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Multiple Solutions of a Nonlinear Elliptic Equation Involving Neumann Conditions and a Critical Sobolev Exponent.

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ABSTRACT - In this paper we prove the existence of two solutions of the nonhomogeneous Neumann problem (1.1) involving a critical Sobolev exponent. It is assumed that the coefficient Q is positive and smooth on Ω and $\lambda > 0$ is a parameter which does not belong to the spectrum of $-\Delta$. We examine the common effect of the mean curvature of the boundary $\partial\Omega$ and the shape of the graph of the coefficient Q on the existence of a second solution.

1. Introduction.

In this paper, we study the existence of multiple solutions of the superlinear problem

$$(1.1)\qquad \begin{cases} -\Delta u = \lambda u + Q(x)\, u_+^{2^*-1} + f(x) & \text{in } \Omega \\ \dfrac{\partial}{\partial \nu} u(x) = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $2^* = \dfrac{2N}{N-2}$, $N \geqslant 3$ is the critical Sobolev exponent, $\lambda \geqslant 0$ is a parameter and $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary

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$\partial\Omega$. We assume that the coefficient Q is smooth and positive on $\overline{\Omega}$ and $f \in L^r(\Omega)$ with $r > N$. We use the notation $u_+ = \max(u, 0)$.

This problem belongs to a class of problems referred to as the Ambrosetti-Prodi type. More precisely, in the case of the Dirichlet problem

$$\begin{cases} -\Delta u = g(u) + f(x) & \text{in} \quad \Omega, \\ u \qquad = 0 & \text{on } \partial\Omega \end{cases}$$

the limits

$$g_- = \lim_{s \to -\infty} \frac{g(s)}{s} \qquad \text{and} \qquad g_+ = \lim_{s \to \infty} \frac{g(s)}{s}$$

play an important role. We can basically distinguish three types of problems using the location of g_- and g_+ with respect to the spectrum of the operator $-\Delta$ with the Dirichlet boundary conditions. Denoting by $\{\lambda_k\}$ the sequence of the eigenvalues of $-\Delta$ with the Dirichlet boundary conditions, the following types of problems have been considered:

(I) $-\infty \leqslant g_- < \lambda_1 < g_+ \leqslant +\infty$,

(II) g_- and g_+ are both finite and the interval (g_-, g_+) contains an eigenvalue. In this case the problem is asymptotically linear,

(III) g_- lies between two consecutive eigenvalues and $g_+ = +\infty$.

We refer to the paper [12] where the extensive bibliography concerning these problems can be found. We point out here that conditions (I) and (III) cover the cases of subcritical, critical and supercritical growth for g. In the case of the Neumann problem the literature is rather scarce. In this paper we consider the nonlinear Neumann problem of type (III) with the nonlinearity of one-sided critical growth. We follow some ideas from [12], which considered a similar problem with the Dirichlet boundary conditions. First we consider the case $\lambda > 0$. The case $\lambda = 0$ will be treated separately.

Problem (1.1) may have constant solutions in contrast to the Dirichlet problem. We now discuss a number of conditions guaranteeing that a positive solution of (1.1) is not constant. If for some $\lambda > 0$ and a constant $c > 0$, the functions Q and f satisfy the equation

$$(*) \qquad \lambda c + Q(x)\, c^{2^*-1} + f(x) = 0$$

for every $x \in \Omega$, then $u = c$ is a solution of (1.1). If f and Q are differentiable on some open subset of Ω then the following condition

(a) $\nabla f(\overline{x})$ is not parallel to $\nabla Q(\overline{x})$ for some $\overline{x} \in \Omega$

ensures that a positive solution of (1.1) is not constant. If f and Q are not differentiable we can proceed as follows. Integrating the equation $(*)$ we get

$$(**)\qquad \lambda c|\Omega| + c^{2^*-1}\int_\Omega Q(x)\,dx + \int_\Omega f(x)\,dx = 0,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. From $(*)$ and $(**)$ we derive the equation

$$c^{2^*-1}\left(Q(x)|\Omega| - \int_\Omega Q(x)\,dx\right) + \left(f(x)|\Omega| - \int_\Omega f(x)\,dx\right) = 0.$$

We immediately obtain a contradiction if

(b) either $Q(x) = \text{const}$ and $f(x) \neq \text{const}$, or $Q(x) \neq \text{const}$ and $f(x) = \text{const}$.

