REND. SEM. MAT. UNIV. PADOVA, Vol. 111 (2004)

On Unions of Scrolls Along Linear Spaces.

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Introduction.

According to the classification resulting from the successive contributions by Bertini [2], Del Pezzo [4] and Xambó [9], the equidimensional varieties of minimal degree which are connected in codimension one are of three types: quadric hypersurfaces, cones over the Veronese surface in $\boldsymbol{P}^5$ and unions of scrolls embedded in linear subspaces. In this paper we give a complete constructive characterization of the ideals defining varieties of the latter type, which were presented in [9]. We also show that for these varieties, equidimensionality and minimal degree imply connectivity in codimension one, which provides a better understanding of the results in [9]. Finally we give a complete description of all rulings of a scroll. Throughout the paper we deal with projective varieties not contained in any hyperplane.

1. Preliminaries.

Let K be an algebraically closed field, and let $\underline{T} = \{T_0, \ldots, T_n\}$ be a finite set of variables over K. Let $R = K[\underline{T}]$ be the corresponding polynomial ring. For a subset S of R, by $\langle S \rangle$ we shall denote the linear subspace of R generated by S. If A is a matrix with entries in R, we shall use the notation $\langle A \rangle$ for the linear subspace generated by the set of all entries of A. We recall some basic definitions.

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A simple scroll matrix will be a matrix of the form

$$\begin{pmatrix} l_0 & l_1 & \dots & l_{m-1} \\ l_1 & l_2 & \dots & l_m \end{pmatrix},$$

where $l_0, \dots, l_m$ are linearly independent linear elements of R.

A *scroll matrix* will be a matrix of the form

$$(\beta_1 | \beta_2 | \dots | \beta_s),$$

where for all $i = 1, \dots, s$, the submatrix β_i is a simple scroll matrix and

$$\langle \beta_i \rangle \cap \left(\sum_{j \neq i} \langle \beta_j \rangle \right) = 0 .$$

A projective variety defined by the vanishing of the 2-minors of a scroll matrix will be called a *scroll.*

Let J be a reduced ideal of R having an irredundant prime decomposition

$$J = \bigcap_{i=1}^{r} J_i.$$

DEFINITION 1.1. A reducible variety $X \subset \boldsymbol{P}^n$ is a scroller if $\boldsymbol{P}^n$ contains linear subspaces $L_1, \dots, L_r$ and there are scrolls $X_i \subseteq L_i$ such that $X = \bigcup_{i=1}^{r} X_i$, and for each $k = 1, \dots, r$ we have

$$(*) \qquad X_k \cap (X_1 \cup \dots \cup X_{k-1}) = L_k \cap (\overline{L_1 \cup \dots \cup L_{k-1}}).$$

Our first aim is to provide an explicit description of the ideal $J \subset R$ of a scroller X. This extends our previous result in [1]. By definition J can be written as $J = \bigcap_{i=1}^{r} J_i$, where J_i is the defining ideal of X_i, for all $i = 1, \dots, r$. Then

$$J_i = (M_i, Q_i),$$

where

– Q_i is a set of linear forms defining L_i, and

– M_i is the set of all 2-minors of a scroll matrix B_i consisting of c_i columns: (M_i) is the defining ideal of the scroll X_i in its space of immersion L_i. If $c_i = 1$, the set M_i is empty. In this case the ideal J_i is generated by linear forms.

Note that the entries of B_i can be considered as a system of coordinates of L_i. In particular

$$\langle B_i \rangle \cap \langle Q_i \rangle = 0 .$$

Up to replacing R with a polynomial ring $S \subseteq R$ we may assume that

$$\bigcap_{i=1}^{r} \langle Q_i \rangle = 0 .$$

Of course the entries of B_i are defined up to linear combination with the elements of Q_i. Any such modification – which of course leaves the ideal J_i untouched – will be called an *admissible change*.

Condition $(*)$ in Definition 1.1 can be re-formulated as follows:

$$(**) \qquad J_k + \bigcap_{i=1}^{k-1} J_i = (Q_k) + \left(\bigcap_{i=1}^{k-1} \langle Q_i \rangle \right)$$

for all $k = 1, \dots, r-1$. In the sequel we shall stick to the notation just introduced. We are now ready to state our main theorem.

2. The Main Theorem.

THEOREM 2.1. *The following two conditions are equivalent*:

(I) *J is the ideal of a scroller $X \subset \boldsymbol{P}^n$.*

(II) *There exist, for all $i = 1, \dots, r$, two subsets D_i, P_i of $\langle Q_i \rangle$ such that*

$$\langle P_i \rangle \oplus \langle D_i \rangle = \langle Q_i \rangle ,$$

and the following axioms are satisfied:

(a) *$D_1 \supseteq D_2 \supseteq \dots \supseteq D_r = \emptyset$, and $\langle D_1 \rangle = \langle Q_1 \rangle$; and, up to admissible changes for $B_1, \dots, B_r$, one has that*

(b) *$M_i \subseteq (D_{i-1})$ for all $i = 2, \dots, r$, and*

(c) *$M_i \subseteq (P_j)$ for all $i = 1 \dots, r-1$ and all $j = i+1, \dots, r$.*

(d) *$\bigcap_{i=1}^{k-1} (Q_i) \subseteq (P_k, D_{k-1})$ for all $k = 2, \dots, r$.*

We prove this Theorem in several steps. The first auxiliary result generalizes Prop. 5.1 in [6], and is taken from [1]. We refer to that paper for the proof.

