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Fields of *CR* Meromorphic Functions.

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ABSTRACT - Let M be a smooth compact CR manifold of CR dimension n and CR codimension k, which has a certain local extension property E. In particular, if M is pseudoconcave, it has property E. Then the field $\mathcal{K}(M)$ of CR meromorphic functions on M has transcendence degree d, with $d \leqslant n+k$. If $f_1, f_2, \ldots, f_d$ is a maximal set of algebraically independent CR meromorphic functions on M, then $\mathcal{K}(M)$ is a simple finite algebraic extension of the field $\mathbb{C}(f_1, f_2, \ldots, f_d)$ of rational functions of the $f_1, f_2, \ldots, f_d$. When M has a projective embedding, there is an analogue of Chow's theorem, and $\mathcal{K}(M)$ is isomorphic to the field $\mathcal{R}(Y)$ of rational functions on an irreducible projective algebraic variety Y, and M has a CR embedding in $\operatorname{reg} Y$. The equivalence between algebraic dependence and analytic dependence fails when condition E is dropped.

Introduction.

In a beautiful paper Siegel [Si], improving upon an idea of Serre [Se], managed to give simple proofs of the basic theorems concerning algebraic dependence and transcendence degree for the field of meromorphic functions on an arbitrary compact complex manifold; thereby generalizing classical results about the field of Abelian functions on a complex n dimensional torus. For a detailed discussion of the now nearly 150 year history of these matters, see the paper of Siegel. His proofs were based on his extension to n dimensions of the classical Schwarz lemma. Later,

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following almost exactly Siegel's argument, Andreotti and Grauert [AG] were able to show that the Siegel modular group, which plays a pivotal role in the study of algebraic fields of automorphic functions, is pseudoconcave. Later Andreotti [A] generalized these kinds of results to general pseudoconcave complex manifolds and spaces; again following Siegel's method.

In the present work we replace the compact complex manifold of Siegel by a smooth compact pseudoconcave *CR* manifold M of general *CR* dimension n and *CR* codimension k, and study algebraic dependence, transcendence degree and related matters for the field $\mathcal{K}(M)$ of *CR* meromorphic functions on M. Again we follow the method of Siegel, based on the Schwarz lemma, and we incorporate some ideas used by Andreotti. Actually we are able to obtain results under a condition on the *CR* manifold that is weaker than pseudoconcavity, which we call condition E. In particular we obtain an analogue of Chow's theorem [C] for compact *CR* manifolds. In the situation where M has a projective embedding, we are able to identify $\mathcal{K}(M)$ with the field $\mathfrak{R}(Y)$ of rational functions on an irreducible algebraic variety Y, in which M has a generic *CR* embedding that avoids the singularities of Y. We show that the possibility for M to have a projective embedding is equivalent to the existence of a complex *CR* line bundle over M having certain properties. In this context, it is interesting to note that the general abstract notion of a complex *CR* line bundle F over a *CR* manifold is such that F may fail to be locally *CR* trivializable, even in the case where M is *CR* embeddable [HN8].

For more information about pseudoconcave *CR* manifolds, we refer the reader to the foundational paper [HN3], to the many examples in [HN8], and to [HN1], [HN2], $\ldots$, [HN11], as well as [BHN], [DCN], and [L].

1. Preliminaries.

An abstract smooth almost *CR* manifold of type (n, k) consists of: a connected smooth paracompact manifold M of dimension $2n+k$, a smooth subbundle HM of TM of rank $2n$, that we call the *holomorphic tangent space* of M, and a smooth complex structure J on the fibers of HM.

Let $T^{0,1}M$ be the complex subbundle of the complexification $\mathbb{C}HM$ of HM, which corresponds to the $-\sqrt{-1}$ eigenspace of J:

$$T^{0,1}M = \{X + \sqrt{-1}JX \mid X \in HM\}. \tag{1.1}$$

We say that M is a *CR* manifold if, moreover, the formal integrability condition

$$[\mathcal{C}^\infty(M, T^{0,1}M), \mathcal{C}^\infty(M, T^{0,1}M)] \subset \mathcal{C}^\infty(M, T^{0,1}M) \tag{1.2}$$

holds. When $k = 0$, via the Newlander-Nirenberg theorem, we recover the definition of a complex manifold.

Next we define $T^{*1,0}M$ as the annihilator of $T^{0,1}M$ in the complexified cotangent bundle $\mathbb{C}T^*M$. We denote by $Q^{0,1}M$ the quotient bundle $\mathbb{C}T^*M/T^{*1,0}M$, with projection π_Q. It is a rank n complex vector bundle on M, dual to $T^{0,1}M$. The $\bar\partial_M$-operator acting on smooth functions is defined by $\bar\partial_M = \pi_Q \circ d$. A local trivialization of the bundle $Q^{0,1}M$ on an open set U in M defines n smooth sections $\bar L_1, \bar L_2, \dots, \bar L_n$ of $T^{0,1}M$ in U; hence

$$\bar\partial_M u = (\bar L_1 u, \bar L_2 u, \dots, \bar L_n u), \tag{1.3}$$

where u is a function in U. Solutions u of $\bar\partial_M u = 0$ are called *CR* functions. We denote by $\mathcal{CR}(U)$ the space of *smooth* ($\mathcal{C}^\infty$) *functions* on an open subset U of M that satisfy $\bar\partial_M u = 0$. Note that $\bar\partial_M$ is a homogeneous first order partial differential operator and hence the space $\mathcal{CR}(U)$ is a commutative algebra with respect to the multiplication of functions. We denote by $\mathcal{CR}_{M,a} = \varinjlim_{U \ni a} \mathcal{CR}(U)$ the local ring of germs of smooth $\mathcal{CR}$ functions at $a \in M$.

Let M_1, M_2 be two abstract smooth *CR* manifolds, with holomorphic tangent spaces HM_1, HM_2, and partial complex structures J_1, J_2, respectively. A smooth map $f : M_1 \to M_2$ is *CR* if $f_*(HM_1) \subset HM_2$, and $f_*(J_1 v) = J_2 f_*(v)$ for every $v \in HM_1$.

A *CR embedding*(1) ϕ of an abstract *CR* manifold M into a complex manifold X, with complex structure J_X, is a *CR* map which is a smooth embedding satisfying $\phi_*(H_a M) = \phi_*(T_a M) \cap J_X(\phi_*(T_a M))$ for every $a \in M$. We say that the embedding is *generic* if the complex dimension of X is $(n+k)$, where (n, k) is the type of M.

Let M be a smooth abstract *CR* manifold of type (n, k). We say that M is *locally embeddable* at $a \in M$, if a has an open neighborhood ω_a in M which admits a *CR* embedding into some complex manifold X_a. In this case we can always take for X_a an open subset of $\mathbb{C}^{n+k}$ and assume that the embedding $\omega_a \hookrightarrow X_a$ is generic. The property of being locally embed-

(1) In this case we shall often identify M with the submanifold $\phi(M)$ of X.

dable at a is equivalent to the fact that there exist an open neighborhood ω_a of a and functions $f_1, f_2, \ldots, f_{n+k} \in \mathcal{CR}(\omega_a)$ such that

$$df_1(a) \wedge df_2(a) \wedge \ldots \wedge df_{n+k}(a) \neq 0 \,. \tag{1.4}$$

The functions $f_1, f_2, \ldots, f_{n+k}$ can be taken to be the restrictions to ω_a of the coordinate functions $z_1, z_2, \ldots, z_{n+k}$ of $X_a \subset \mathbb{C}^{n+k}$. For this reason one can say that they provide *CR coordinates* on M near a.

The *characteristic bundle* $H^0 M$ is defined to be the annihilator of HM in T^*M. Its purpose it to parametrize the Levi form: recall that the *Levi form* of M at x is defined for $\xi \in H^0_x M$ and $X \in H_x M$ by

$$\mathcal{L}(\xi; X) = d\tilde{\xi}(X, JX) = \langle \xi, [J\tilde{X}, \tilde{X}] \rangle, \tag{1.5}$$

where $\tilde{\xi} \in \mathcal{C}^\infty(M, H^0 M)$ and $\tilde{X} \in \mathcal{C}^\infty(M, HM)$ are smooth extensions of ξ and X. For each fixed ξ it is a Hermitian quadratic form for the complex structure J_x on $H_x M$.

A *CR* manifold M is said to be *q-pseudoconcave* if the Levi form $\mathcal{L}(\xi; \cdot)$ has at least q negative and q positive eigenvalues for every $a \in M$ and every nonzero $\xi \in H^0_a M$.

By the term *pseudoconcave CR manifold M* we mean an abstract *CR* manifold which is: (i) locally embeddable at each point, and (ii) 1-pseudoconcave.