If both functions $Q(x)$ and $f(x)$ are not constant we define a set

$$\Omega_0 = \left\{x;\ \frac{1}{|\Omega|}\int_\Omega Q(x)\,dx = Q(x)\right\},$$

which is nonempty. Then a positive solution cannot be constant if

(c) either $f(x) = \frac{1}{|\Omega|}\int_\Omega f(x)\,dx$ for all $x \in \Omega - \Omega_0$, or $f(x) \neq \frac{1}{|\Omega|}\int_\Omega f(x)\,dx$ for some $x \in \Omega_0$.

Finally, if (c) does not hold we require

(d) the ratio

$$\frac{f(x)|\Omega| - \int_\Omega f(x)\,dx}{\int_\Omega Q(x)\,dx - Q(x)|\Omega|}$$

is either not constant on $\Omega - \Omega_0$, or it is constant and nonpositive on $\Omega - \Omega_0$.

Therefore one of these conditions will be assumed throughout this work.

We assume that $f(x) = t + h(x)$, where t is a constant and $h \in L^r(\Omega)$ with $r > N$. We start by finding a negative solution of (1.1). We denote by $\lambda_1 = 0 < \lambda_2 < \ldots$ the sequence of eigenvalues for $-\Delta$ with the Neumann boundary conditions. The first eigenvalue is simple and has constant eigenfunctions.

Let $\lambda \neq \lambda_k$ for every k. Then there exists a unique solution $u_0 \in \in H^1(\Omega) \cap L^\infty(\Omega)$ of the problem

$$\begin{cases} -\Delta u = \lambda u + h(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

The function $u_t = -\frac{t}{\lambda} + u_0$, with $t > \lambda \sup_\Omega |u_0(x)|$ is negative and satisfies (1.1). We look for a second solution of the form $u = v + u_t$, where v satisfies

$$\begin{cases} -\Delta v = \lambda v + Q(x)(v + u_t)_+^{2^*-1} & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Problem (1.2) will be solved through the min-max based on a topological linking. To this end, we define a variational functional

$$J(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 - \lambda v^2)\, dx - \frac{1}{2^*}\int_\Omega Q(x)(v + u_t)_+^{2^*}\, dx$$

for $v \in H^1(\Omega)$. In the next section we examine Palais-Smale sequences for J. In particular, we find the energy level of the functional J below which the Palais-Smale condition holds. In Section 3 we verify that the functional J has the geometry of a topological linking. Conditions guaranteeing the existence of critical points of J will be given in Sections 4 and 5. The existence results of this section depend on a relation between $Q_m = \max_{x \in \partial\Omega} Q(x)$ and $Q_M = \max_{x \in \overline{\Omega}} Q(x)$. Section 6 is devoted to the case $\lambda = 0$. The existence of a critical point in this case is obtained through the implicit function theorem. The distinction of two cases involving the quantities Q_M and Q_m envisaged in Section 4 disappears in the case $\lambda = 0$.

2. The Palais-Smale condition.

We need two quantities:

$$Q_m = \max_{x \in \partial\Omega} Q(x) \quad \text{and} \quad Q_M = \max_{x \in \overline{\Omega}} Q(x).$$

We set

$$S_\infty = \min\left(\frac{S^{N/2}}{NQ_M^{(N-2)/2}}, \frac{S^{N/2}}{2NQ_m^{(N-2)/2}} \right),$$

where S denotes the best Sobolev constant, that is,

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N) - \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^N} |u(x)|^{2^*} dx\right)^{2/2^*}}.$$

Here $D^{1,2}(\mathbb{R}^N)$ denotes a Sobolev space obtained as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2_{D^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx.$$

In what follows, $\|\cdot\|$ denotes the norm in $H^1(\Omega)$, which is given by

$$\|u\|^2 = \int_\Omega (|\nabla u|^2 + u^2) \, dx.$$

In this paper we frequently use the Sobolev inequality:

$$\left(\int_\Omega |u|^{2^*} dx \right)^{2/2^*} \leq C_s \int_\Omega (|\nabla u|^2 + u^2) \, dx$$

for all $u \in H^1(\Omega)$, where $C_s > 0$ is a constant.

