \leqslant q, & F_2(p,q) \leqslant C|D^2 w|_p^a |w|_q^{1-a}, \\ p > q, & F_2(p,q) \leqslant \begin{cases} C|D^2 w|_p^a |w|_q^{1-a} & \text{if } p < n, \\ C|D^2 w|_p^a |w|_q^{1-a} + \eta |\nabla w|_r & \text{if } p \geqslant n. \end{cases} \end{cases}$$

Since $|\nabla w|_r \leqslant |\nabla w_1^R|_r + |\nabla w_2^R|_r$, choosing $0 < \eta \leqslant 1/3$ in $(4.27)_3$ and $(4.31)_3$, from (4.20) and (4.29) we deduce (4.19) under condition $(4.21)_2$.

We now consider the case $r = +\infty$. First, we apply Lemma 4.1 to ∇w, then we apply (2.2) to $|\nabla w|_s$ with $s < \infty$. So, we have

$$(4.32)\qquad |\nabla w|_\infty \leqslant C|D^2 w|_p^b |\nabla w|_s^{1-b} \leqslant C|D^2 w|_p^b (|D^2 w|_p^c |w|_q^{1-c})^{1-b} = \\ = C|D^2 w|_p^{b+c(1-b)} |w|_q^{(1-c)(1-b)},$$

with $b = np/(sp - ns + np)$, because of (4.9), and $c = p(sq - nq + + ns)/s(2pq - nq + np)$, because of (2.3). Now, taking into account the value of a in (2.3), a simple computation gives $a = b + c(1-b)$.

Finally, let $r < 0$. Since $p > n$, by Lemma 3.5 we get

$$(4.33)\qquad |\nabla w|_{np/(n-p)} \leqslant C|D^2 w|_p .$$

Moreover, inequality (4.32) implies that ∇w belongs to $L^\infty(\Omega)$. Hence, using the interpolation lemma 3.6 we have

$$|\nabla w|_r \leqslant |\nabla w|_{np/(n-p)}^e |\nabla w|_\infty^{1-e}, \qquad \frac{1}{r} = e\left(\frac{1}{p} - \frac{1}{n}\right).$$

Then majorizing the right-hand side of the previous inequality by (4.32) and (4.33) we obtain

$$|\nabla w|_r \leqslant C|D^2 w|_p^e (|D^2 w|_p^d |w|_q^{1-d})^{1-e} = C|D^2 w|_p^{e+d(1-e)} |w|_q^{(1-e)(1-d)},$$

where $d(1-e) + e = a$, with a given in (2.3). Thus, we have completed the proof of the case $k = 1$, $m = 2$.

In order to establish inequality (2.2) in the general case, we proceed by induction on k, showing that if the result holds for $m - 1$ and for any $0 \leqslant k < m - 1$, it also holds for m and for $k = m - 1$. Subsequently, we will easily show that the inequality holds for m and for any k with $0 \leqslant \leqslant k \leqslant m - 2$. Hence, we assume the validity of the following inequality for all $0 \leqslant k < m-1$, $m \geqslant 3$ and for a suitable $\overline{p} \geqslant 1$

$$|D^k w|_{r_k} \leqslant C|D^{m-1} w|_{\overline{p}}^{b} |w|_q^{1-b}, \tag{4.34}$$

and show that

$$|D^{m-1} w|_{r_{m-1}} \leqslant C|D^m w|_p^a |w|_q^{1-a}, \tag{4.35}$$

where

$$\frac{1}{r_{m-1}} = \frac{m-1}{n} + a\left(\frac{1}{p} - \frac{m}{n}\right) + (1-a)\frac{1}{q}, \qquad a \in \left[\frac{m-1}{m}, 1\right]. \tag{4.36}$$