In this paper we shall be concerned with *CR* manifolds M of type (n, k) which have a certain property E (E is for *extension*). *M is said to have property E* iff there is an E-pair (M, X). By an E-pair we mean that

(i) M is a generic *CR* submanifold of the complex manifold X, and

(ii) for each $a \in M$, the restriction map induces an isomorphism $\mathcal{O}_{X,a} \to \mathcal{CR}_{M,a}$.

Remark. *If M is a pseudoconcave CR manifold, then M has property E.*

In fact, property (i) for a pseudoconcave M was proved in Proposition 3.1 of [HN3]; however, Theorem 1.3 below gives a new simplified proof of this fact. Property (ii) for a pseudoconcave M was proved in [BP], [NV]; however, a very short proof of this fact is also given by Theorem 13.2 in [HN7]. Thus property E is to be regarded as a somewhat weaker hypothesis on M than pseudoconcavity.

When $k=0$, so M is of type $(n,0)$, then M is an n-dimensional complex manifold, and we obtain an E pair by choosing $X=M$. Hence we adopt the convention that *any complex manifold has property E*.

When $n=0$, so M is of type $(0,k)$, then M is a smooth totally real k-dimensional manifold, and we can never obtain an E-pair, (unless $M=X=$ a point), because then any smooth function belongs to $\mathcal{CR}(M)$.

THEOREM 1.1. *Let (M,X) be an E-pair. Then for any open set $\omega\subset M$ there is a corresponding open set $\Omega\subset X$ such that*

(*i*) $\Omega\cap M=\omega$, *and*

(*ii*) $r:\mathcal{O}(\Omega)\to\mathcal{CR}(\omega)$ *is an isomorphism.*

PROOF. We fix a Hermitian metric g on X, with associated distance $d(x,y)$. Let $a\in\omega$ and consider

$$\mathcal{F}_n=\left\{(f,\tilde{f})\in\mathcal{CR}(\omega)\times\mathcal{O}\left(B\left(a,\frac{1}{n}\right)\right)\,\middle|\,\tilde{f}=f\text{ on }B\left(a,\frac{1}{n}\right)\cap\omega\right\}.\tag{1.6}$$

Here $B\left(a,\frac{1}{n}\right)$ denotes the ball of radius $\frac{1}{n}$ in X, centered at a. Note that each $\mathcal{F}_n$ is a closed subspace of a Fréchet-Schwartz space, and hence a Fréchet-Schwartz space itself. For each n, the map

$$\pi_n\colon\mathcal{F}_n\ni(f,\tilde{f})\to f\in\mathcal{CR}(\omega)\tag{1.7}$$

is linear and continuous. By our hypothesis,

$$\bigcup_{n=1}^{\infty}\pi_n(\mathcal{F}_n)=\mathcal{CR}(\omega).\tag{1.8}$$

Hence by the Baire category theorem, some $\pi_{n_0}(\mathcal{F}_{n_0})$ is of the second category. It follows from a theorem of Banach that $\pi_{n_0}\colon\mathcal{F}_{n_0}\to\mathcal{CR}(\omega)$ is surjective. Now we denote $B\left(a,\frac{1}{n_0}\right)$ by B_a.

Next we fix a tubular neighborhood U of M in X, with $\pi:U\to M$ denoting the orthogonal projection. By letting $\varrho\in\mathcal{C}^\infty(\omega,\mathbb{R}^+)$ vary, we produce a fundamental system of open neighborhoods

$$\Omega_\varrho=\{z\in U\,|\,\pi(z)\in\omega,\ \text{and}\ d(z,\pi(z))<\varrho(\pi(z))\}\tag{1.9}$$

of ω in X. We choose $\varrho_0\in\mathcal{C}^\infty(\omega,\mathbb{R}^+)$ such that

$$\Omega_{\varrho_0}\subset\bigcup_{a\in\omega}B_a.\tag{1.10}$$

Since M is a deformation retract of Ω_{ϱ_0}, and the local holomorphic extension of CR functions from the generic M is unique, the different extensions to each B_a of a given $f \in \mathcal{CR}(\omega)$ match at points of Ω_{ϱ_0}. This completes the proof with $\Omega = \Omega_{\varrho_0}$.

COROLLARY 1.2. *In the situation of Theorem* 1.1 *we have, in addition, that*

(*iii*) *If* $f \in \mathcal{CR}(\omega)$, *and* f *vanishes of infinite order at* $a \in \omega$, *then* $f \equiv 0$ *in the connected component of* a *in* ω.

(*iv*) $(r^* f)(\Omega) = f(\omega)$.

(*v*) *If* $|f|$ *has a local maximum at a point* $a \in \omega$, *then* f *is constant on the connected component of* a *in* ω.

PROOF. If $f \in \mathcal{CR}(\omega)$ vanishes of infinite order at $a \in \omega$, then also $r^* f$ vanishes of infinite order at a and, by the strong unique continuation of holomorphic functions, $r^* f$ vanishes identically in the connected component of Ω containing a, and we obtain (*iii*).

To prove (*iv*), we assume by contradiction that $r^* f$ takes some value $z_0 \in \mathbb{C}$ at some point of Ω, but that f does not assume that value at any point of ω. Then the function

$$g = \frac{1}{f - z_0} \tag{1.11}$$

belongs to $\mathcal{CR}(\omega)$, and has no holomorphic extension to Ω, contradicting (*ii*).

By (*ii*) and (*iv*), a local maximum of $f \in \mathcal{CR}(\omega)$ at $a \in \omega$, is also a local maximum of $r^* f$ at $a \in \Omega$; thereby $r^* f$ is constant on the connected component of a in Ω and we obtain (*v*).

COROLLARY 1.3. *Let* (M, X) *and* (N, Y) *be E-pairs, and let* $f : M \to$ $\to N$ *be a smooth CR isomorphism. Then there are E-pairs* (M, X') *and* (N, Y'), *with* $X' \subset X$ *and* $Y' \subset Y$, *such that* f *extends to a biholomorphic diffeomorphism* $\tilde{f} : X' \to Y'$.

PROOF. We first consider the case where X and Y are open sets in $\mathbb{C}^{n+k}$. By Theorem 1.1, there is an open neighborhood Ω of M in X where f has a holomorphic extension $\tilde{f} : \Omega \to \mathbb{C}^{n+k}$. By shrinking Ω to Ω', we can arrange that $\tilde{f}(\Omega') \subset Y$ and the Jacobian determinant of $\tilde{f}$ is different from zero on Ω'. Likewise there is an open neighborhood Ω'' of N in

Y where $f^{-1} = g$ extends to $\tilde{g}$, with $g(\Omega'') \subset \Omega'$ and the Jacobian determinant of $\tilde{g}$ being nonzero in Ω''. By uniqueness of holomorphic extension of CR functions from M to $\tilde{g}(\Omega'')$, it follows that $\tilde{g} \circ \tilde{f} =$ identity on a neighborhood of M in X.

Now we consider the general case. Introducing local holomorphic coordinates charts on X and Y, we may use the special case above to produce local holomorphic extensions. The local holomorphic extensions patch together, by unique continuation, to give the desired $\tilde{f}$.

We may now use Corollary 1.3 to show that M having property E is actually a local property of M.

THEOREM 1.4. *M has property E if and only if for each $a \in M$, there is an open neighborhood ω_a of a in M such that ω_a has property E.*

PROOF. By hypothesis we have an E-pair (ω_a, X_a), for each $a \in M$. We can assume that $\omega_a \Subset M$, and that $\pi_a: X_a \to \omega_a$ is the orthogonal projection from a tubular neighborhood, with a distance function $d_a(x, y)$. By Corollary 1.3, whenever $\omega_a \cap \omega_b \neq \emptyset$, there are open neighborhoods X_{ab} of $\omega_a \cap \omega_b$ in X_a and X_{ba} of $\omega_a \cap \omega_b$ in X_b, and a unique biholomorphic map $\tilde{f}_{ab}: X_{ab} \to X_{ba}$, extending the identity map on $\omega_a \cap \omega_b$. We may select a locally finite open covering $\{\omega_a\}$ of M, parametrized by $a \in A \subset M$. By shrinking, we refine the $\{\omega_a\}$ to an open covering $\{\omega'_a\}$, with $\omega'_a \Subset \omega_a$. With $\varepsilon_a > 0$ sufficiently small, we define

$$X'_a = \pi_a^{-1}(\omega'_a) \cap \{d_a(x, \omega'_a) < \varepsilon_a\}, \tag{1.12}$$

so as to have

$$\pi_a^{-1}(\omega'_a \cap \omega'_b) \cap X'_a \subset X_{ab}, \tag{1.13}$$

for all $b \in A$ such that $\omega'_b \cap \omega'_a \neq \emptyset$. Set

$$X'_{ab} = f_{ab}^{-1}(X_{ab}) \cap X'_a. \tag{1.14}$$

Then X is obtained by gluing together the X'_a's, by

$$X'_a \supset X'_{ab} \xrightarrow{\tilde{f}_{ab}} X'_{ba} \subset X'_b. \tag{1.15}$$

This completes the proof.