LEMMA 2.2. *Assume that the axioms* (a), (b) *and* (c) *of* 2.1 *are satisfied. Then*

$$\bigcap_{i=1}^{k} J_i = \left(M_1, \ldots, M_k, \bigcap_{i=1}^{k} (Q_i)\right)$$

for all $k = 1, \ldots r$.

PROOF OF 2.1 «(II)$\Rightarrow$(I)». In view of 2.2, for all $k = 1, \ldots, r-1$ we have

$$J_k + \bigcap_{i=1}^{k-1} J_i = \left(M_1, \ldots, M_{k-1}, M_k, \bigcap_{i=1}^{k-1} (Q_i), P_k, D_k\right) = (M_k, P_k, D_k) + \bigcap_{i=1}^{k-1} (Q_i)$$

$$\subseteq (P_k, D_k, D_{k-1}) = (P_k, D_{k-1})$$

where the equalities follow easily from (a), (b) and (c), and the inclusion is a consequence of (d). Since by (a)

$$D_{k-1} \subseteq \bigcap_{i=1}^{k-1} (Q_i) \subseteq \bigcap_{i=1}^{k-1} J_i,$$

and $P_k \subseteq J_k$, the above inclusion can be reversed, so that

$$J_k + \bigcap_{i=1}^{k-1} J_i = (P_k, D_{k-1}) = (Q_k) + (D_{k-1}) \subseteq (Q_k) + \left(\bigcap_{i=1}^{k-1} \langle Q_i \rangle\right).$$

By d) this suffices to conclude that $(**)$ is fulfilled.

Now we turn to the proof of the other implication.

DEFINITION 2.3. Let $D_1 \supseteq D_2 \supseteq \ldots \supseteq D_r$ be a chain of sets such that for all $i = 1, \ldots, r$, the set D_i is a basis of $\bigcap_{j=1}^{i} \langle Q_i \rangle$. In particular $D_r = \emptyset$. For all $i = 2, \ldots, r$ let Δ_i be a set completing D_i to a basis of $\langle D_{i-1} \rangle$, and for all $i = 1, \ldots, r$ let P_i be a set of linear forms completing D_i to a basis of $\langle Q_i \rangle$. In particular $P_1 = \emptyset$.

COROLLARY 2.4. *For all* $i = 2, \ldots, r$, $\langle P_i \rangle \cap \langle D_{i-1} \rangle = 0$.

PROOF. By Definition 2.3 one has that

$$\langle P_i \rangle \cap \langle D_{i-1} \rangle \subseteq \langle Q_i \rangle \cap \left(\bigcap_{j=1}^{i-1} \langle Q_j \rangle\right) = \bigcap_{j=1}^{i} \langle Q_j \rangle = \langle D_i \rangle,$$

but $\langle P_i \rangle \cap \langle D_i \rangle = 0$. ∎

As a consequence of Corollary 2.4, for all $i = 2, \dots, r$ there is a decomposition

$$\langle Q_i, D_{i-1} \rangle = \langle D_i \rangle \oplus \langle \Delta_i \rangle \oplus \langle P_i \rangle,$$

where $D_i \cup \Delta_i$ is a basis of $\langle D_{i-1} \rangle$. It follows that

$$(1) \qquad \langle D_i \rangle = \bigoplus_{j=i+1}^{r} \langle \Delta_j \rangle \quad \text{for all } i = 1, \dots, r-1.$$

Moreover, we can rewrite condition $(**)$ as follows:

$$(**) \qquad J_k + \bigcap_{i=1}^{k-1} J_i = (Q_k, D_{k-1}) = (P_k, D_{k-1}) = (P_k, \Delta_k, D_k)$$

REMARK 2.5. Axioms (a) and (d) immediately follow from Definition 2.3 and $(**)$ respectively.

LEMMA 2.6. *Assume condition* $(**)$ *is fulfilled. Then there are admissible changes for* $B_1, \dots, B_r$ *such that* $M_i \subseteq (\Delta_i)$ *for all* $i = 2, \dots, r$. *In particular axiom* (b) *is satisfied.*

PROOF. Fix an index $i \in \{2, \dots, r\}$. Condition $(**)$ implies that

$$(2) \qquad M_i \subseteq J_i \subseteq (D_i, \Delta_i, P_i).$$

Choose a system of variables $\underline{T}$ such that $D_i \cup \Delta_i \cup P_i \subseteq \underline{T}$. For all entries x of B_i write $x = u + v$, where

$$u = \sum_{T \in D_i \cup P_i} \alpha_T T \quad \text{and} \quad v = \sum_{T \in \underline{T} \setminus (D_i \cup P_i)} \alpha_T T, \quad (\alpha_T \in K).$$

Replace x by v. This is an admissible change for B_i, because $u \in \langle Q_i \rangle$. After this operation, condition (2) for the modified set M_i implies that $M_i \subseteq (\Delta_i)$, because no variable of $D_i \cup P_i$ appears in the polynomials belonging to M_i. ■

Now the sets D_i are completely determined. Next we modify the sets P_i introduced above. To this end we shall again resort to admissible changes of $B_1, \dots, B_r$. Simultaneously we shall have to modify the sets Δ_i (and, consequently, the sets D_i) in order to preserve the validity of the claim of Lemma 2.6. These modifications will also be called *admissible changes*.