PROPOSITION 2.1. *Let* $\lambda_k < \lambda < \lambda_{k+1}$. *If*

$$J(u_n) \to c < S_\infty \quad \text{and} \quad J'(u_n) \to 0 \quad \text{in} \quad H^{-1}(\Omega)$$

then $\{u_n\}$ *is relatively compact in* $H^1(\Omega)$.

PROOF. We commence by showing that $\{u_m\}$ is bounded in $H^1(\Omega)$. We write

$$u_n = u_n^- + u_n^+, \quad u_n^- \in E^- \quad \text{and} \quad u_n^+ \in E^+,$$

where

$E^- =$ span of all eigenfunctions corresponding to $\lambda_1, \ldots, \lambda_k$, and $E^+ = (E^-)^\perp$. If $\phi \in H^1(\Omega)$, then

$$\int_\Omega \nabla u_n \nabla \phi\, dx - \lambda \int_\Omega u_n \phi\, dx = \int_\Omega Q(x)(u_n + u_t)_+^{2^*-1} \phi\, dx + \varepsilon_n \|\phi\| \tag{2.1}$$

with $\varepsilon_n \to 0$. Taking $\phi = u_n^+$, we get

$$\int_\Omega |\nabla u_n^+|^2 - \lambda \int_\Omega (u_n^+)^2 = \int_\Omega Q(x)(u_n + u_t)_+^{2^*-1} u_n^+\, dx + \varepsilon_n \|u_n^+\|.$$

Let $\delta > 0$ be such that $\lambda + \delta < \lambda_{k+1}$. Then

$$\left(1 - \frac{\lambda + \delta}{\lambda_{k+1}}\right) \int_\Omega |\nabla u_n^+|^2\, dx + \delta \int_\Omega (u_n^+)^2\, dx \leqslant \tag{2.2}$$

$$\leqslant \int_\Omega Q(x)(u_n + u_t)_+^{2^*-1} u_n^+\, dx + \varepsilon_n \|u_n^+\|.$$

We now use (2.1) with $\phi = u_n^-$ and let $\delta_1 > 0$ be such that $\lambda - \delta_1 > \lambda_k$. Then

$$\left(\frac{\lambda - \delta_1}{\lambda_k} - 1\right) \int_\Omega |\nabla u_n^-|^2\, dx + \delta_1 \int_\Omega (u_n^-)^2\, dx \leqslant \tag{2.3}$$

$$\leqslant - \int_\Omega Q(x)(u_n + u_t)_+^{2^*-1} u_n^-\, dx + \varepsilon_n \|u_n^-\|.$$

On the other hand for $n \geqslant n_0$, we can write

$$\begin{aligned} c + \varepsilon_n \|u_n\| + 1 &\geqslant J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle \\ &= \frac{1}{2}\int_\Omega Q(x)(u_n + u_t)_+^{2^*-1} u_n\, dx - \frac{1}{2^*}\int_\Omega Q(x)(u_n + u_t)_+^{2^*}\, dx \\ &= \frac{1}{N}\int_\Omega Q(x)(u_n + u_t)_+^{2^*}\, dx - \frac{1}{2}\int_\Omega Q(x)(u_n + u_t)_+^{2^*-1} u_t\, dx \\ &\geqslant \frac{1}{N}\int_\Omega Q(x)(u_n + u_t)_+^{2^*}\, dx. \end{aligned}$$