We now turn to the object of main concern in this paper, which are the CR meromorphic functions on an M satisfying property E. The ring $\mathcal{CR}(\omega)$ of smooth ($\mathcal{C}^\infty$) CR functions on $\omega \subset M$ is an integral domain if ω is connected. Let $\Delta(\omega)$ be the subset of $\mathcal{CR}(\omega)$ of divisors of zero; i.e.

$\Delta(\omega)$ is the set of those *CR* functions on ω which vanish in some connected component of ω. Let $\mathcal{M}(\omega)$ be the quotient ring of $\mathcal{CR}(\omega)$ with respect to $\mathcal{CR}(\omega)\setminus\Delta(\omega)$. This means that $\mathcal{M}(\omega)$ is the set of the equivalence classes of pairs (p, q) with $p \in \mathcal{CR}(\omega)$ and $q \in \mathcal{CR}(\omega)\setminus\Delta(\omega)$. The equivalence relation $(p, q) \sim (p', q')$ is defined by $pq' = p'q$. If $\omega' \subset \omega$ is an inclusion of open subsets of M, the restriction map $r_\omega^{\omega'}: \mathcal{CR}(\omega) \to$ $\to \mathcal{CR}(\omega')$ sends $\mathcal{CR}(\omega)\setminus\Delta(\omega)$ into $\mathcal{CR}(\omega')\setminus\Delta(\omega')$ and thus induces a homomorphism of rings:

$$r_\omega^{\omega'}: \mathcal{M}(\omega) \to \mathcal{M}(\omega'). \tag{1.16}$$

We obtain in this way a presheaf of rings. We shall call the corresponding sheaf $\mathcal{M}$ the *sheaf of CR meromorphic functions on M*. By a *CR* meromorphic function on an open set $\omega \subset M$, we mean a continuous section f of $\mathcal{M}$ over ω. If ω is connected, the space of all such sections $\mathcal{K}(\omega)$ forms a field. Since we always assume that M is connected, we have in particular $\mathcal{K}(M)$, *the field of CR meromorphic functions on M*.

We recall these standard notions: Let $\mathbb{F}$ be a field and $\mathbb{F}_0 \subset \mathbb{F}$ a subfield. Then $f_1, f_2, \dots, f_\ell \in \mathbb{F}$ are said to be *algebraically dependent* over $\mathbb{F}_0$ iff there is a nonzero polynomial $P \in \mathbb{F}_0[x_1, x_2, \dots, x_\ell]$ with coefficients in $\mathbb{F}_0$ such that

$$P(f_1, f_2, \dots, f_\ell) = 0\,; \tag{1.17}$$

otherwise they are called *algebraically independent*. The *transcendence degree* of $\mathbb{F}$ over $\mathbb{F}_0$ is the cardinality of a maximal set $S \subset \mathbb{F}$ such that every finite subset of S is algebraically independent over $\mathbb{F}_0$. If the transcendence degree of $\mathbb{F}$ over $\mathbb{F}_0$ is zero, we say that $\mathbb{F}$ is *algebraic* over (or is an *algebraic extension of*) $\mathbb{F}_0$. The cardinal $[\mathbb{F}: \mathbb{F}_0]$ denotes the dimension of $\mathbb{F}$ over $\mathbb{F}_0$, as a vector space. The field $\mathbb{F}$ is said to be a *simple algebraic extension of* $\mathbb{F}_0$ if there exists an element $\theta \in \mathbb{F}$ such that any $f \in \mathbb{F}$ can be written as a polynomial in θ with coefficients in $\mathbb{F}_0$. When $\mathbb{F}_0$ has characteristic zero, the primitive element theorem says that any finite algebraic extension of $\mathbb{F}_0$ is simple.

Finally we discuss the general notion of a smooth *complex CR line bundle* $F \xrightarrow{\pi} M$, which was introduced in [HN8]. By this we mean that F is a smooth complex line bundle over M such that:

(*i*) F and M are smooth abstract *CR* manifolds of type $(n+1, k)$ and (n, k), respectively,

(*ii*) $\pi: F \to M$ is a smooth *CR* submersion, *(iii)* $F\oplus F\ni(\xi_1,\xi_2)\to\xi_1+\xi_2\in F$ and $\mathbb{C}\times F\ni(\lambda,\xi)\to\lambda\cdot\xi\in F$ are *CR* maps.

Note that the Whitney sums $F\oplus F$ and $\mathbb{C}\times F$ have natural structures of smooth *CR* manifolds of type $(n+2,k)$; see [HN8]. There we also introduced the notion of the tangential *CR* operator $\overline{\partial}_M^F$, acting on smooth sections of F. We may take a smooth (not necessarily *CR*) trivialization (U_α,σ_α) of F, where σ_α is a smooth non vanishing section of F on U_α. Then a smooth section s of F has a local representation $s=s_\alpha\sigma_\alpha$ in U_α, where s_α is a smooth complex valued function in U_α, and $s_\alpha=g_{\alpha\beta}s_\beta$ in $U_\alpha\cap U_\beta$, with $g_{\alpha\beta}=s_\beta/s_\alpha$. In each U_α the tangential *CR* operator acting on s has a representation of the form:

$$\overline{\partial}_M^F s=(\overline{\partial}_M s_\alpha+A_\alpha s_\alpha)\otimes\sigma_\alpha, \tag{1.18}$$

where $A_\alpha\in\mathcal{C}^\infty(U_\alpha,\mathcal{Q}^{0,1}M)$ and $\overline{\partial}_M A_\alpha=0$. On $U_\alpha\cap U_\beta$ we have:

$$A_\beta-A_\alpha=g_{\beta\alpha}\overline{\partial}_M g_{\alpha\beta},\quad\text{with }g_{\beta\alpha}=g_{\alpha\beta}^{-1}. \tag{1.19}$$

If s satisfies $\overline{\partial}_M^F s=0$, it is called a *CR* section of F.

The ℓ-th tensor power F^ℓ of F is still a smooth complex *CR* line bundle over M, which can be defined in the same trivialization, and we have

$$\overline{\partial}_M^{F^\ell} t=(\overline{\partial}_M t_\alpha+\ell A_\alpha t_\alpha)\otimes\sigma_\alpha^\ell \tag{1.20}$$

in U_α, where t is a smooth section of F^ℓ, with $t=t_\alpha\sigma_\alpha^l$ in U_α.

If the local trivialization is a *CR* trivialization, then the $(0,1)$ forms A_α in (1.18) and (1.20) are equal to zero. On the other hand, if the $\overline{\partial}_M$-closed forms A_α are locally $\overline{\partial}_M$-exact, then F and F^ℓ are locally *CR* trivializable.

Let $f\in\mathcal{K}(M)$, where we now assume that M has property E. Then we can associate to f a smooth complex line bundle $F\xrightarrow{\pi}M$, which is locally *CR* trivializable. By definition, f has local representations

$$f=\frac{p_\alpha}{q_\alpha}\quad\text{on}\quad\omega_\alpha, \tag{1.21}$$

with $p_\alpha,q_\alpha\in\mathcal{CR}(\omega_\alpha)$. Moreover we may arrange that their holomorphic extensions $\tilde{p}_\alpha$ and $\tilde{q}_\alpha$ to a neighborhood Ω_α of ω_α in X have no nontrivial common factor at each point of Ω_α. Then there are uniquely determined non vanishing functions $g_{ab} \in \mathcal{CR}(\omega_a \cap \omega_b)$ such that

$$q_a = g_{ab} q_b \quad \text{on} \quad \omega_a \cap \omega_b. \tag{1.22}$$

The $\{g_{ab}\}$ are then the transition functions of a smooth complex *CR* line bundle F over M, and F is therefore locally *CR* trivializable. The $\{p_a\}$ and $\{q_a\}$ give global smooth sections p and q of F over M, whose quotient p/q is the *CR* meromorphic function f.