For the sake of simplicity, we replace r_{m-1} by r. Since w_1^R is defined on $\mathbb{R}^n$, by Theorem 3.1 it follows that

$$|D^{m-1} w_1^R|_r \leqslant C|D^m w_1^R|_p^a |w_1^R|_q^{1-a} \leqslant$$

$$\leqslant C\left(|D^m w|_p + \sum_{j=0}^{m-1} |D^{m-j} h^R|_p |D^j w|_p\right)^a |w|_q^{1-a} \leqslant$$

$$\leqslant C\left(|D^m w|_p^a + \sum_{j=0}^{m-1} \frac{1}{R^{(m-j)a}} |D^j w|_{L^p(K_R)}^a\right) |w|_q^{1-a}.$$

Now, applying the Ehrling-Gagliardo-Nirenberg theorem and taking into account the geometry of the set K_R, we get

$$|D^{m-1} w_1^R|_r \leqslant C(|D^m w|_p^a |w|_q^{1-a} + \frac{1}{R^{ma}} |w|_{L^p(K_R)}^a |w|_q^{1-a}) = \tag{4.37}$$

$$= \tilde{I}_1(p, q) + \tilde{I}_2(p, q, R).$$

Again we select two cases:

$$\frac{|w|_q}{|D^m w|_p} \leqslant d^{m-n(q-p)/qp} \quad \text{or} \quad \frac{|w|_q}{|D^m w|_p} > d^{m-n(q-p)/qp}. \tag{4.38}$$

In the former case we adopt the same arguments used for the second order derivatives case. Hence, we show inequality (4.35) only under condition $(4.38)_2$. In this case we choose R such that

$$R^{m-n(q-p)/qp} = \frac{|w|_q}{|D^m w|_p}. \tag{4.39}$$

Then, we estimate the term $|D^{m-1} w_1^R|_r$ under condition $(4.38)_2$. We start by considering the cases $p \leqslant q$, $q < p < n$. We shall discuss the other case $p \geqslant n$ and $p > q$, in the sequel in a slightly different way. If $p \leqslant q$, using Hölder's inequality and (4.39), we obtain for $\tilde{I}_2(p, q, R)$ in (4.37) the following estimate

$$\tilde{I}_2(p, q, R) \leqslant CR^{-(m - n(q-p)/qp)a} |w|_q^a |w|_q^{1-a} \leqslant C|D^m w|_p^a |w|_q^{1-a}.$$

If $p > q$ and $p < n$ we first use the induction hypothesis, subsequently Sobolev's inequality of Lemma 3.1, and then we have

$$\tilde{I}_2(p, q, R) \leqslant \frac{C}{R^{ma}} |w|_p^a |w|_q^{1-a} \leqslant \frac{C}{R^{ma}} (|D^{m-1} w|^d_{\frac{np}{n-p}} |w|_q^{1-d})^a |w|_q^{1-a} \leqslant$$

$$\leqslant \frac{C}{R^{ma}} |D^m w|_p^{da} |w|_q^{(1-d)a} |w|_q^{1-a}.$$

Now, using (4.39) we obtain

$$\tilde{I}_2(p, q, R) \leqslant \frac{C}{R^{ma}} |D^m w|_p^{da} |D^m w|_p^{(1-d)a} R^{(m - n(q-p)/qp)(1-d)a} |w|_q^{1-a} =$$

$$= C|D^m w|_p^a |w|_q^{1-a},$$

where $d = n(p-q)/(np + mpq - nq)$ (because of (2.3)), and so $(m - - n(q-p)/qp)(1-d) = m$.