Applying the Young inequality, we deduce from (2.2) and the above estimate that for $\eta > 0$ we have

$$(2.4)\quad \left(1 - \frac{\lambda + \delta}{\lambda_{k+1}}\right) \int_\Omega |\nabla u_n^+|^2 dx + \delta \int_\Omega (u_n^+)^2 dx \leqslant$$

$$\leqslant \int_\Omega Q(x)(u_n + u_t)_+^{2^*-1} u_n^+ dx + \varepsilon_n \|u_n^+\| \leqslant$$

$$\leqslant \eta \left(\int_\Omega Q(x) |u_n^+|^{2^*} dx \right)^{2/2^*} + C_\eta \left(\int_\Omega Q(x)(u_n + u_t)_+^{2^*} dx \right)^{2(2^*-1)/2^*} dx + \varepsilon_n \|u_n^+\|$$

$$\leqslant C_s Q_M^{2/2^*} \eta \|u_n^+\|^2 + C_1 \left(\int_\Omega (u_n + u_t)_+^{2^*} dx \right)^{(N+2)/N} + \varepsilon_n \|u_n^+\|$$

$$\leqslant C_s Q_M^{2/2^*} \eta \|u_n^+\|^2 + C_1 \|u_n\|^{(N+2)/N} + C_2 \|u_n^+\| + C_3$$

for some constants $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$. In a similar way, we obtain

$$(2.5)\quad \left(\frac{\lambda - \delta_1}{\lambda_k} - 1 \right) \int_\Omega |\nabla u_n^-|^2 dx + \delta_1 \int_\Omega (u_n^-)^2 dx \leqslant$$

$$\leqslant C_s Q_M^{2/2^*} \eta \|u_n^-\|^2 + C_4 \left(\|u_n\|^{(N+2)/N} + \|u_n^-\| + 1 \right)$$

for some constant $C_4 > 0$. Estimates (2.4) and (2.5) imply that $\{u_n\}$ is bounded in $H^1(\Omega)$. We may therefore assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$. By the concentration-compactness principle there exist sequences of points $\{x_j\} \subset \mathbb{R}^N$, sequences of numbers $\{\nu_j\}$ and $\{\mu_j\}$ such that

$$|u_n|^{2^*} \overset{*}{\rightharpoonup} |u|^{2^*} + \sum_j \nu_j \delta_{x_j}$$

and

$$|\nabla u_n|^2 \overset{*}{\rightharpoonup} |\nabla u|^2 + \sum_j \mu_j \delta_{x_j}$$

in the sense of measures, where

$$S \nu_j^{2/2^*} \leqslant \mu_j \quad \text{if } x_j \in \Omega$$

and

$$\frac{S\nu_j^{2/2^*}}{2^{2/N}} \leqslant \mu_j \quad \text{if } x_j \in \partial\Omega.$$

Fix x_j. Let $\{\phi_\delta\}$ be a family of smooth and positive functions concentrating at x_j as $\delta \to 0$. Then using the Brézis-Lieb Lemma, we obtain

$$\int_\Omega |\nabla u_n|^2 \phi_\delta \, dx + \int_\Omega \nabla u_n u_n \nabla\phi_\delta \, dx + \lambda \int_\Omega u_n^2 \phi_\delta \, dx$$

$$= \int_\Omega Q(x)(u_n + u_t)_+^{2^*-1} u_n \phi_\delta \, dx + o(1)$$

$$= \int_\Omega Q(x)(u_n + u_t)_+^{2^*} \phi_\delta \, dx - \int_\Omega Q(x)(u_n + u_t)_+^{2^*-1} u_t \phi_\delta \, dx + o(1)$$

$$\leqslant \int_\Omega Q(x)|u_n + u_t|^{2^*} \phi_\delta \, dx - \int_\Omega Q(x)(u_n + u_t)_+^{2^*-1} u_t \phi_\delta \, dx + o(1)$$

$$= \int_\Omega Q(x)|u_n|^{2^*} \phi_\delta dx - \int_\Omega Q(x)|u|^{2^*} \phi_\delta dx + \int_\Omega Q(x)|u + u_t|^{2^*} \phi_\delta dx$$

$$- \int_\Omega Q(x)(u_n + u_t)_+^{2^*-1} u_t \phi_\delta \, dx + o(1).$$

Letting $n \to \infty$ and then $\delta \to 0$ we deduce that in both cases $x_j \in \partial\Omega$ and $x_j \in \Omega$,

$$\mu_j \leqslant Q(x_j)\, \nu_j.$$

If $\mu_j > 0$ for some x_j, then

$$\mu_j \geqslant \frac{S^{N/2}}{Q(x_j)^{(N-2)/2}} \quad \text{if } x_j \in \Omega \text{ and } \mu_j \geqslant \frac{S^{N/2}}{2\,Q(x_j)^{(N-2)/2}} \quad \text{if } x_j \in \partial\Omega.$$