Let us return now to the smooth complex *CR* line bundle $F \xrightarrow{\pi} M$, which may not be locally *CR* trivializable. In this context, it is natural to consider smooth abstract *CR* manifolds M, which may not have property E, but which are *essentially pseudoconcave*, as defined (2) in [HN8]. The important consequence of the assumption of essential pseudoconcavity on M is that one has the weak unique continuation property for *CR* sections of F. Note that 1-pseudoconcave abstract *CR* manifolds are essentially pseudoconcave. Under these assumptions we can give a more general notion of what is a *CR* meromorphic function on M: We associate a *CR* meromorphic function f to any pair (p, q), where p and q are smooth global *CR* sections of a smooth complex *CR* line bundle $F \xrightarrow{\pi} M$, with $q \not\equiv 0$. Another pair (p', q'), which are smooth *CR* global sections of another such $F' \xrightarrow{\pi'} M$, with $q' \not\equiv 0$, define the same f iff $pq' = p'q$ as sections of $F \otimes F'$. Note that $f = p/q$ is a well defined smooth *CR* function where $q \neq 0$. With this more general definition, we get a new collection $\widehat{\mathcal{K}}(M)$ of objects called *CR* meromorphic functions on M. Observe that $\widehat{\mathcal{K}}(M)$ is a field. For an essentially pseudoconcave M, which has property E, $\mathcal{K}(M)$ is a subfield of $\widehat{\mathcal{K}}(M)$. If in addition M is 2-pseudoconcave, then all smooth complex *CR* line bundles over M are locally *CR* trivializable, and then $\mathcal{K}(M) = \widehat{\mathcal{K}}(M)$.

2. *CR* meromorphic functions on compact *CR* manifolds.

Let M be a connected smooth compact *CR* manifold of type (n, k), having property E. Then:

(2) M is *essentially pseudoconcave* iff it is *minimal*, i.e. does not contain germs of *CR* manifolds with the same *CR* dimension and a smaller *CR* codimension, and admits a Hermitian metric on HM for which the traces of the Levi forms are zero at each point.

THEOREM 2.1. *The field $\mathcal{K}(M)$ of CR meromorphic functions on M has transcendence degree over $\mathbb{C}$ less or equal to $n+k$.*

Setting $k=0$ above, we recover Satz 1 in Siegel [Si].

PROOF. According to the discussion in § 1, the statement means: Given $n+k+1$ CR meromorphic functions $f_0, f_1, \dots, f_{n+k}$ on M, there exists a non zero polynomial with complex coefficients $F(x_0, x_1, \dots, x_{n+k})$ such that

$$F(f_0, f_1, \dots, f_{n+k}) \equiv 0 \quad \text{on} \quad M. \tag{2.1}$$

From the preceding section, we may regard M as a generic CR submanifold of an $n+k$ dimensional complex manifold X.

For each point $a \in M$ there is a connected open coordinate neighborhood Ω_a, in which the holomorphic coordinate z_a is centered at a. We choose Ω_a in such a way that $\omega_a = \Omega_a \cap M$ is a connected neighborhood of a in M. Moreover we can arrange that, for $j = 0, 1, \dots, n+k$, each f_j has a representation

$$f_j = \frac{p_{ja}}{q_{ja}} \quad \text{on} \quad \omega_a \tag{2.2}$$

with p_{ja} and q_{ja} being smooth CR functions in ω_a. According to Theorem 1.1 we may also assume that the restriction map $\mathcal{O}(\Omega_a) \to \mathcal{CR}(\omega_a)$ is an isomorphism. For each CR function g on ω_a, we denote its unique holomorphic extension to Ω_a by $\tilde{g}$. By a careful choice of the p_{ja} and q_{ja}, and an additional shrinking of ω_a, Ω_a, we can also arrange that

$$\tilde{f}_j = \frac{\tilde{p}_{ja}}{\tilde{q}_{ja}} \quad \text{on} \quad \Omega_a, \tag{2.3}$$

with the functions $\tilde{p}_{ja}$ and $\tilde{q}_{ja}$ being holomorphic and having no nontrivial common factor at each point in a neighborhood of $\overline{\Omega}_a$. For each pair of points a, b on M we have the transition functions

$$\tilde{q}_{ja} = g_{jab}\, \tilde{q}_{jb}, \tag{2.4}$$

which are holomorphic and non vanishing on a neighborhood of $\overline{\Omega}_a \cap \overline{\Omega}_b$. Again, for each $a \in M$ we consider the polydiscs:

$$K_a = \{ |z_a| \leq r_a \} \quad \text{and} \quad L_a = \{ |z_a| < e^{-1} r_a \}, \tag{2.5}$$

where $|z_a|$ denotes the max norm in $\mathbb{C}^{n+k}$, and $r_a > 0$ is chosen so that $K_a \Subset \Omega_a$. By the compactness of M, we may fix a finite number of points $a_1, a_2, \ldots, a_m$ on M, such that the $L_{a_1}, L_{a_2}, \ldots, L_{a_m}$ provide an open covering of M. Then we choose positive real numbers μ and ν to provide the bounds:

$$|g_{0ab}| < e^{\mu} \quad \text{and} \quad \left| \prod_{j=1}^{n+k} g_{jab} \right| < e^{\nu} \tag{2.6}$$

on $\overline{\Omega}_a \cap \overline{\Omega}_b$ for $a, b = a_1, a_2, \ldots, a_m$.

Consider a polynomial with complex coefficients to be determined later, $F(x_0, x_1, \ldots, x_{n+k})$ of degree s with respect to x_0 and of degree t with respect to each x_i for $i = 1, 2, \ldots, n+k$. The number of coefficients to be determined is

$$A = (s+1) \cdot (t+1)^{n+k}. \tag{2.7}$$

Now, letting a stand for any one of the $a_1, a_2, \ldots, a_m$, we introduce the functions

$$Q_a = \tilde{q}_{0a}^{s} \prod_{j=1}^{n+k} \tilde{q}_{ja}^{t}, \qquad P_a = Q_a F(\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_{n+k}) \tag{2.8}$$

which are holomorphic on a neighborhood of $\overline{\Omega}_a$. For a positive integer h, to be made precise later, we wish to impose the condition, for $a = a_1, a_2, \ldots, a_m$, that P_a vanishes to order h at a. In terms of our local coordinates z_a, this means that all partial derivatives of order $\leqslant h-1$ must vanish at $z_a = 0$. This imposes a certain number of linear homogeneous conditions on the unknown coefficients of the polynomial F. The number of such conditions is

$$B = m \binom{n+k+h-1}{n+k} \leqslant m h^{n+k}. \tag{2.9}$$

If we can arrange that $B < A$, then this system of linear homogeneous equations has a non trivial solution.

However, in order to apply the Schwarz lemma later, we need also to arrange that s, t and h satisfy

$$\mu s + \nu t < h\,. \tag{2.10}$$

To this end we fix s to be an integer with $s > m\nu^{n+k}$. Thus, for each positive h, we denote by t_h the largest positive integer satisfying $s t_h^{n+k} <$

$< mh^{n+k}$. In this way we obtain that

$$B \leqslant mh^{n+k} \leqslant s(t_h+1)^{n+k} < (s+1)(t_h+1)^{n+k} = A\,. \tag{2.11}$$

On the other hand, since $t_h \to \infty$ as $h \to \infty$, by choosing h sufficiently large we have

$$m\left(\frac{\mu s}{t_h}+\nu\right)^{n+k} < s\,, \tag{2.12}$$

which implies (2.10) for $t = t_h$. Set

$$\Upsilon = \max_{1 \leqslant i \leqslant m} \max_{K_{a_i}} |P_{a_i}|\,. \tag{2.13}$$

This maximum is obtained at some point z^* belonging to some K_{a^*}, for a^* equal to some one of $a_1, a_2, \dots, a_m$. Since $z^* \in K_{a^*} \subset \Omega_{a^*}$, because of our choices of the ω_a, Ω_a, according to (*iv*) in Corollary 1.2, there is another point $z^{**} \in \omega_{a^*}$ such that

$$P_{a^*}(z^*) = P_{a^*}(z^{**})\,. \tag{2.14}$$

But the point z^{**} belongs to some $L_{a^{**}} \subset K_{a^{**}}$, where a^{**} is one of the a_1, $a_2, \dots, a_m$. Hence by the Schwartz lemma of Siegel [Si] we obtain

$$|P_{a^{**}}(z^{**})| \leqslant \Upsilon e^{-h}. \tag{2.15}$$

However

$$P_{a^*}(z^{**}) = P_{a^{**}}(z^{**})\left[g^s_{0a^*a^{**}}(z^{**})\prod_{j=1}^{n+k} g^t_{ja^*a^{**}}(z^{**})\right]. \tag{2.16}$$

Hence from (2.6), (2.14), (2.15) we obtain

$$\Upsilon = |P_{a^*}(z^{**})| \leqslant \Upsilon\, e^{\mu s + \nu t - h}. \tag{2.17}$$

By (2.10) this implies that $\Upsilon = 0$. Hence each $P_{a_j} \equiv 0$, which in turn yields $F(\tilde f_0, \tilde f_1, \dots, \tilde f_{n+k}) \equiv 0$. Therefore restricting to M we get (2.1). This completes the proof.