LEMMA 2.7. *Assume* (I) *is true. For all* $i = 1, \dots, r$ *we can choose* P_i *in such a way that after suitable admissible changes for* $B_1, \dots, B_r$ *and* $\Delta_2, \dots, \Delta_r$, $M_i \subseteq (P_j)$ *for all* $i = 1, \dots, r-1$ *and all* $j = i+1, \dots, r$.

PROOF. We define P_i and the subsets Δ_i for $i=2, \dots, r$ by recursion. We proceed by induction on $k=1, \dots, r-1$ showing that there are subsets $P_1, \dots, P_k$ and admissible changes for $B_1, \dots, B_k$ and $\Delta_2, \dots, \Delta_k$ such that the claim is fulfilled for $i \leqslant k-1$ and $i<j \leqslant k$. At the k-th step we assume that this is true, and choose P_{k+1} in such a way that the claim is true for $i=k$ and $j=k+1$. To this end we shall perform admissible changes on $B_1, \dots, B_k$ and $\Delta_2, \dots, \Delta_k$, and also suitable modifications on $P_1, \dots, P_k$, so that the claim will finally be true for $i \leqslant k$ and $i<j \leqslant k+1$.

Let P_{k+1} be as required in 2.3. In view of Corollary 2.4, there is a system of variables $\underline{T}$ such that $P_{k+1} \cup D_k \subseteq \underline{T}$. Let $i \in \{2, \dots, k\}$ and $x \in P_i$. Write $x=u+v$, where

$$u = \sum_{T \in D_k} \alpha_T T \quad \text{and} \quad v = \sum_{T \in \underline{T} \setminus D_k} \alpha_T T, \quad (\alpha_T \in K).$$

Replace x by v in P_i. Perform this substitution for all $x \in P_i$ and all entries x of B_i, for $i=1, \dots, k$. Since $D_k \subseteq D_i \subseteq (Q_i)$, these substitutions are admissible changes and respect the definition of P_i. By Lemma 2.6, $M_i \subseteq \subseteq (D_{i-1}) \subseteq (D_j) \subseteq J_j$ for all $j=1, \dots, i-1$, and this condition is preserved by the changes, since $(D_k) \subseteq (D_{i-1})$. Similarly, it remains true that $M_i \subseteq \subseteq (P_j) \subseteq J_j$, for all $j=i+1, \dots, k$, so that, in view of $(**)$:

$$M_i \subseteq \bigcap_{j=1}^{k} J_j \subseteq (P_{k+1}, D_k)\,.$$

But by construction the elements of M_i do not contain any variable from D_k. Hence $M_i \subseteq (P_{k+1})$. Furthermore, by Lemma 2.6, for all $i=1, \dots, k-1$, it holds that $M_k \subseteq (D_{k-1}) \subseteq (D_i)$. Moreover $M_k \subseteq J_k$. By virtue of $(**)$ it follows that

$$M_k \subseteq \bigcap_{j=1}^{k} J_j \subseteq (P_{k+1}, D_k)\,,$$

which implies that $M_k \subseteq (P_{k+1})$, since the elements of M_k do not contain any variable of the subset D_k. This proves the claim for $i \leqslant k$ and $j=k+1$. ■

Theorem 2.1 is completely proven now: the required subsets D_i and P_i are those fulfilling the claim of Lemma 2.7.

Next we prove that every equidimensional scroller of minimal degree is connected in codimension 1. Recall that, according to the definition given by Hartshorne [8], J is *connected in codimension 1* if – up to rearranging the indices – for all $k = 1, \dots, r$ it holds:

$$\operatorname{codim}\left(J_k + \bigcap_{i=1}^{k-1} J_i\right) = \operatorname{codim} J_k + 1 .$$

LEMMA 2.8. *If M is the set of maximal minors of a scroll matrix with $c \geqslant 2$ columns, and $L_1, \dots, L_n$ are n linear forms such that $M \subseteq \subseteq (L_1, \dots, L_n)$, then $n \geqslant c$.*

PROOF. If $M \subset (L_1, \dots, L_n)$, then $V(L_1, \dots, L_n) \subset V(M)$, where $V(M)$ is the variety defined by M. But $\operatorname{codim} V(M) = c - 1$, and $\operatorname{codim} V(L_1, \dots, L_n) = n$. It follows that $n \geqslant c - 1$. In fact, since $V(M)$ is irreducible and $V(M) \neq V(L_1, \dots, L_n)$, we must have $n > c - 1$, so that $n \geqslant c$. ■

This lemma also follows from the results we will give in Section 4.

The following result is quoted from Eisenbud-Goto ([5], Th. 4.2 and 4.3):

THEOREM 2.9 (Del Pezzo-Bertini-Xambó). *Let J be a reduced ideal of R. Suppose it is connected in codimension 1, and it has pure dimension d. Let X be the variety of $\boldsymbol{P}^n$ defined by J. Let* $\deg J$ *denote the degree of J. If* $\deg J \leqslant \operatorname{codim} J + 1$, *then J is Cohen-Macaulay, and either*:

(1) *X is a quadric hypersurface; $R/J = K[T_0, \dots, T_n]/(Q)$, for some quadratic polynomial Q;*

(2) *X is a cone over the Veronese surface in $\boldsymbol{P}^5$; R/J is isomorphic to a polynomial ring over $K[T_{00}, T_{01}, T_{02}, T_{11}, T_{12}, T_{22}]$ modulo the ideal of 2-minors of the generic symmetric matrix:*

$$\begin{bmatrix} T_{00} & T_{01} & T_{02} \\ T_{01} & T_{11} & T_{12} \\ T_{02} & T_{12} & T_{22} \end{bmatrix},$$

or

(3) *$\boldsymbol{P}^n$ contains linear subspaces $L_1, \dots, L_r$ and there are d-dimensional scrolls $X_i \subseteq L_i$ such that $X = \bigcup_{i=1}^{r} X_i$, and for each $k = 1, \dots, r$ we*

have

$$(*)\qquad X_k \cap (X_1 \cup \ldots \cup X_{k-1}) = L_k \cap (\overline{L_1 \cup \ldots \cup L_{k-1}}),$$

which is a linear subspace of dimension $d-1$.