Now, we estimate $|D^{m-1} w_2^R|_r$. First, we consider the case $p \in \left[1, \dfrac{n}{m}\right)$ and $q > \dfrac{np}{n - mp}$, which implies $q > p$. If we consider r as in (2.3) as a function of a, it takes its minimum value for $a = 1$, its maximum value for $a = k/m$. We denote the maximum value of r, corresponding to the kth order derivative, by r_k. We observe that $r_k > r_{k+1}$. From the study of the interpolation inequality for second order derivatives, we deduce

$$|D^{m-1} w_2^R|_{r_{m-1}} \leqslant C|D^m w_2^R|_p^{1/2} |D^{m-2} w_2^R|_{r_{m-2}}^{1/2} \tag{4.40}$$

where $r_{m-1} = \dfrac{mpq}{(m-1)q + p}$ and $r_{m-2} = \dfrac{mpq}{(m-2)q + 2p}$. From the induction hypothesis we also have

$$|D^{m-2} w_2^R|_{r_{m-2}} \leqslant C|D^{m-1} w_2^R|_{r_{m-1}}^{(m-2)/(m-1)} |w_2^R|_q^{1/(m-1)}. \tag{4.41}$$

Then, substituting (4.41) in (4.40), after a simple computation we obtain

(4.42) $$|D^{m-1} w_2^R|_{r_{m-1}} \leqslant C|D^m w_2^R|_p^{(m-1)/m} |w_2^R|_q^{1/m}.$$

Consider the Sobolev's exponent $np/(n-p)$, corresponding to the value $a = 1$ in (4.36). Using Sobolev's inequality of Lemma 3.1 we have

(4.43) $$|D^{m-1} w_2^R|_{\frac{np}{n-p}} \leqslant C|D^m w_2^R|_p.$$

Hence, applying the convexity theorem and using (4.42) and (4.43) we get

$$|D^{m-1} w_2^R|_r \leqslant |D^{m-1} w_2^R|_{r_{m-1}}^b |D^{m-1} w_2^R|_{\frac{np}{n-p}}^{1-b} \leqslant$$

$$\leqslant C|D^m w_2^R|_p^{(m-b)/m} |w_2^R|_q^{b/m}, \qquad b = \frac{mq}{r} \frac{np + rp - nr}{np + mpq - nq},$$

where $(m-b)/m = a$ given in (4.36). Now, applying the Ehrling-Gagliardo-Nirenberg theorem and taking into account the geometry of the set K_R, we get

$$|D^{m-1} w_2^R|_r \leqslant C|D^m w_2^R|_p^a |w_2^R|_q^{1-a} \leqslant$$

$$\leqslant C(|D^m w|_p + \frac{1}{R^m} |w|_{L^p(K_R)})^a |w_2^R|_q^{1-a}.$$

Since $p < q$, we can apply Hölder's inequality and then use (4.39). Thus, we obtain

$$|D^{m-1} w_2^R|_r \leqslant C|D^m w|_p^a |w|_q^{1-a} + R^{-m(n-(q-p)/qp)a} |w|_q^a |w|_q^{1-a} \leqslant$$

$$\leqslant C|D^m w|_p^a |w|_q^{1-a}.$$

If $p \in \left[1, \frac{n}{m}\right)$ and $q \leqslant \frac{np}{n-mp}$ or $p \in \left[\frac{n}{m}, n\right)$ we have $r_{m-1} \leqslant np/(n-p)$, and we can apply Hölder's inequality and Sobolev's inequality of Lemma 3.1 and we get

$$|D^{m-1} w_2^R|_r \leqslant CR^{(np+rp-nr)/rp} |D^{m-1} w_2^R|_{\frac{np}{n-p}} \leqslant$$

$$\leqslant CR^{(np+rp-nr)/rp} |D^m w_2^R|_p.$$

In the remaining case, namely if $p \geqslant n$, we apply Hölder's inequality with exponents p/s, $(s-p)/s$, $s > r$ and then we use Lemma 4.1. Hence, the following estimate holds

$$(4.44)\qquad |D^{m-1}w_2^R|_r \leqslant CR^{n(s-r)/rs}|D^{m-1}w_2^R|_s \leqslant \leqslant CR^{n(s-r)/rs}|D^m w_2^R|_p^e\,|D^{m-1}w_2^R|_r^{1-e}.$$