3. Analytic and algebraic dependence of *CR* meromorphic functions.

Let $f_0, f_1, \dots, f_\ell \in \mathcal{K}(M)$. We say that they are *analytically dependent* if

$$df_0 \wedge df_1 \wedge \dots \wedge df_\ell = 0 \quad \text{where it is defined}. \tag{3.1}$$

THEOREM 3.1. *Let M be a connected smooth compact CR manifold of type (n, k), having property E. Let $f_0, f_1, \dots, f_\ell \in \mathcal{K}(M)$. Then they are algebraically dependent over $\mathbb{C}$ if and only if they are analytically dependent.*

PROOF. First we observe that algebraic dependence implies analytic dependence. Assume that there is a nontrivial polynomial F, with complex coefficients, of minimal total degree, such that $F(f_0, f_1, \dots, f_\ell) \equiv 0$. Then

$$\sum_{j=0}^{\ell} \frac{\partial F}{\partial x_j}(f_0, f_1, \dots, f_\ell)\, df_j = 0 \tag{3.2}$$

where it is defined. It follows that some coefficient in (3.2) is a nonzero *CR* meromorphic function on M. This implies (3.1) on an open dense subset of M, and hence whenever it is defined.

For the proof in the other direction, we can assume that $f_1, \dots, f_\ell$ are analytically independent. Our task is to show that there exists a nonzero polynomial with complex coefficients $F(x_0, x_1, \dots, x_\ell)$ such that

$$F(f_0, f_1, \dots, f_\ell) \equiv 0 \quad \text{on} \quad M\,. \tag{3.3}$$

To this end we repeat the proof of Theorem 2.1, with $n+k$ replaced by ℓ, down to the line below (2.8). We shall replace s, t, ν, A, B by new s', t', ν', A', B'. After that we choose additional points $a_1', a_2', \dots, a_m'$ with $a_j' \in \omega_{a_j}$ and a_j' sufficiently close to a_j, for $j = 1, 2, \dots, m$. These points are chosen so that

$$K_{a_j'} = \{\, |z_{a_j} - z_{a_j}(a_j')| \leqslant r_{a_j} \} \Subset \Omega_{a_j}, \tag{3.4}$$

the $L_{a_j'} = \{\, |z_{a_j} - z_{a_j}(a_j')| < e^{-1} r_{a_j} \}$ still give an open covering of M, the $\tilde f_0, \tilde f_1, \dots, \tilde f_\ell$ are holomorphic at each a_j', and $\tilde f_1, \dots, \tilde f_\ell$ can be completed to a system of holomorphic coordinates in a neighborhood of each a_j'. This is possible because the set of points on M, where the $\tilde f_1, \dots, \tilde f_\ell$ are holomorphic, and the $d\tilde f_1, \dots, d\tilde f_\ell$ are linearly independent, is open and dense. Our assumption (3.1) that the $f_0, f_1, \dots, f_\ell$ are analytically dependent implies that, near each point a_j', $\tilde f_0$ is a holomorphic function of $\tilde f_1, \dots, \tilde f_\ell$. We modify the proof of Theorem 2.1 by requiring that the holomorphic functions P_{a_j} vanish to order h at a_j', for $j = 1, \dots, m$. To accomplish this, we require that $F(\tilde f_0, \tilde f_1, \dots, \tilde f_\ell)$ vanish to order h at each a_j'. This amount to requiring that all partial derivatives of order $\leqslant (h -$

$-1)$ of $F(\tilde f_0, \tilde f_1, \dots, \tilde f_\ell)$ with respect to $\tilde f_1, \dots, \tilde f_\ell$ should vanish at a_j'. The number of homogeneous linear equations is now

$$B' = m\binom{\ell + h - 1}{\ell} \leqslant mh^\ell . \tag{3.5}$$

Now we fix an integer s' with $s' > m(\nu')^\ell$. Just as in the proof of Theorem 2.1, we choose h sufficiently large, and take t' to be the largest positive integer satisfying $s'(t')^\ell < mh^\ell$, so as to obtain

$$B' \leqslant mh^\ell < (s'+1)(t'+1)^\ell = A', \tag{3.6}$$

$$\mu s' + \nu' t' < h . \tag{3.7}$$

By (3.6) we can choose a nontrivial F, of degree s' in x_0 and of degree t' in each $x_1, \dots, x_\ell$, such that all P_{a_j} vanish to order h at a_j'. Set

$$\Upsilon = \max_{1 \leqslant j \leqslant m} \max_{K_{a_j'}} |P_{a_j}| . \tag{3.8}$$

This maximum is attained at some point z' belonging to some $K_{a_{j0}'}$. Since $z' \in K_{a_{j0}'} \subset \Omega_{a_{j0}}$, as before by (*iv*) in Corollary 1.2 there is another point $z'' \in \omega_{a_{j0}}$ such that

$$P_{a_{j0}}(z') = P_{a_{j0}}(z'') . \tag{3.9}$$

This point z'' belongs to some $L_{a_{j1}'} \subset K_{a_{j1}'}$. So by the Schwarz lemma we obtain

$$|P_{a_{j1}}(z'')| \leqslant \Upsilon e^{\mu s' + \nu t' - h}. \tag{3.10}$$

Thus as before we obtain

$$\Upsilon = |P_{a_{j1}}(z'')| \leqslant \Upsilon e^{\mu s' + \nu t' - h}, \tag{3.11}$$

showing that $\Upsilon = 0$. This implies (3.3), completing the proof.

In order to make the exposition more clear, we have divided the discussion into two parts; however Theorem 2.1 is a direct consequence of Theorem 3.1.

Algebraic dependence always implies analytic dependence. However, in the absence of property E, the converse may be false. We give a general counterexample:

PROPOSITION 3.2. *Let M be a connected smooth compact CR manifold of type (n, k). Assume that M has a smooth CR immersion into some Stein manifold. Then*

(1) *condition E is violated, and*

(2) *there exists an infinite sequence of smooth CR functions on M, any two of which are analytically dependent, and which are algebraically independent over $\mathbb{C}$.*

PROOF. By the embedding theorem for Stein manifolds, we can assume that M has a smooth CR immersion in some $\mathbb{C}^N$. Then by a result in [HN1], there exists a point x_0 in M and a $\xi \in H^0_{x_0}M$ such that the Levi form $\mathcal{L}_{x_0}(\xi, \cdot)$ is positive definite on $H_{x_0}M$. This implies that a small neighborhood of x_0 in M is contained in the smooth boundary of a strictly pseudoconvex domain Ω in $\mathbb{C}^N$. It is well known that there are holomorphic functions in Ω, which are smooth on $\overline{\Omega}$, and cannot be holomorphically extended beyond x_0. Thus condition E is violated at x_0.

To prove (2) we first observe that some coordinate z_1 must be non constant on M. Consider the sequence of holomorphic functions $\{f_\lambda(z) = e^{z_1^\lambda} \mid \lambda \in \mathbb{N}\} \subset \mathcal{O}(\mathbb{C}^N)$, and denote their pull-backs to M by $\{f_\lambda^*\}$. Clearly any two of them are analytically dependent. Assume, by contradiction, that some finite collection $f^*_{\lambda_1}, f^*_{\lambda_2}, \dots, f^*_{\lambda_m}$, are algebraically dependent. Then there is a nontrivial polynomial P, with complex coefficients, such that

$$P(f^*_{\lambda_1}, f^*_{\lambda_2}, \dots, f^*_{\lambda_m}) = 0 \quad \text{on} \quad M\,. \tag{3.12}$$

Since $e^{z_1^{\lambda_1}}, e^{z_1^{\lambda_2}}, \dots, e^{z_1^{\lambda_m}}$ are algebraically independent in $\mathcal{O}(\mathbb{C})$, the entire function $z_1 \to P(e^{z_1^{\lambda_1}}, e^{z_1^{\lambda_2}}, \dots, e^{z_1^{\lambda_m}})$ is not constant and has isolated zeroes. Because M is connected, it follows that the pullback on M of the function z_1 is constant, contradicting our assumption that the coordinate z_1 was not constant on M.

4. The field of *CR* meromorphic functions.

Suppose M is a connected compact CR manifold of type (n, k), having property E. We have:

THEOREM 4.1. *Let d be the transcendence degree of $\mathcal{K}(M)$ over $\mathbb{C}$, and let $f_1, f_2, \dots, f_d$ be a maximal set of algebraically independent CR meromorphic functions in $\mathcal{K}(M)$. Then $\mathcal{K}(M)$ is a simple finite algebra-*

ic extension of the field $\mathbb{C}(f_1, f_2, \ldots, f_d)$ *of rational functions of* $f_1, f_2, \ldots, f_d$.