We refer to the paper of Xambó [9] for a proof.

The next result shows that the hypothesis of connectivity in codimension 1 in the above characterization is not restrictive for varieties of type (3). In fact we have:

COROLLARY 2.10. *A scroller* $X \subset \boldsymbol{P}^n$ *of pure dimension and minimal degree is connected in codimension 1.*

PROOF. Let d be the pure dimension of X, and let $J = \bigcap_{i=1}^{r} J_i$ be its defining ideal. Then, for all $i = 1, \ldots r$,

$$(3)\qquad n-d = \operatorname{codim} J_i = |P_i| + |D_i| + c_i - 1\,,$$

and, in particular,

$$(4)\qquad n-d = \operatorname{codim} J_1 = |Q_1| + c_1 - 1\,,$$

for all $i = 1, \ldots, r$. It is well known that the degree of the cylinder over a scroll is equal to the number of its columns (see [7]). Hence for all $i = 1, \ldots, r$ one has that

$$\deg J_i = c_i,$$

and

$$(5)\qquad \deg J = \sum_{i=1}^{r} \deg J_i = \sum_{i=1}^{r} c_i.$$

Now, by Lemma 2.6 and Lemma 2.8, $|\Delta_i| \geqslant c_i$ for all $i = 2, \ldots, r$. Hence, by (4), and in view of Definition 2.3,

$$n-d+1 = |Q_1| + c_1 = \sum_{i=2}^{r} |\Delta_i| + c_1 \geqslant \sum_{i=1}^{r} c_i.$$

This, together with (5) and minimality of degree, yields

$$|\Delta_i| = c_i$$

for all $i = 2, \ldots, r$. Finally, by (**), and (3)

$$\operatorname{codim}\left(J_k + \bigcap_{i=1}^{k-1} J_i\right) = |P_k| + |D_k| + |\Delta_k| = n-d+1-c_k+c_k = n-d+1\,,$$

which proves connectivity in codimension 1. ■

EXAMPLE 2.11. Let

$$B_1 = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad B_2 = \begin{pmatrix} d & e \\ e & g \end{pmatrix}, \quad B_3 = \begin{pmatrix} h & i & j \\ i & j & k \end{pmatrix}.$$

Then

$$\begin{aligned} J_1 &= (ac - b^2, d, e, h, i, j, k) \\ J_2 &= (dg - e^2, b, c, h, i, j, k) \\ J_3 &= (hj - i^2, hk - ij, ik - j^2, a, b, d, e) \end{aligned}$$

are linear scrolls associated to B_1, B_2 and B_3 respectively, where

$$\begin{aligned} & && D_1 = (d, e, h, i, j, k) \\ & P_2 = (b, c) && D_2 = (h, i, j, k) \\ & P_3 = (a, b, d, e) && \end{aligned}$$

It is easy to verify that part (II) of Theorem 2.1 is fulfilled. Hence ideal

$$\begin{aligned} J = J_1 \cap J_2 \cap J_3 = \\ = (be, ce, eh, ei, ej, ek, bd, cd, dh, di, dj, dk, bh, bi, bj, bk, ah, ai, \\ aj, ak, b^2 - ac, e^2 - dg, i^2 - hj, ij - hk, j^2 - ik) \end{aligned}$$

defines a scroller $X \subset \boldsymbol{P}^{10}$. Note that J_1 and J_2 are of degree 2 and codimension 7, J_3 is of degree 3 and codimension 6, so that J is of degree 3 and codimension 6.

In the next Section we will show that the ideals of all scrollers are generated by elements of degree 2.

3. A constructive method.

Our next aim is to give an explicit constructive method for the defining ideals of scrollers. This will be a generalization of the one described in our previous paper [1], to which we refer for a proof of the next auxiliary result.

LEMMA 3.1. *Assume* (I) *is true. For all* $i = 1, \ldots, r$, *a set* P_i *fulfilling Definition* 2.3 *can be chosen in such a way that after suitable admissible changes for* $B_1, \ldots, B_r$ *and* $\Delta_2, \ldots, \Delta_r$ *preserving the statement of Lemma* 2.6, *the following properties are satisfied.*

(i) *For all* $j = 2, \ldots, r$ *let*

$$P_j \times \Delta_j = \{p\delta \mid p \in P_j,\ \delta \in \Delta_j\}.$$

Set $G_1 = \emptyset$, *and*

$$G_i = \bigcup_{j=1}^{i} P_j \times \Delta_j$$

for all $j = 2, \ldots, r$. *Then* $G_i \subsetneq (P_{i+1})$ *for all* $i = 1, \ldots, r-1$

(ii) *For all* $i = 1, \ldots, r-1$ *there is an index* $l(i+1)$, $1 \leq l(i+1) \leq i$, *such that*

$$\langle P_{i+1} \rangle \supseteq \langle \Pi_{i+1} \rangle + \langle P_{l(i+1)} \rangle \oplus \langle \Delta_{l(i+1)+1} \rangle \oplus \ldots \oplus \langle \Delta_i \rangle,$$

where Π_{i+1} *is a subset of* $\langle Q_{i+1} \rangle$ *for which* $M_{l(i+1)} \subsetneq (\Pi_{i+1})$.