We now write

$$
\begin{aligned}
J(u_n) - \frac{1}{2^*}\langle J'(u_n), u_n\rangle
&= \frac{1}{N}\int_\Omega (|\nabla u_n|^2 - \lambda u_n^2)\, dx - \frac{1}{2^*}\int_\Omega Q(x)(u_n + u_t)_+^{2^*}\, dx \\
&+ \frac{1}{2^*}\int_\Omega Q(x)(u_n + u_t)_+^{2^*-1} u_n + o(1) \\
&= \frac{1}{N}\int_\Omega (|\nabla u_n|^2 - \lambda u_n^2)\, dx - \frac{1}{2^*}\int_\Omega Q(x)(u_n + u_t)_+^{2^*-1} u_t\, dx + o(1) \\
&\geqslant \frac{1}{N}\int_\Omega (|\nabla u_n|^2 - \lambda u_n^2)\, dx + o(1).
\end{aligned}
$$

Since u is a solution of (1.1) we also have

$$
\int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx = \int_\Omega Q(x)(u + u_t)_+^{2^*-1} u\, dx =
$$

$$
= \int_\Omega Q(x)(u + u_t)_+^{2^*-1} u_+\, dx \geqslant 0.
$$

We aim to show that $\mu_j = 0$ for every j. If not, the concentration-compactness principle implies that

$$
c \geqslant \frac{1}{N}\int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx + \frac{1}{N}\sum_j \mu_j \geqslant \frac{1}{N}\sum_j \mu_j .
$$

If $\mu_j > 0$ for some j with $x_j \in \partial\Omega$, then

$$
c \geqslant \frac{1}{2N}\,\frac{S^{N/2}}{Q(x_j)^{(N-2)/2}} \geqslant \frac{1}{2N}\,\frac{S^{N/2}}{Q_m^{(N-2)/2}} .
$$

This is obviously impossible. Similarly if $\mu_j > 0$ for some j with $x_j \in \Omega$. Thus

$$
\int_\Omega Q(x)(u_n + u_t)_+^{2^*}\, dx \to \int_\Omega Q(x)(u + u_t)_+^{2^*}\, dx
$$

and also

$$\int_\Omega |\nabla u_n|^2\,dx \to \int_\Omega |\nabla u|^2\,dx$$

and the result follows. ■

3. Topological linking.

We assume that $\lambda \in (\lambda_k, \lambda_{k+1})$. Let

$$E^- = \operatorname{span}\{e_1, \dots, e_l\},$$

where $e_1, \dots, e_l$ are eigenfunctions corresponding to $\lambda_1, \dots, \lambda_k$. We set $E^+ = (E^-)^\perp$. Let

$$S_\varrho = \partial B_\varrho \cap E^+ \quad \text{and} \quad D = [0, Re] \oplus (B_r \cap E^-), \qquad e \in E^+,$$

where B_r denotes the ball of radius r with centre at 0. To apply a topological linking we need to verify that

$$J|_{S_\varrho} \geq \alpha > 0, \qquad \varrho < R,$$

$$J|_{\partial D} < \alpha \quad \text{and} \quad \max_{u \in D} J(u) < S_\infty.$$

LEMMA 3.1. *There exist $\varrho_0 > 0$ and $\alpha : (0, \varrho_0] \to (0, \infty)$ such that*

$$J(u) \geq \alpha(\varrho) \quad \text{for every} \quad v \in S_\varrho.$$

PROOF. We choose $\eta > 0$ so that $\lambda_k < \lambda + \eta < \lambda_{k+1}$. Then

$$\int_\Omega |\nabla u|^2\,dx \geq \lambda_{k+1} \int_\Omega u^2\,dx$$

for every $u \in E^+$. Since $u_t < 0$ on Ω, we have

$$J(u) \geq \int_\Omega \left(1 - \frac{\lambda + \eta}{\lambda_{k+1}}\right) |\nabla u|^2 dx + \eta \int_\Omega u^2 dx - C_s^{-2^*/2} Q_M \left(\int_\Omega (|\nabla u|^2 + u^2)\,dx\right)^{2^*/2}$$