Taking into account that (4.2) gives $e = np(s-r)/s(rp+np-nr)$, from (4.44) we have

$$|D^{m-1}w_2^R|_r \leqslant CR^{n(s-r)/rse}|D^m w_2^R|_p = = CR^{(np+rp-nr)/rp}|D^m w_2^R|_p, \quad p \geqslant n,\ s > r,$$

so, in both cases we get

$$|D^{m-1}w_2^R|_r \leqslant CR^{(np+rp-nr)/rp}|D^m w_2^R|_p.$$

Finally, using the Ehrling-Gagliardo-Nirenberg theorem and taking into account the geometry of the set K_R, we obtain

$$(4.45)\qquad |D^{m-1}w_2^R|_r \leqslant CR^{(np+rp-nr)/rp}\left(|D^m w|_p + \frac{1}{R^m}|w|_{L^p(K_R)}\right) = = \widetilde{F}_1(p,q) + \widetilde{F}_2(p,q).$$

Setting

$$(4.46)\qquad y = m - \frac{n(q-p)}{qp},$$

from (4.39) we obtain for $\widetilde{F}_1(p,q)$:

$$\widetilde{F}_1(p,q) = C|D^m w|_p^{1-(np+rp-nr)/rpy}|w|_q^{(np+rp-nr)/rpy} = C|D^m w|_p^a|w|_q^{1-a}.$$

We now estimate $\widetilde{F}_2(p,q)$. If $p \leqslant q$ we use Hölder's inequality and condition $(4.38)_2$, and we have

$$\widetilde{F}_2(p,q) \leqslant CR^{-y+(np+rp-nr)/rp}|w|_q \leqslant C|D^m w|_p^a|w|_q^{1-a}.$$

If $p > q$ and $p < n$, we apply the induction hypothesis, with a suitable exponent of summability for $D^{m-1}w$, in order to apply Lemma 3.1. Hence, we get

$$\widetilde{F}_2(p,q) \leqslant CR^{(np-rp-nr)/rp}|w|_p \leqslant CR^{(np-rp-nr)/rp}|D^{m-1}w|^d_{\frac{np}{n-p}}|w|_q^{1-d} \leqslant$$

$$\leqslant CR^{(np-rp-nr)/rp}|D^m w|_p^d |w|_q^{1-d},$$

where $d = n(p-q)/(mpq+np-nq)$ is given in (2.3). Now, using (4.39) we obtain

$$\widetilde{F}_2(p,q) \leqslant C|D^m w|_p^a |w|_q^{1-a}.$$

We now consider the last case when $p > q$ and $p \geqslant n$. We choose $r_{m-1} = r_{\min}$, where $r_{\min}$ is the minimum exponent of summability for $D^{m-1}w$. It corresponds to the value $a = (m-1)/m$. For simplicity, in the sequel we will denote r_{min} by $\tilde{r}$. Applying the induction hypothesis and then Lemma 4.1 to the term $\tilde{I}_2(p,q,R)$ in (4.37) with $r = \tilde{r}$, we obtain

$$\tilde{I}_2(p,q,R) \leqslant$$

$$\leqslant R^{(\frac{n}{p}-m)a}|w|^a_{L^\infty(K_R)}|w|_q^{1-a} \leqslant R^{(\frac{n}{p}-m)a}(|D^{m-1}w|_p^t|w|_q^{1-t})^a|w|_q^{1-a} \leqslant$$

$$\leqslant R^{(\frac{n}{p}-m)a}(|D^m w|_p^v|D^{m-1}w|_{\tilde{r}}^{1-v})^{ta}|w|_q^{(1-t)a}|w|_q^{1-a}.$$