We are now ready for the required characterization: we show that, given scroll matrices $B_1, \ldots, B_r$, whenever the sets Δ_i, D_i, P_i and Q_i are chosen so as to fulfil Definition 2.3, Lemma 2.6 and Lemma 3.1 (ii), then the resulting ideal J defines a scroller. By virtue of Theorem 2.1 and Remark 2.5, there remains to prove that (d) and (e) are satisfied. But this can be easily settled by induction on i, as we have done in [1].

The next result, together with Lemma 2.2, yields an explicit description of the generators of ideal J.

PROPOSITION 3.2. *For all* $k = 2, \ldots r+1$ *one has that* $G_{k-1} \subsetneq (P_k)$. *Moreover it holds*

$$\bigcap_{i=1}^{k-1} (Q_i) = (G_{k-1}, D_{k-1}).$$

As in [1], we conclude that

COROLLARY 3.3. *For all* $i = 1, \ldots, r$ *let* $J_i = (M_i, Q_i)$. *Let* $J = J_1 \cap \cap \ldots \cap J_r$. *Then*

$$J = (M_1, \ldots, M_r, G_r).$$

In particular the ideal J *is generated by elements of degree* 2.

Finally we give a constructive method for the defining ideal of any scroller, which is a generalization of Theorem 3.1 in [1]. We fix the number r of scroll matrices, with $c_1, \ldots, c_r$ columns respectively; then we construct, for all $i = 1, \ldots, r$, the scroll matrix B_i and the set of linear forms Q_i fulfilling the required conditions. As a consequence of Lemma 2.8, for all indices $i \geq 2$, the sets Δ_i and Π_i must be chosen in such a way that $|\Delta_i| \geq c_i$ and $|\Pi_i| \geq c_{l(i-1)}$, whenever $c_i \geq 2$ and $c_{l(i-1)} \geq 2$ respectively. Otherwise Δ_i and Π_i are subject to no restriction and may even be empty. In either case, the sets will be called *suitable*.

THEOREM 3.4. *Let $R = K[T_0, \ldots, T_n]$ be a polynomial ring over the field K. Let $r, c_1, \ldots, c_r$ be positive integers. Let $\Delta_1, \Delta_2, \ldots, \Delta_r$ be suitable independent sets of linear forms of R. Set*

$$D_i = \Delta_{i+1} \cup \ldots \cup \Delta_r,$$

for all $i = 1, \ldots, r-1$, and let $D_r = \emptyset$. Then set $P_1 = \emptyset$, and $P_2 = \Delta_1$. Apply the following recursive construction.

1. *Let $i = 3$, $\Pi_1 = \Delta_1$.*

2. *Choose an index $l = l(i)$, $1 \leq l \leq i-1$. If $l = l(j)$ and $\langle \Pi_j \rangle \neq \langle \Delta_l \rangle$, for some $j < i$, then goto 5.*

3. *Choose a suitable set Π_i of independent linear forms such that*

$$(\star) \qquad \langle \Pi_i \rangle \cap (\langle P_l \rangle \oplus \langle \Delta_{l+1} \rangle \oplus \ldots \oplus \langle \Delta_r \rangle) = 0.$$

If $\langle \Pi_i \rangle = \langle \Delta_l \rangle$ or $c_l = 1$, then goto 6.

4. *Choose a scroll matrix B_l with c_l columns such that the set M_l of its 2-minors is contained in (Δ_l) and (Π_i), and $\Delta_l, \Pi_i \subsetneq \langle B_l \rangle$. Goto 6.*

5. *Choose a suitable set Π_i of c_l independent linear forms such that $M_l \subsetneq (\Pi_i)$, and $(\star)$ is true.*

6. *Set*

$$P_i = P_l \cup \Pi_i \cup \Delta_{l+1} \cup \ldots \cup \Delta_{i-1}.$$

If $i < r$, replace i with $i+1$ and goto 2.

7. *For all $i = 1, \ldots, r$, if $i \neq l(j)$ for all indices j, choose B_i to be a scroll matrix with c_i columns such that the set M_i of its minors is contained in (Δ_i), $\Delta_i \subsetneq \langle B_i \rangle$, and $\langle B_i \rangle \cap \langle P_i, D_i \rangle = 0$. End.*

Then axioms (a), (c), (d), (e), (f) *of Theorem* 2.1 *are satisfied.*

REMARK 3.5. Note that the above recursive construction is always possible. The only step that really has to be justified is 4. But one easily sees that it is possible to find a scroll matrix B_l such that $\langle \Delta_l \rangle$ and $\langle \Pi_l \rangle$ (which are supposed to be distinct) are generated by the set of entries of the first and the second row of B_l respectively. And in step 5 one can always choose $\Pi_i = \Delta_l$.

The ideal constructed according to Theorem 3.4 is equidimensional and of minimal degree iff, for all $i = 1, \dots, r$, $|\Delta_i| = |\Pi_i| = c_i$. It follows that every scroller can be obtained from a minimal variety by adding linear forms to some of the sets Δ_i and Π_i, with the only restriction that linear independence be preserved, and/or by eliminating linear forms corresponding to one-column matrices.