Setting $k = 0$ above, and taking the special case where $d = n$, we recover Satz 2 of Siegel [Si].

The theorem is a consequence of the following:

PROPOSITION 4.2. *Let* $f_1, f_2, \ldots, f_\ell$ *be CR meromorphic functions in* $\mathcal{K}(M)$. *Then there exists a positive integer* $\kappa = \kappa(f_1, f_2, \ldots, f_\ell)$ *such that every* $f_0 \in \mathcal{K}(M)$, *which is algebraically dependent on* $f_1, f_2, \ldots, f_\ell$, *satisfies a nontrivial polynomial equation of degree* $\leqslant \kappa$ *whose coefficients are rational functions of* $f_1, f_2, \ldots, f_\ell$.

PROOF. Without any loss of generality we may assume that $f_1, f_2, \ldots, f_\ell$ are algebraically independent. By Theorem 3.1 they are also analytically independent. This puts us in the situation of the second half of Theorem 3.1. The difference, however, is that we use *only* the functions $f_1, f_2, \ldots, f_\ell$ in (2.2), (2.3) and (2.4) to determine the ω_a, Ω_a. In this way the numbers m and ν' depend only on $f_1, f_2, \ldots, f_\ell$. We fix the integer $s' > m(\nu')^\ell$ as before. The proof of Theorem 3.1 shows that any *CR* meromorphic function $f \in \mathcal{K}(M)$, which can be represented on each ω_{a_i}, for $i = 1, 2, \ldots, m$, as the quotient p_i/q_i of two *CR* functions globally defined on ω_{a_i}, satisfies a nontrivial polynomial equation of degree less or equal to $\kappa = s'$, with coefficients in $\mathbb{C}(f_1, f_2, \ldots, f_\ell)$. This reduces our task to showing that f_0 has such a representation.

By hypothesis our given f_0 satisfies an equation

$$G_0 f_0^\lambda + G_1 f_0^{\lambda - 1} + \ldots + G_\lambda = 0 \tag{4.1}$$

where $G_0, G_1, \ldots, G_\lambda$ are polynomials in $f_1, f_2, \ldots, f_\ell$, and G_0 is not identically 0 in M. Let σ be an upper bound for the degrees of the $G_0, G_1, \ldots, G_\lambda$, with respect to each of the $f_1, f_2, \ldots, f_\ell$. We set

$$\widetilde{Q}_a = \prod_{j=1}^{\ell} \tilde{q}_{ja}^{\sigma}, \tag{4.2}$$

$$\widetilde{H}_{\alpha a} = \widetilde{Q}_a^\alpha \widetilde{G}_0^{\alpha - 1} \widetilde{G}_\alpha \quad (\alpha = 1, 2, \ldots, \lambda), \tag{4.3}$$

$$\widetilde{S}_a = \widetilde{Q}_a \widetilde{G}_0 \tilde{f}_0. \tag{4.4}$$

Multiplying (4.1) by $\widetilde{Q}_a^{\lambda}\widetilde{G}_0^{\lambda-1}$, we obtain that

$$\widetilde{S}^{\lambda} + \widetilde{H}_{a1}\widetilde{S}^{\lambda-1} + \ldots + \widetilde{H}_{a\lambda} = 0 \quad \text{on} \quad \Omega_a. \tag{4.5}$$

Note that $\widetilde{Q}_a\widetilde{G}_\beta$ $(\beta = 0, 1, \ldots, \lambda)$ and the $\widetilde{H}_{a\alpha}$ $(\alpha = 1, 2, \ldots, \lambda)$ are holomorphic functions on Ω_a, and $\widetilde{S}_a$ is meromorphic on an open neighborhood of ω_a in Ω_a. Since $\widetilde{S}_a$ satisfies (4.5), it is locally bounded, and hence actually holomorphic. Then the restrictions

$$p_{0a} = \widetilde{S}_a\,|_{\omega_a} \quad \text{and} \quad q_{0a} = \widetilde{Q}_a\widetilde{G}_0\,|_{\omega_a}. \tag{4.6}$$

are CR functions on ω_a, and

$$f_0 = \frac{p_{0a}}{q_{0a}} \quad \text{on} \quad \omega_a. \tag{4.7}$$

The proof is complete.

Now we explain what is the point of Theorem 4.1. Consider a maximal set $f_1, f_2, \ldots, f_d$ of algebraically independent CR meromorphic functions on M, where d is the transcendence degree of $\mathcal{K}(M)$. Consider an $f \in \mathcal{K}(M)$. Then f is algebraically dependent on $f_1, f_2, \ldots, f_d$; i.e. it satisfies an equation

$$f^{\lambda} + g_1 f^{\lambda-1} + \ldots + g_\lambda = 0\,, \tag{4.8}$$

where $g_1, g_2, \ldots, g_\lambda \in \mathbb{C}(f_1, f_2, \ldots, f_d)$. The minimal λ for which such an equation holds is called the *degree* of f over $\mathbb{C}(f_1, f_2, \ldots, f_d)$. The content of Proposition 4.2 is that this degree is bounded from above by $\kappa = \kappa(f_1, f_2, \ldots, f_d)$. Now choose an element $\Theta \in \mathcal{K}(M)$ so that its degree α is maximal. For any $f \in \mathcal{K}(M)$ consider the algebraic extension field $\mathbb{C}(f_1, f_2, \ldots, f_d, \Theta, f)$. By the primitive element theorem this extension is simple; i.e. there exists an element $h \in \mathbb{C}(f_1, f_2, \ldots, f_d, \Theta, f)$ such that $\mathbb{C}(f_1, f_2, \ldots, f_d, \Theta, f) = \mathbb{C}(f_1, f_2, \ldots, f_d, h)$. Then

$$\begin{aligned} \alpha \geqslant [\mathbb{C}(f_1, f_2, \ldots, f_d, h) : \mathbb{C}(f_1, f_2, \ldots, f_d)] &= \\ = [\mathbb{C}(f_1, f_2, \ldots, f_d, \Theta, f) : \mathbb{C}(f_1, f_2, \ldots, f_d, \Theta)] &\times \\ \times [\mathbb{C}(f_1, f_2, \ldots, f_d, \Theta) : \mathbb{C}(f_1, f_2, \ldots, f_d)] &\geqslant \alpha\,. \end{aligned} \tag{4.9}$$

Hence the first factor on the right must be one; therefore $f \in \mathbb{C}(f_1, f_2, \ldots, f_d, \Theta)$. The conclusion is that

$$\mathcal{K}(M) = \mathbb{C}(f_1, f_2, \ldots, f_d, \Theta) = \mathbb{C}(f_1, f_2, \ldots, f_d)[\Theta]\,, \tag{4.10}$$

and *any* $f \in \mathcal{K}(M)$ *can be written as a polynomial of degree* $< a$ *having coefficients that are rational functions of* $f_1, f_2, \ldots, f_d$.

From the above remark we derive the

PROPOSITION 4.3. *There is an open neighborhood* U *of* M *in* X *such that the restriction map*

$$\mathcal{K}(U) \to K(M) \tag{4.11}$$

is an isomorphism. Here $\mathcal{K}(U)$ denotes the field of meromorphic functions on U.

Let M be a connected smooth abstract *CR* manifold of type (n, k). Consider a complex *CR* line bundle $F \xrightarrow{\pi} M$ over M. Introduce the graded ring

$$\mathcal{A}(M, F) = \bigcup_{\ell=0}^{\infty} \mathcal{CR}(M, F^{\ell}), \tag{4.12}$$

where $\mathcal{CR}(M, F^{\ell})$ are the smooth global *CR* sections of the ℓ-th tensor power of F. Note that if $\sigma_1 \in \mathcal{CR}(M, F^{\ell_1})$ and $\sigma_2 \in \mathcal{CR}(M, F^{\ell_2})$, then $\sigma_1 \sigma_2 \in \mathcal{CR}(M, F^{\ell_1 + \ell_2})$.

Assume that we are in a situation where smooth sections of F have the weak unique continuation property; e.g. we could take M to be essentially pseudoconcave (see [HN8]). Then $\mathcal{A}(M, F)$ is an integral domain because M is connected. Let

$$\mathcal{Q}(M, F) = \left\{ \frac{\sigma_1}{\sigma_2} \;\middle|\; \sigma_1, \sigma_2 \in \mathcal{CR}(M, F^{\ell}) \text{ for some } \ell, \text{ and } \sigma_0 \not\equiv 0 \right\} \tag{4.13}$$

denote the field of quotients.

Then $\mathcal{Q}(M, F) \subset \widehat{\mathcal{K}}(M)$, and $\mathcal{CR}(M) = \mathcal{A}(M, \text{trivial bundle})$.