$$\geq \beta \int_\Omega (|\nabla u|^2 + u^2)\,dx - C_s^{-2^*/2} Q_M \left(\int_\Omega (|\nabla u|^2 + u^2)\,dx\right)^{2^*/2},$$

where

$$\beta = \min\left(1 - \frac{\lambda + \eta}{\lambda_{k+1}}, \eta\right).$$

Letting $\varrho = \|u\|$ we obtain the following estimate

$$J(u) \geqslant \beta\varrho^2 - C_s^{-2^*/2} Q_M \varrho^{2^*}.$$

To complete the proof we set

$$\alpha(\varrho) = \beta\varrho^2 - C_s^{-2^*/2} Q_M \varrho^{2^*},$$

with $\varrho_0 > 0$ such that $\varrho_0^2 - C_s^{-2^*/2} Q_M \varrho_0^{2^*} > 0$. ■

From now on, we use the instanton

$$U_\varepsilon(x) = \frac{c_N \varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{N-2/2}},$$

in the definition of the set D, where $c_N > 0$ is a constant and we set $e_\varepsilon = P^+ U_\varepsilon$. It is well-known that $U = U_1$ satisfies the equation

$$-\Delta U = U^{2^*-1} \quad \text{in } \mathbb{R}^N$$

and moreover

$$\int_{\mathbb{R}^N} |\nabla U|^2 \, dx = \int_{\mathbb{R}^N} U^{2^*} \, dx = S^{N/2}.$$

With the choice of $e = e_\varepsilon$ we verify the remaining conditions of the topological linking. Without loss of generality we may assume in Lemma 3.2 below that $0 \in \Omega$.

LEMMA 3.2. *There exist* $r_0 > 0$, $R_0 > 0$ *and* $\varepsilon_0 > 0$ *such that for* $r \geqslant r_0$, $R \geqslant R_0$ *and* $0 < \varepsilon \leqslant \varepsilon_0$ *we have*

$$J(u) < \alpha \quad \textit{for every } u \in \partial D.$$

PROOF. We set

$$\partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

where

$$\Gamma_1 = \overline{B}_r \cap E^-,$$

$$\Gamma_2 = \{v \in H^1(\Omega);\ v = w + se_\varepsilon,\ w \in E^-,\ \|w\| = r,\ 0 \leqslant s \leqslant R\},$$

$$\Gamma_3 = \{v \in H^1(\Omega);\ v = w + Re_\varepsilon,\ w \in E^- \cap B_r\}.$$

For $v \in E^-$ we have

$$\int_\Omega |\nabla v|^2\,dx \leqslant \lambda_k \int_\Omega v^2\,dx$$

and

$$J(v) \leqslant \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\int_\Omega |\nabla v|^2\,dx - \frac{1}{2^*}\int_\Omega Q(x)(v + u_t)^{2^*}\,dx \leqslant 0.$$

We now consider Γ_2. Let $v \in \Gamma_2$ and define

$$\delta^2 = \sup_{0 < \varepsilon \leqslant 1} \int_\Omega |\nabla e_\varepsilon|^2\,dx.$$

Let $r^2 = \|\nabla w\|^2$ and choose $\eta_1 > 0$ so that $\lambda_k < \lambda - \eta_1$. Then for $v = w + {}$ ${}+ se_\varepsilon$ we have

$$\begin{aligned}
J(v) &\leqslant \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \frac{\lambda}{2}\int_\Omega v^2\,dx \\
&\leqslant \frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \frac{\lambda}{2}\int_\Omega w^2\,dx + \frac{s^2}{2}\int_\Omega |\nabla e_\varepsilon|^2\,dx \\
&\leqslant \frac{1}{2}\left(1 - \frac{\lambda - \eta_1}{\lambda_k}\right)\int_\Omega |\nabla w|^2\,dx - \frac{\eta_1}{2}\int_\Omega w^2\,dx + \frac{s^2}{2}\int_\Omega |\nabla e_\varepsilon|^2\,dx.
\end{aligned}$$

