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A Note on Abelian Varieties Embedded in Quadrics.

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ABSTRACT - We show that if A is a d-dimensional abelian variety in a smooth quadric of dimension $2d$ then $d=1$ and A is an elliptic curve of bidegree $(2, 2)$ on a quadric. This extends a result of Van de Ven which says that A only can be embedded in $\boldsymbol{P}^{2d}$ when $d=1$ or 2.

1. Introduction.

Let A be a d-dimensional abelian variety embedded in $\boldsymbol{P}^N$. It is well known that $2d \leqslant N$. Moreover, in [8] Van de Ven proved that the equality holds only when $d=1$ or 2.

It is a natural question to study the possibilities for d when the abelian variety A is embedded in any other smooth $2d$-dimensional variety V. In particular, here we study the embedding in smooth quadrics. We obtain the following result:

THEOREM 1.1. *If A is a d-dimensional abelian variety in a smooth quadric of dimension $2d$ then $d=1$ and A is an elliptic curve of bidegree $(2, 2)$ on a quadric.*

We will use similar methods to Van de Ven's proof. The calculation of the self intersection of A in the quadric and the Riemann-Roch theorem for abelian varieties allow only the cases $d=1, 2, 3$.

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The case $d=1$ is the classical elliptic curve of type $(2, 2)$ contained in the smooth quadric of $\boldsymbol{P}^3$.

When $d=2$, A is an abelian surface in $\boldsymbol{P}^5$. We see that is the projection of an abelian surface $A' \subset \boldsymbol{P}^6$ given by a $(1, 7)$ polarization. By a result [7] due to R. Lazarsfeld, this is projectively normal and it is not contained in quadrics. Therefore, A is not contained in quadrics either.

Finally, two results [5], [6] of J. N. Iyer allow us to discard the case $d=3$.

2. Proof of the Theorem.

Let $j : A \hookrightarrow Q$ be an embedding of a d-dimensional abelian variety into a $2d$-dimensional smooth quadric, with $d > 1$. The Chow ring of the smooth quadric in codimension d is generated by cocycles α and β with the relations $\alpha^2 = \beta^2 = 1$, $\alpha . \beta = 0$. Thus, A will be equivalent to $a\alpha + b\beta$ and

$$A . A = a^2 + b^2. \tag{1}$$

On the other hand, by the self-intersection formula ([4], pag 431) we have $A . A = j_* c_d(N_{A, Q})$. To obtain $c_d(N_{A, Q})$, let us consider the normal bundle sequence:

$$0 \to T_A \to j^* T_Q \to N_{A, Q} \to 0$$

Since the tangent bundle of an abelian variety is trivial, we see that $c(N_{A, Q}) = j^*(c(T_Q))$. We compute the class of the tangent bundle of a quadric in the following lemma:

LEMMA 2.1. *Let* $i : Q \hookrightarrow \boldsymbol{P}^{n+1}$ *be an n-dimensional smooth quadric in* $\boldsymbol{P}^{n+1}$. *Then*

$$c(T_Q) = (1 + \overline{H})^{n+2}(1 + 2\overline{H})^{-1}$$

where $\overline{H} = i^* H$ *and* H *is a hyperplane in* $\boldsymbol{P}^{n+1}$.

PROOF. We have an exact sequence:

$$0 \to T_Q \to i^* T_{\boldsymbol{P}^{n+1}} \to N_{Q, \boldsymbol{P}^{n+1}} \to 0$$

Since Q is a hypersurface $N_{Q, \boldsymbol{P}^{n+1}} \cong \mathcal{O}_Q(Q) \cong \mathcal{O}_Q(2\overline{H})$ and the total class of the normal bundle is $c(N_{Q, \boldsymbol{P}^{n+1}}) = 1 + 2\overline{H}$. On the other hand, it is well known that $c(T_{\boldsymbol{P}^{n+1}}) = (1+H)^{n+2}$. Now, from the splitting principle the claim follows. ■

Let us apply this lemma to the previous situation. We obtain

$$c(N_{A,Q}) = (1+h)^{2d+2}(1+2h)^{-1} = \sum_{k=0}^{2d+2} \binom{2d+2}{k} h^k \sum_{l=0}^{\infty} (-2h)^l$$

where $h = j^*\overline{H}$. In particular, the top class is

$$c_d = F_d h^d, \text{ with } F_d = \sum_{k=0}^{d} \binom{2d+2}{k} (-2)^{(d-k)}.$$