PROPOSITION 4.4. *Assume that* M *is compact and has property* E.

(1) *If* F *is locally CR trivializable, then* $\mathcal{Q}(M, F)$ *is an algebraically closed subfield of* $\mathcal{K}(M)$.

(2) *There exists a choice of a locally CR trivializable* F *such that* $\mathcal{Q}(M, F) = \mathcal{K}(M)$.

Assume that M *is compact and essentially pseudoconcave. Then*

(3) $\mathcal{Q}(M, F)$ *is algebraically closed in* $\widehat{\mathcal{K}}(M)$.

In case M *is compact and satisfies both hypothesis, then*

(4) $\mathcal{K}(M)$ *is algebraically closed in* $\widehat{\mathcal{K}}(M)$.

PROOF. To prove (1) [or (3)] we take an $h \in \mathcal{K}(M)$ [or $h \in \widehat{\mathcal{K}}(M)$] which is algebraic over $\mathcal{Q}(M, F)$; i.e. h satisfies an equation of minimal degree

$$h^\mu + k_1 h^{\mu-1} + \ldots + k_\mu \equiv 0 , \tag{4.14}$$

where $k_i \in \mathcal{Q}(M, F)$. Let $k_i = \frac{s_i}{t_i}$ with $s_i, t_i \in \mathcal{CR}(M, F^{\ell_i})$. Multiplying by $\sigma_0 = \prod_{i=1}^{\mu} t_i$ we obtain

$$\sigma_0 h^\mu + \sigma_1 h^{\mu-1} + \ldots + \sigma_\mu \equiv 0 , \tag{4.15}$$

where $\sigma_0 \not\equiv 0$ and $\sigma_i \in \mathcal{CR}(M, F^\ell)$ for $\ell = \sum_{i=1}^{\mu} \ell_i$. Multiplying by $\sigma_0^{\mu-1}$ we have:

$$\begin{aligned} P(\sigma_0 h) &= \\ &= (\sigma_0 h)^\mu + \sigma_1 (\sigma_0 h)^{\mu-1} + \ldots + \sigma_0^{\mu-2} \sigma_{\mu-2} (\sigma_0 h) + \sigma_0^{\mu-1} \sigma_\mu \equiv 0 . \end{aligned} \tag{4.16}$$

Note that $(\sigma_0 h)$ is bounded, and hence is a smooth section of F^ℓ over M. In a local smooth trivialization of the bundle F, the tangential Cauchy-Riemann operator on sections s of F has the form (see [HN8])

$$\bar\partial_M^F s = \bar\partial_M s + As , \tag{4.17}$$

where A is a smooth $\bar\partial_M$-closed $(0, 1)$ form on M. For the ℓ-th tensor power F^ℓ of F, in the same trivialization, we have

$$\bar\partial_M^{F^\ell} s = \bar\partial_M s + \ell As . \tag{4.18}$$

We apply $\bar\partial_M^{F^{\ell\mu}}$ to both sides of (4.16) and obtain

$$\begin{aligned} 0 = \bar\partial_M^{F^{\ell\mu}} P(\sigma_0 h) &= P'(\sigma_0 h)[\bar\partial_M (\sigma_0 h) + \ell A \cdot (\sigma_0 h)] = \\ &= P'(\sigma_0 h)\, \bar\partial_M^{F^\ell} (\sigma_0 h) . \end{aligned} \tag{4.19}$$

Because μ is minimal in (4.14), we obtain that $P'(\sigma_0 h) \not\equiv 0$, and therefore $\tau = \sigma_0 h \in \mathcal{CR}(M, F^\ell)$. Since $\sigma_0 \in \mathcal{CR}(M, F^\ell)$, we obtain that $h = \dfrac{\tau}{\sigma_0} \in \mathcal{Q}(M, F)$. This completes the proof of (1) and (3).

To prove (2) it suffices to observe that it is possible to choose a locally trivializable smooth complex CR line bundle F over M such that the $f_1, f_2, \ldots, f_d, \Theta$ appearing in (4.10) belong to $\mathcal{Q}(M, F)$. Then by (4.10), $\mathcal{Q}(M, F) = \mathcal{K}(M)$.

Finally (4) is a consequence of (2) and (3). This completes the proof of the proposition.

Let M be a connected smooth compact *CR* manifold of type (n, k), having property E. Then

THEOREM 4.5. *Let* $F \xrightarrow{\pi} M$ *be a locally* $\mathcal{CR}$ *trivializable smooth complex CR line bundle over* M. *Then*

$$\dim_{\mathbb{C}} \mathcal{CR}(M, F) < \infty. \tag{4.20}$$

PROOF. For each point $a \in M$, we fix an open neighborhood ω_a of a in M, and an open coordinate neighborhood Ω_a of a in X, such that $\omega_a = \Omega_a \cap M$ and the restriction map $\mathcal{O}(\Omega_a) \to \mathcal{CR}(\omega_a)$ is an isomorphism. These ω_a are to be chosen to give a local *CR* trivialization of F. Denote the $\mathcal{CR}$ transition functions by g_{ab}. We may assume that the g_{ab} are bounded on $\omega_a \cap \omega_b$. Introduce the polydiscs K_a and L_a as in (2.5), and choose $a_1, a_2, \dots, a_m$ on M so that the $L_{a_1}, L_{a_2}, \dots, L_{a_m}$ provide an open covering of M. Choose an integer μ such that

$$|g_{ab}| < e^{\mu} \quad \text{on} \quad \omega_a \cap \omega_b \tag{4.21}$$

for $a, b = a_1, a_2, \dots, a_m$. Consider a section $s \in \mathcal{CR}(M, F)$ which vanishes at each point a_i of order $\geqslant \mu + 1$. The section s is represented by a smooth *CR* function s_i on each ω_{a_i}. Let

$$\Upsilon = \max_{1 \leqslant i \leqslant m} \max_{K_{a_i}} |\tilde{s}_i|, \tag{4.22}$$

as before. This maximum is attained at some point z^* belonging to some K_{a^*}, where a^* is one of the $a_1, a_2, \dots, a_m$. Then by (iv) in Corollary 1.2, there is some $z^{**} \in \omega_{a^*}$ such that

$$\tilde{s}_{a^*}(z^*) = s_{a^*}(z^{**}). \tag{4.23}$$

But z^{**} belongs to some $L_{a^{**}}$, where a^{**} is one of the $a_1, a_2, \dots, a_m$. By the Schwarz lemma,

$$|s_{a^{**}}(z^{**})| \leqslant \Upsilon e^{-(\mu+1)}. \tag{4.24}$$

On the other hand we have

$$\Upsilon = |s_{a^*}(z^{**})| = |g_{a^* a^{**}}(z^{**})| \cdot |s_{a^{**}}(z^{**})| \leqslant M e^{-1}. \tag{4.25}$$

This shows that the Taylor expansion of the representatives s_i, at a_i, up to order μ, completely determine a section s. Hence $\dim_{\mathbb{C}} \mathcal{CR}(M, F) \leqslant m \binom{n+k+\mu+1}{\mu+1} < \infty$.

5. The Chow theorem for *CR* manifolds.

Again M is a connected smooth compact *CR* manifold of type (n, k), having property E. We have:

THEOREM 5.1. *Let $\tau: M \to \mathbb{CP}^N$ be a smooth CR map. Suppose that τ has maximal rank $2n+k$ at one point of M. Then $\tau(M)$ is contained in an irreducible algebraic subvariety of complex dimension $n+k$, and the transcendence degree of $\mathcal{K}(M)$ over $\mathbb{C}$ is $n+k$.*

PROOF. Let Y be the smallest algebraic subvariety of $\mathbb{C}P^N$ containing $\tau(M)$. Certainly Y exists and is irreducible; it is defined by the homogeneous prime ideal

$$\mathcal{P}_Y = \{p \in \mathbb{C}_0[z_0, z_1, \ldots, z_N] \,|\, p \circ \tau = 0\}, \tag{5.1}$$

where $\mathbb{C}_0[z_0, z_1, \ldots, z_N]$ is the graded ring of homogeneous polynomials on $\mathbb{CP}^N$.

Let $\mathcal{R}(Y)$ be the field of rational functions on Y. Any element $f \in \mathcal{R}(Y)$ is represented as the quotient of two homogeneous polynomials $f = p/q$, with $q \notin \mathcal{P}_Y$. If $f = p/q = p'/q'$, then $pq' - p'q \in \mathcal{P}_Y$. By (5.1) this shows that $f \circ \tau$ is a well defined *CR* meromorphic function in $\mathcal{K}(M)$, and that the homomorphism

$$\tau^*: \mathcal{R}(Y) \to K(M) \tag{5.2}$$

is injective. By the assumption that τ has maximum rank at one point, and by Theorem 2.1, we obtain

$$\begin{aligned} n + k \leqslant \dim_{\mathbb{C}} Y &= \text{transcendence degree of } \mathcal{R}(Y) \\ &\leqslant \text{transcendence degree of } \mathcal{K}(M) \leqslant n + k\,. \end{aligned} \tag{5.3}$$

This completes the proof of the theorem.