EXAMPLE 3.6. In $K[a, \dots, q]$ we consider the ideal

$$J = J_1 \cap \dots \cap J_5,$$

where $J_i = (M_i, Q_i)$, and

– $M_2 = M_5 = \emptyset$ and M_1, M_3, M_4 are the sets of 2-minors of the scroll matrices B_1, B_3, B_4 given below;

– for all $i = 1, \dots, 5$ the set Q_i is constructed as follows.

	B_1	B_2	B_3	B_4	B_5
	$\begin{matrix} a & c & d \\ b & d & e \end{matrix}$		$\begin{matrix} h & i \\ i & j \end{matrix}$	$\begin{matrix} k & m \\ l & a+j \end{matrix}$	
Q_1		$f\ \boldsymbol{n}$	$h\quad i$	$m\quad a+j\ \boldsymbol{o}$	$l+b$
		Δ_2	Δ_3	Δ_4	Δ_5
Q_2	$a+b\quad c+d\quad d+e$		$h\quad i$	$m\quad a+j\ \boldsymbol{o}$	$l+b$
	Π_2		Δ_3	Δ_4	Δ_5
Q_3	$a+b\quad c+d\quad d+e$	g		$m\quad a+j\ \boldsymbol{o}$	$l+b$
	P_2	Π_3		Δ_4	Δ_5
Q_4	$c\quad d\quad e\ \boldsymbol{p}+\boldsymbol{kq}+\boldsymbol{n}$	$f\ \boldsymbol{n}$	$h\quad i$		$l+b$
	Π_4	Δ_2	Δ_3		Δ_5
Q_5	$a+b\quad c+d\quad d+e$	g	$i\quad j$	$m\quad a+j\ \boldsymbol{o}$	
	P_3		Π_5	Δ_4	

Eliminating the linear forms in boldface we get a minimal variety.

Another scroller can be obtained by choosing Π_3 to be the empty set.

4. More on rulings.

In this section we give a complete characterization of the set of linear subspaces of a scroll. In [7] it is proven that a d-dimensional scroll $X \in \boldsymbol{P}^n$ is a *ruled variety*: if B is the associated scroll matrix, then X is the union of all $(n-d)$-planes (*rulings*) defined by the annulation of a non trivial linear combination of the row vectors of B. Our Corollary 4.2 is a stronger version of this result.

LEMMA 4.1. *Let B be a scroll matrix with $c > 1$ columns. Let M be the set of its 2-minors. Let Q be a set of independent linear forms such that $M \subseteq (Q)$. Let L_1, L_2 be the row vectors of B. Then one of the following cases occurs*:

(1) *There is $(\lambda, \mu) \in K^2 \setminus \{(0, 0)\}$ such that $\langle \lambda L_1 + \mu L_2 \rangle \subseteq \langle Q \rangle$*,

(2) *B contains an isolated column $\widehat{\beta}$ such that $\langle \widehat{\beta} \rangle \cap \langle Q \rangle = 0$, and for every other small block β of B either*

(i) *$\langle \beta \rangle \subseteq \langle Q \rangle$, or*

(ii) *β is an isolated column and $\langle \beta + \alpha \widehat{\beta} \rangle \subseteq \langle Q \rangle$ for some $\alpha \in K$.*

PROOF. We first prove the claim in the case where B is a simple scroll matrix, say

$$B = \begin{pmatrix} l_0 & l_1 & \dots & l_{c-1} \\ l_1 & l_2 & \dots & l_c \end{pmatrix}.$$

Let $\underline{T}$ be a set of variables such that $\underline{T} \supseteq Q$. For all $i = 0, \dots, c$ write $l_i = l_i' + l_i''$, where

$$l_i' = \sum_{T \in Q} \alpha_T T \quad \text{and} \quad l_i'' = \sum_{T \in \underline{T} \setminus Q} \alpha_T T \quad (\alpha_T \in K).$$

Let

$$\overline{B} = \begin{pmatrix} l_0'' & l_1'' & \dots & l_{c-1}'' \\ l_1'' & l_2'' & \dots & l_c'' \end{pmatrix}$$

be the image of B in the polynomial ring $\overline{R} = K[\underline{T}]/(Q)$. All 2-minors of $\overline{B}$

are zero in $\overline{R}$. First assume that $l_1'' = 0$. Then

$$0 = \begin{vmatrix} l_1'' & l_2'' \\ l_2'' & l_3'' \end{vmatrix} = l_1'' l_3'' - l_2''^2 = l_2''^2,$$

so that $l_2''^2 = 0$. Suppose that $l_0'' \neq 0$. Let $i \in \{3, \dots, c\}$. One has that

$$0 = \begin{vmatrix} l_0'' & l_{i-1}'' \\ l_1'' & l_i'' \end{vmatrix} = l_0'' l_i'' - l_1'' l_{i-1}'' = l_0'' l_i'',$$

so that $l_i'' = 0$. It follows that $L_2 = (l_1', l_2', \dots, l_c')$, whence $\langle L_2 \rangle \subsetneq \langle Q \rangle$. Now suppose that $l_0'' = 0$. We prove that $l_i'' = 0$ for all $i = 3, \dots, c-1$. Fix such an index i and assume that $l_{i-1}'' = 0$. We have that

$$0 = \begin{vmatrix} l_{i-1}'' & l_i'' \\ l_i'' & l_{i+1}'' \end{vmatrix} = l_{i-1}'' l_{i+1}'' - l_i''^2 = -l_i''^2,$$

so that $l_i'' = 0$. Hence $L_1 = (l_0', l_1', \dots, l_{c-1}')$, so that $\langle L_1 \rangle \subsetneq \langle Q \rangle$. Now assume that $l_1'' \neq 0$. We prove that for all $i = 0, \dots, c$ there is $\alpha_i \in K \setminus \{0\}$ such that $l_i'' = \alpha_i l_1''$. The claim is obviously fulfilled for $i = 1$ with $\alpha_1 = 1$. We proceed by induction on $i = 0, \dots, c$. First note that

$$0 = \begin{vmatrix} l_0'' & l_1'' \\ l_1'' & l_2'' \end{vmatrix} = l_0'' l_2'' - l_1''^2.$$