Let $\eta_2 = \max\left(1 - \frac{\lambda - \eta_1}{\lambda_k},\ -\frac{\eta_1}{2}\right) < 0$. We then have

$$J(v) \leqslant \eta_2 r^2 + \frac{s^2}{2}\int_\Omega |\nabla e_\varepsilon|^2\,dx.$$

We set $s_0 = \frac{\sqrt{2\alpha}}{\delta}$. Then $J(v) \leqslant \alpha$ for $0 \leqslant s \leqslant s_0$. We now consider the case $s > s_0$. Put

$$K = \sup\left\{\left\|\frac{w+u_t}{s}\right\|_\infty;\ s_0 \leqslant s \leqslant R,\ \|w\| = r,\ w \in E^-\right\}.$$

We now estimate $P^- U_\varepsilon$. Let

$$P^- U_\varepsilon = \sum_{j=1}^{l} \alpha_j e_j, \qquad \alpha_j = \int_\Omega U_\varepsilon e_j(x)\, dx.$$

Since the first eigenfunction corresponding to $\lambda = 0$ is constant, we see that $P^- U_\varepsilon \not\equiv 0$. Hence

$$\|P^- U_\varepsilon\|_2^2 = \sum_{j=1}^{l} \alpha_j^2 = \sum_{j=1}^{l} \left(\int_\Omega U_\varepsilon e_j\, dx\right)^2$$

$$\leqslant \sum_{j=1}^{l} \|e_j\|_\infty^2 \, \|U_\varepsilon\|_1^2 \leqslant C\varepsilon^{N-2}.$$

Therefore

$$\begin{aligned} P^+ U_\varepsilon(0) &= U_\varepsilon(0) - P^- U_\varepsilon(0) \\ &\geqslant C\varepsilon^{-(N-2)/2} - \|P^- U_\varepsilon\|_\infty \geqslant C\varepsilon^{-(N-2)/2}. \end{aligned}$$

By the continuity of $P^+ U_\varepsilon$ there exists a $\delta = \delta(K)$ such that

$$B_\delta(0) \subset \{x \in \Omega;\ P^+ U_\varepsilon(x) > K\}.$$

We also need the following inequality: if $\omega \subset \Omega$ and $u + v > 0$ on ω, then

$$\left|\int_\omega (u+v)^p\, dx - \int_\omega |u|^p\, dx - \int_\omega |v|^p\, dx\right| \leqslant C\int_\omega (|u|^{p-1}|v| + |u|\,|v|^{p-1})\, dx,$$

where $C = C(p)$. We apply this estimate with $\omega = \Omega_\varepsilon$, where

$$\Omega_\varepsilon = \{x \in \Omega;\ P^+ U_\varepsilon(x) > K\}.$$

Letting $Q_* = \min_{x \in \Omega} Q(x)$ we get

$$\int_\Omega Q(x)\left(\frac{w+u_t}{s}+e_\varepsilon\right)_+^{2^*} dx \geqslant Q_* \int_{\Omega_\varepsilon}\left(e_\varepsilon + \frac{w+u_t}{s}\right)_+^{2^*} dx$$

$$\geqslant Q_*\left(\int_{\Omega_\varepsilon} |e_\varepsilon|^{2^*} dx + \int_{\Omega_\varepsilon} \left|\frac{w+u_t}{s}\right|^{2^*} dx\right.$$

$$\left. - C\int_{\Omega_\varepsilon}\left(|e_\varepsilon|^{2^*-1}\left|\frac{w+u_t}{s}\right| + |e_\varepsilon|\left|\frac{w+u_t}{s}\right|^{2^*-1}\right)dx\right)$$

$$\geqslant Q_*\left(\int_\Omega |e_\varepsilon|^{2^*} dx + \int_{\Omega_\varepsilon}\left|\frac{w+u_t}{s}\right|^{2^*} dx\right)$$

$$- C_1\left(\|e_\varepsilon\|^{2^*-1}_{L^{2^*-1}(\Omega_\varepsilon)} + \|e_\varepsilon\|_{L^1(\Omega_\varepsilon)}\right).$$