Let

$$\alpha_2 = \frac{(m-n/q)a}{1-(1-v)ta}, \quad \beta_2 = \frac{vta}{1-(1-v)ta}, \quad \gamma_2 = \frac{1-ta}{1-(1-v)ta},$$

where $t = np/(np+(m-1)pq-nq)$ because of (2.3), $v = n(p-\tilde{r})/(np+p\tilde{r}-n\tilde{r})$ because of (4.2). Applying Cauchy inequality and using (4.46) we obtain

$$\tilde{I}_2(p,q,R) \leqslant R^{-\alpha_2}|D^m w|_p^{\beta_2}|w|_q^{\gamma_2} + \varepsilon|D^{m-1}w|_{\tilde{r}} =$$

$$= |D^m w|_p^{\beta_2+\alpha_2/y}|w|_q^{\gamma_2-\alpha_2/y} + \varepsilon|D^{m-1}w|_{\tilde{r}}, \quad \forall \varepsilon > 0,$$

where $\beta_2 + \alpha_2/y = a$, $\gamma_2 - \alpha_2/y = 1-a$. In a similar way, applying the induction hypothesis and then Lemma 4.1 to the term $\widetilde{F}_2(p,q)$ in (4.45), by simple computations we obtain

$$(4.47)\qquad \widetilde{F}_2(p,q)\leqslant$$

$$\leqslant CR^{(n+\tilde r(1-m))/\tilde r}\,|w|_\infty \leqslant CR^{(n+\tilde r(1-m))/\tilde r}\,|D^{m-1}w|_p^t\,|w|_q^{1-t}\leqslant$$

$$\leqslant CR^{(n+\tilde r(1-m))/\tilde r}(\,|D^m w|_p^v\,|D^{m-1}w|_{\tilde r}^{1-v})^t\,|w|_q^{1-t}.$$

Let

$$\alpha_3=\frac{(n+\tilde r(1-m))/\tilde r}{1-(1-v)t},\quad \beta_3=\frac{vt}{1-(1-v)t},\quad \gamma_3=\frac{1-t}{1-(1-v)t},$$

where $t=np/(np+(m-1)pq-nq)$ because of (2.3) and $v=n(p-\tilde r)/(np+p\tilde r-n\tilde r)$ because of (4.2). Applying Cauchy inequality and using (4.46), inequality (4.47) becomes

$$\widetilde{F}_2(p,q)\leqslant CR^{\alpha_3}\,|D^m w|_p^{\beta_3}\,|w|_q^{\gamma_3}+\varepsilon|D^{m-1}w|_{\tilde r}=$$

$$=C|D^m w|_p^{\beta_3-\alpha_3/y}\,|w|_q^{\gamma_3+\alpha_3/y}+\varepsilon|D^{m-1}w|_{\tilde r},\quad \forall\varepsilon>0,$$

where $\beta_3-\alpha_3/y=a$, $\gamma_3+\alpha_3/y=1-a$.

Since we have shown the inequality for $\tilde r=r_{\min}$, in the case $p\geqslant n$ and $p>q$, it is immediate to get the inequality for any $r=r_{m-1}$ satisfying the dimensional balance (4.36). Indeed, it suffices to use Lemma 4.1 and the inequality already obtained for $|D^{m-1}w|_{\tilde r}$:

$$|D^{m-1}w|_{\tilde r}\leqslant C|D^m w|_p^c\,|w|_q^{1-c},$$

in order to get

$$|D^{m-1}w|_r\leqslant C|D^m w|_p^d\,|D^{m-1}w|_{\tilde r}^{1-d}\leqslant C|D^m w|_p^{d+c(1-d)}\,|w|_q^{(1-c)(1-d)}.$$

Now, it is easy to check that $d+c(1-d)=a$, with a obtained from (4.36).

Finally, the general case follows from inequalities (4.34) and (4.35).

Acknowledgement. The paper was made under the auspices of GN-FM (INdAM) and on a MURST Contract. The work of the first author was supported by a grant at Università degli Studi di Napoli «Federico II».

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Manoscritto pervenuto in redazione il 15 aprile 2003.