Since $\overline{R}$ is a UFD, and the elements l_i'' are all linear forms, it follows that $l_0'' = \alpha_0 l_1''$ and $l_2'' = \alpha_2 l_1''$ for some $\alpha_0, \alpha_2 \in K \setminus \{0\}$. Hence the claim is true for $i = 0, 1, 2$. Now let $i \in \{3, \dots, c\}$ and suppose that $l_{i-2}'' = \alpha_{i-2} l_1''$, $l_{i-1}'' = \alpha_{i-1} l_1''$ for some $\alpha_{i-2}, \alpha_{i-1} \in K \setminus \{0\}$. It holds

$$0 = \begin{vmatrix} l_{i-2}'' & l_{i-1}'' \\ l_{i-1}'' & l_i'' \end{vmatrix} = l_{i-2}'' l_i'' - l_{i-1}''^2 = \alpha_{i-2} l_1'' l_i'' - \alpha_{i-1}^2 l_1''^2.$$

Hence

$$l_i'' = \frac{\alpha_{i-1}^2}{\alpha_{i-2}} l_1''.$$

This completes the induction.

We have just proven that

$$\overline{B} = \begin{pmatrix} \alpha_0 l_1'' & \alpha_1 l_1'' & \dots & \alpha_{c-1} l_1'' \\ \alpha_1 l_1'' & \alpha_2 l_1'' & \dots & \alpha_c l_1'' \end{pmatrix}.$$

Since all minors of $\overline{B}$ are zero in $\overline{R}$, its rows are proportional: there is $\lambda \in K \setminus \{0\}$ such that $\alpha_i = \lambda \alpha_{i+1}$ for all $i = 0, \dots, c-1$. Therefore $L_1 - \lambda L_2 = (l_0' - \lambda l_1', \dots, l_{c-1}' - \lambda l_c')$, whose entries all belong to $\langle Q \rangle$. This completes the proof of the claim in the case where B is simple: in this case (1) holds.

Now suppose that B consists of more than one small block. Let

$$\tilde{\beta} = \begin{pmatrix} \widetilde{L}_1 \\ \widetilde{L}_2 \end{pmatrix}$$

be a small block of B. If $\tilde{\beta}$ is not an isolated column and $\widetilde{M}$ is the set of its minors, then

$$\widetilde{M} \subseteq M \subseteq (Q),$$

so that by the first part of the proof there is $(\tilde{\lambda}, \tilde{\mu}) \in K^2 \setminus \{(0, 0)\}$ for which

$$\langle \tilde{\lambda} \widetilde{L}_1 + \tilde{\mu} \widetilde{L}_2 \rangle \subseteq \langle Q \rangle.$$

Let $B' = (\beta_1 | \dots | \beta_s)$ be the submatrix of B formed by all small blocks for which (1) holds. Then $B' = B$ if B has no isolated column. We show that the claim is true for B'. For all $i = 1, \dots, s$ let $L_1^{(i)}$, $L_2^{(i)}$ be the row vectors of β_i. For all $i = 1 \dots, s$ let $(\lambda^{(i)}, \mu^{(i)}) \in K^2 \setminus \{(0, 0)\}$ be such that

$$\langle \lambda^{(i)} L_1^{(i)} + \mu^{(i)} L_2^{(i)} \rangle \subseteq \langle Q \rangle.$$

Suppose for a contradiction that (1) is not true for B'. Then one can easily prove that there are two indices i, j, $1 \leqslant j < i \leqslant s$ such that there is an entry x of $\lambda^{(i)} L_1^{(j)} + \mu^{(i)} L_2^{(j)}$ and an entry y of $\lambda^{(j)} L_1^{(i)} + \mu^{(j)} L_2^{(i)}$ such that $x, y \notin \langle Q \rangle$. This implies that

$$\begin{vmatrix} \lambda^{(i)} & \mu^{(i)} \\ \lambda^{(j)} & \mu^{(j)} \end{vmatrix} \neq 0 .$$

Therefore there is an invertible row transformation mapping B into the matrix

$$\widetilde{B} = \begin{pmatrix} \lambda^{(i)} L_1 + \mu^{(i)} L_2 \\ \lambda^{(j)} L_1 + \mu^{(j)} L_2 \end{pmatrix}.$$

In particular the ideals of the 2-minors of B and $\widetilde{B}$ coincide. Consider the following minor of $\widetilde{B}$:

$$m = \begin{vmatrix} x & y' \\ x' & y \end{vmatrix} = xy - x'y'.$$

Then $m \in (M) \subseteq (Q)$. Since x' is an entry of $\lambda^{(j)} L_1^{(j)} + \mu^{(j)} L_2^{(j)}$, it follows that $x' \in \langle Q \rangle$. But then $xy \in (Q)$. Since (Q) is a prime ideal and $x, y \notin (Q)$, this provides the required contradiction.