As a corollary we obtain for compact *CR* manifolds an analogue of Chow's theorem:

THEOREM 5.2. *Let M be a connected smooth compact CR embedded submanifold of $\mathbb{CP}^N$, of type (n, k) and having property E. Then M is a generically embedded CR submanifold of an irreducible algebraic subvariety Y of $\mathbb{CP}^N$; moreover M is contained in the set* $\operatorname{reg} Y$ *of regular points of Y, and the map* (5.2) *is an isomorphism.*

Setting $k = 0$ above, we recover the theorem of Chow [C].

PROOF. We verify that M avoids any singularity of Y. Consider a point $x_0 \in M$. We can assume that near x_0, the inhomogeneous coordinates are $z_1, z_2, \dots, z_N$, centered at x_0, and that $z_1, z_2, \dots, z_{n+k}$ are coordinates for the smallest affine complex linear subspace containing $T_{x_0}M$. Let $\pi : \mathbb{C}^N \to \mathbb{C}^{n+k}$ be the holomorphic projection onto this affine subspace. Then locally near x_0, $M' = \pi(M)$ is a smooth generic *CR* submanifold of $\mathbb{C}^{n+k}$, having property E. But $\phi = (\pi|_M)^{-1}$ is a *CR* map on M', so by property E it has a local holomorphic extension $\tilde{\phi}$ to a neighborhood ω of $\pi(x_0)$ in $\mathbb{C}^{n+k}$. Then $z_j - \tilde{\phi}_j(z_1, z_2, \dots, z_{n+k})$, for $j = n+k+1, \dots, N$, give $N-(n+k)$ holomorphic functions in $\pi^{-1}(\omega)$ which vanish locally on M near x_0, and have independent differentials. In particular they define germs in $\mathcal{P}_Y \otimes \mathcal{O}_{x_0}$. Using the fact that $\mathcal{O}_{x_0}$ is faithfully flat over the ring $\mathbb{C}[z_1, z_2, \dots, z_N]$, we obtain that Y is locally a smooth complete intersection at x_0. Thus $M \subset \operatorname{reg} Y$.

Since (5.2) is injective, $\mathcal{K}(M)$ has a transcendence basis $f_1, f_2, \dots, f_{n+k}$ with each $f_i = \tilde{f}_i|_M$, where $f_i \in \mathcal{R}(Y)$. Thus any $f \in \mathcal{K}(M)$ is the solution of an irreducible algebraic equation of the form

$$f^\lambda + g_1 f^{\lambda-1} + \dots + g_\lambda = 0\,, \tag{5.4}$$

with coefficients $g_1, g_2, \dots, g_\lambda \in \mathcal{R}(Y)$. Take a point $x_0 \in M$ where $g_1, g_2, \dots, g_\lambda$ are smooth *CR* functions, and

$$\xi^\lambda + g_1 \xi^{\lambda-1} + \dots + g_\lambda = 0\,, \tag{5.5}$$

has λ distinct complex roots. Since $\mathcal{R}(Y)$ is algebraically closed, there is some root $\tilde{f} \in \mathcal{R}(Y)$ of (5.4) such that $\tilde{f}(x_0) = f(x_0)$. As the roots of (5.5) depend continuously on the coefficients, we have that $\tilde{f}(x) = f(x)$ for x in a neighborhood of x_0 on M. By unique continuation, $f = \tilde{f}|_M$. This completes the proof of the theorem.

6. Projective embedding.

Let M be a connected smooth compact *CR* manifold of type (n, k), having property E. Then

THEOREM 6.1. *The following are equivalent*:

(1) *M has a smooth CR embedding as a CR submanifold of some* $\mathbb{CP}^N$.

(2) *There exists over M a smooth complex CR line bundle F such that the graded ring $\mathcal{A}(M, F) = \bigcup_{\ell=0}^{\infty} \mathcal{CR}(M, F^{\ell})$ separates points and gives «local coordinates» at each point of M.*

PROOF. First we explain the meaning of (2). To say that $\mathcal{A}(M, F)$ separates points means that, given $x \neq y$ on M, there exists an integer $\ell = \ell(x, y) > 0$, and smooth sections $s_0, s_1 \in \mathcal{CR}(M, F^{\ell})$ such that

$$\det \begin{bmatrix} s_0(x) & s_0(y) \\ s_1(x) & s_1(y) \end{bmatrix} \neq 0 . \tag{6.1}$$

To say that $\mathcal{A}(M, F)$ gives «local coordinates» means that, given x in M, there exists an integer $\ell = \ell(x) > 0$ and smooth sections $s_0, s_1, \dots, s_{n+k} \in \mathcal{CR}(M, F^{\ell})$ such that

$$\sum_{i=0}^{n+k} (-1)^i s_i \, ds_0 \wedge \dots \wedge \widehat{ds_i} \wedge \dots \wedge ds_{n+k} \neq 0 \text{ at } x . \tag{6.2}$$

In other words, the *CR* meromorphic function s_1/s_0 takes *different values* at x and y; and (assuming $s_0(x) \neq 0$) the *CR* meromorphic functions $s_1/s_0, s_2/s_0, \dots, s_{n+k}/s_0$ provide analytically independent local *CR* functions at x. Note that a line bundle F satisfying (2) is locally *CR* trivializable; hence also F^{ℓ}.

First we show that (2) implies (1): By row reduction we may assume that the matrix (6.1) is in diagonal form; i.e. $s_0(y) = s_1(x) = 0$ and $s_0(x) \cdot s_1(y) \neq 0$. Hence, for any $m > 0$,

$$\det \begin{bmatrix} s_0^m(x) & s_0^m(y) \\ s_1^m(x) & s_1^m(y) \end{bmatrix} \neq 0 . \tag{6.3}$$

This shows that if two points x and y are separated by $s_0, s_1 \in \mathcal{CR}(M, F^{\ell})$, they are also separated by $s_0^m, s_1^m \in \mathcal{CR}(M, F^{\ell m})$. Next we consider sections $s_0, s_1, \dots, s_{n+k} \in \mathcal{CR}(M, F^{\ell})$ which satisfy (6.2) at a given point $x \in M$. We can assume $s_0(x) \neq 0$. By changing s_i to $s_i - (s_i(x)/s_0(x)) s_0$, we can also assume that $s_i(x) = 0$ for $i = 1, 2, \dots, n+k$. Then for any $m > 0$, we have that $s_0^m, s_0^{m-1}s_1, \dots, s_0^{m-1}s_{n+k} \in \mathcal{CR}(M, F^{\ell m})$ also satisfy (6.2). For each $\ell > 0$ we define the following subset of $M \times M$:

$$\mathcal{U}_{\ell} = \{(x, y) \mid \exists (s_0, s_1) \in \mathcal{CR}(M, F^{\ell}) \text{ such that (6.1) holds}\} \cup \tag{6.4}$$
$$\cup \{(x, x) \mid \exists s_0, s_1, \dots, s_{n+k} \in \mathcal{CR}(M, F^{\ell}) \text{ such that (6.2) holds}\} .$$

Each $\mathcal{U}_\ell$ is open in $M \times M$, and $\mathcal{U}_\ell \subset \mathcal{U}_{\ell \cdot m}$ for $m > 0$. By our hypothesis (2) the $\{\mathcal{U}_\ell\}$ give an open covering of $M \times M$. By compactness $M \times M$ is covered by $\mathcal{U}_{\ell_1}, \mathcal{U}_{\ell_2}, \dots, \mathcal{U}_{\ell_r}$. Therefore $M \times M = \mathcal{U}_{\ell_0}$ where $\ell_0 = \ell_1 \cdot \ell_2 \dots \ell_r$. Now let $s_0, s_1, \dots, s_N$ be a basis for smooth sections of the complex vector space $\mathcal{CR}(M, F^{\ell_0})$, which is finite dimensional by Theorem 4.5. Consider the map $\tau : M \to \mathbb{CP}^N$ defined by $x \to (s_0(x) : s_1(x) : \dots : s_N(x))$. It is a *CR* map, which is one-to-one, and an immersion at each point; thereby giving a *CR* embedding of M into $\mathbb{CP}^N$.

Next we show that (1) implies (2): It suffices to take F equal to the pull-back to M of the hyperplane section line bundle on $\mathbb{CP}^N$. With this F we may take $\ell = 1$ in (6.1) and (6.2).

This completes the proof.

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