Substituting this into the self-intersection formula, we have:

$$A.A = F_d j_*(j^*\overline{H}^d) = F_d \overline{H}^d . j_* A = F_d(a\alpha + b\beta).\overline{H}^d = F_d(a+b).$$

Combining this expression with (1) we obtain the following relation

$$a^2 + b^2 = F_d(a+b) \tag{2}$$

or equivalently,

$$\left(a - \frac{F_d}{2}\right)^2 + \left(b - \frac{F_d}{2}\right)^2 = \frac{F_d^2}{2}.$$

We are interested in bounding the degree of A, when (a, b) satisfy this equation. Note that this is a circle of center $\left(\frac{F_d}{2}, \frac{F_d}{2}\right)$ and radius $\frac{F_d}{\sqrt{2}}$. Since $deg(A) = a + b$, it is clear that the maximal degree is reached when $(a, b) = (F_d, F_d)$, that is,

$$deg(A) \leqslant 2F_d.$$

On the other hand, the abelian variety is embedded in $Q \subset \boldsymbol{P}^{2d+1}$. When $d > 2$, by Van de Ven's Theorem, it spans $\boldsymbol{P}^{2d+1}$. Furthermore, by the Riemann-Roch theorem for abelian varieties, we know that $h^0(\mathcal{O}_A(h)) = \frac{deg(A)}{d!}$. Thus, we have the following inequality:

$$deg(A) \geqslant 2(d+1)!$$

Comparing the two bounds we see that a sufficient condition for the nonexistence of the embedding j is $F_d < (d+1)!$. Now,

$$F_d = \sum_{k=0}^{d} \binom{2d+2}{k} (-2)^{(d-k)} \leqslant \sum_{k=0}^{d} \binom{2d+2}{k} (2)^d \leqslant 2^d 2^{2d+1} = 2^{3d+1}.$$

We see that $(d+1)! > 2^{3d+1} \geqslant F_d$ when $d = 17$. A simple inductive argument shows that this holds if $d \geqslant 17$.

If $d \leqslant 17$, using the exact value of F_d, we see that $(d+1)! > F_d$ for any $d > 3$.

We conclude that the unique possibilities are $d = 2$ or $d = 3$.

First, suppose that A is an abelian surface contained in a quadric. $F_2 = 7$ and we can check that the unique positive integer solution of the equation (2) is $a = b = 7$. Thus A must be an abelian surface of degree 14 given by the polarization $(1, 7)$. Note that $A \subset Q \subset \boldsymbol{P}^5$ is not linearly normal, that is, it is the projection of a linearly normal abelian surface $A' \subset \boldsymbol{P}^6$. The quadric Q can be lifted to a quadric containing the surface A'.

Lazarsfeld proved in [7] that a very ample divisor of type $(1, d)$ with $d \geqslant 13$ or $d = 7, 8, 9$ is projectively normal. From this the following sequence is exact:

$$0 \to H^0(I_{A', \boldsymbol{P}^6}(2)) \to H^0(\mathcal{O}_{\boldsymbol{P}^6}(2)) \to H^0(\mathcal{O}_{A'}(2)) \to 0\,.$$

Since $h^0(\mathcal{O}_{\boldsymbol{P}^6}(2)) = h^0(\mathcal{O}_{A'}(2)) = 28$, there are not quadrics containing the abelian surface A' and we obtain a contradiction.
Finally, suppose that $d = 3$. Now, $F_3 = 24 = (3+1)!$, so the degree of the abelian variety is exactly $2F_3 = 48$. The line bundle $\mathcal{O}_A(h)$ corresponds to a divisor of type $(1, 1, 8)$ or $(1, 2, 4)$. But J.N.Iyer prove in [5] that a line bundle of type $(1, \dots, 1, 2d+1)$ is never very ample. Moreover, in [6] she studies the map defined by a line bundle of type $(1, 2, 4)$ in a generic abelian threefold. She obtains that it is birational but not an isomorphism onto its image. Note that the very ampleness is an open condition for polarized abelian varieties (see [2]). It follows that a linear system of type $(1, 2, 4)$ cannot be very ample on any abelian threefold and this completes the proof.

REMARK 2.2. *In* [1] *C. Ciliberto and V. Di Gennaro obtain more general results about subvarieties of low codimension. In particular they give a bound for the degree of a d-dimensional abelian variety embedded on a smooth hypersurface of dimension* $2d$.

REMARK 2.3. *The sequence* $F_d = \sum_{k=0}^{d} \binom{2d+2}{k} (-2)^{(d-k)}$ *is related to the Fine numbers. For a reference see* [3].

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