Now suppose that there is a small block $\widehat{\beta}$ such that (1) is not true. Then it is an isolated column

$$\widehat{\beta} = \begin{pmatrix} l_0 \\ l_1 \end{pmatrix},$$

and $\langle \widehat{\beta} \rangle \cap \langle Q \rangle = 0$. We fix another small block of B:

$$\beta = \begin{pmatrix} l_2 & l_3 & \dots & l_{m-1} \\ l_3 & l_4 & \dots & l_m \end{pmatrix}.$$

By the first part of the proof, if β is not an isolated column, up to inverting the rows of B we may assume that $l_i + \nu l_{i+1} \in \langle Q \rangle$ for all $i = 2, \dots, m-1$, and for some $\nu \in K \setminus \{0\}$. For all $i = 0, \dots, m$ let l_i' and l_i'' have the same meaning as above.

In $\overline{R}$ it holds

$$\begin{vmatrix} l_0'' & l_{m-1}'' \\ l_1'' & l_m'' \end{vmatrix} = l_0'' l_m'' - l_1'' l_{m-1}'' = 0 .$$

First we show that l_0'' and l_1'' cannot be proportional. Suppose that $\lambda l_0'' + \mu l_1'' = 0$ for some $(\lambda, \mu) \in K^2 \setminus \{(0, 0)\}$. Then

$$\lambda l_0 + \mu l_1 = \lambda(l_0' + l_0'') + \mu(l_1' + l_1'') = \lambda l_0' + \mu l_1' \in \langle Q \rangle,$$

against our assumption on $\widehat{\beta}$. Since $\overline{R}$ is a UFD, it follows that one of the following cases occurs.

(1) $l_m'' = l_{m-1}'' = 0$. We show that in this case $\langle \beta \rangle \subseteq \langle Q \rangle$. We have that $l_m, l_{m-1} \in \langle Q \rangle$. If β is an isolated column, there is nothing left to prove. Otherwise $l_i + \nu l_{i+1} \in \langle Q \rangle$ for all $i = 2, \dots, m-1$. By finite descending induction one concludes that $l_i \in \langle Q \rangle$ for $i = 2, \dots, m-1$. Hence case (2)(i) holds.

(2) There is $\alpha \in K \setminus \{0\}$ such that $l''_{m-1} + \alpha l''_0 = 0$ and $l''_m + \alpha l''_1 = 0$. Then

$$l_{m-1} + \alpha l_0 = l'_{m-1} + l''_{m-1} + \alpha(l'_0 + l''_0) = l'_{m-1} + \alpha l'_0 \in \langle Q \rangle,$$

and, similarly,

$$l_m + \alpha l_1 = l'_m + \alpha l'_1 \in \langle Q \rangle.$$

We prove that in this case β is an isolated column. Suppose this were not the case. Then $l_{m-1} + \nu l_m \in \langle Q \rangle$, for some $\nu \in K \setminus \{0\}$. But then

$$l_0 + \nu l_1 = l'_0 + \nu l'_1 + l''_0 + \nu l''_1 = l'_0 + \nu l'_1 - \alpha^{-1}(l''_{m-1} + \nu l''_m) \in \langle Q \rangle,$$

against our assumption on $\widehat{\beta}$. In this case (2)(ii) holds. ■

Note that the number of isolated columns of B is equal to the number of linear components of X. Hence the previous result has the following immediate consequence:

COROLLARY 4.2. *If X does not contain any d-plane, then for every linear subspace V contained in X there is a ruling $W \subseteq X$ containing V.*

The next claim describes all $(n-d)$-planes contained in a scroll X. It follows easily from 4.1 and 4.2.

COROLLARY 4.3. *Let B be a scroll matrix with $c > 1$ columns. Let M be the set of its 2-minors. Let Q be a set of linear forms such that $M \subseteq (Q)$. If $|Q| = c$, then either*

(i) *$\langle Q \rangle$ is generated by the entries of a non trivial linear combination of the row vectors of B, or*

(ii) *$B = (\beta_1 \,|\, \overline{B})$, where β_1 is an isolated column, and $\langle \overline{B} \rangle = \langle Q \rangle$, or*

(iii) *$B = (\beta_1 \,|\, \beta_2)$, where β_1 and β_2 are isolated columns and $\langle Q \rangle$ is generated by the entries of a non trivial linear combination of β_1 and β_2.*

PROOF. If B has no isolated columns, by 4.1, case (i) holds. If B consists of two isolated columns, then there is nothing to prove. Suppose $B = (\beta_1 \,|\, \overline{B})$, where β_1 is an isolated column such that $\langle \beta_1 \rangle \cap \langle Q \rangle = 0$, and $\overline{B}$ is not an isolated column. We show that $\overline{B}$ contains no isolated column γ such that $\langle \gamma \rangle \cap \langle Q \rangle = 0$: then, by 4.1, case (ii) is fulfilled. Suppose for a contradiction that $\gamma_2, \ldots, \gamma_s$ are the isolated columns of $\overline{B}$ such that $\langle \beta_1 + \alpha_i \gamma_i \rangle \subseteq \langle Q \rangle$ for suitable $\alpha_i \in K \setminus \{0\}$, for all $i = 1, \ldots, s$. It follows that $\langle Q \rangle$ contains $2s$ linearly independent forms; in addition, by 4.1, $\langle Q \rangle$ contains at least $c-s$ linearly independent entries from the remaining $c-s-1$ columns, but $2s+c-s>c$, against our assumption.

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Manoscritto pervenuto in redazione il 5 luglio 2003.