Since

$$\|P^+ U_\varepsilon\|^{2^*-1}_{L^{2^*-1}} \leqslant C\varepsilon^{\frac{N-2}{2}} \quad \text{and} \quad \|P^+ U_\varepsilon\|_{L^1} \leqslant C\varepsilon^{\frac{N-2}{2}}$$

we deduce from the previous estimate that

$$J(v) \leqslant \eta_2 r^2 + \frac{s^2}{2} S^{N/2} - \frac{s^{2^*}}{2^*} Q_* S^{N/2} + Cs^{2^*}\varepsilon^{(N-2)/2}$$

$$= \eta_2 r^2 + \Phi_\varepsilon(s).$$

It is easy to check that

$$\Phi_\varepsilon(s) \leqslant \frac{1}{2}\left(\frac{S^{N/2}}{Q_* S^{N/2}}\right)^{1/2} + O\left(\varepsilon^{(N-2)/2}\right).$$

Increasing r, if necessary, we get

$$J(v) < 0 \quad \text{for} \quad v \in \Gamma_2.$$

If $v \in \Gamma_3$, then

$$J(v) = \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\int_\Omega |\nabla w|^2 dx + \frac{R^2}{2}\int_\Omega |\nabla e_\varepsilon|^2 dx$$

$$- \frac{R^{2^*}}{2^*}\int_\Omega\left(e_\varepsilon + \frac{w+u_t}{R}\right)_+^{2^*} dx.$$

Let $K > 0$ be such that $\|w + u_t\|_{L^\infty} \leqslant K$. Then there exists an $\varepsilon_0 > 0$ (small enough) such that $P^+ e_\varepsilon(0) > 2K$ for every $0 < \varepsilon \leqslant \varepsilon_0$. Then for every $0 < \varepsilon \leqslant \varepsilon_0$ we can find $R_0 = R_0(\varepsilon)$ and $\eta = \eta(\varepsilon) > 0$ such that

$$\left| \left\{ x \in \Omega;\ e_\varepsilon + \frac{w + e_\varepsilon}{R} > 1 \right\} \right| \geqslant \eta$$

for $R \geqslant R_0$. Hence $J(v) \leqslant 0$ for $v \in \Gamma_3$ for $\varepsilon \leqslant \varepsilon_0$ and $R \geqslant R_0$. ■

4. Case $Q_M < 2^{2/(N-2)} Q_m$.

Let $H(y)$ denote the mean curvature of the boundary $\partial\Omega$ at $y \in \partial\Omega$. Throughout this section we assume that:

(**A**) the coefficient Q satisfies the following conditions:

$$Q_M < 2^{2/(N-2)} Q_m,$$

and $|Q(x) - Q(y)| = o(|x - y|)$ for some $y \in \partial\Omega$ with $Q(y) = Q_m$, $H(y) > 0$ and x close to y.

Obviously in this case we have $S_\infty = \dfrac{S^{N/2}}{2NQ_m^{(N-2)/2}}$.

PROPOSITION 4.1. *Let $N \geqslant 5$ and suppose that* (**A**) *holds. Then*

$$\max_{v \in D} J(v) < \frac{S^{N/2}}{2NQ_m^{(N-2)/2}}. \tag{4.1}$$

PROOF. Without loss of generality we may assume that $y = 0$. Let $v \in D$. Then $v = w + se_\varepsilon$ and

$$J(w + se_\varepsilon) = \frac{1}{2} \int_\Omega (|\nabla w|^2 - \lambda w^2)\, dx + \frac{s^2}{2} \int_\Omega (|\nabla e_\varepsilon|^2 - \lambda e_\varepsilon^2)\, dx$$
$$- \frac{1}{2^*} \int_\Omega Q(x)(w + se_\varepsilon + u_t)_+^{2^*}\, dx.$$

For $0 < s \leqslant s_0$, with s_0 sufficiently small, we have

$$J(w + se_\varepsilon) \leqslant \frac{s^2}{2} \int_\Omega |\nabla e_\varepsilon|^2\, dx \leqslant \frac{S^{N/2}}{2NQ_m^{(N-2)/2}}.$$

If $s \geqslant s_0$, then repeating the estimates from Lemma 3.2 we get

$$J(w + se_\varepsilon) \leqslant \frac{s^2}{2} \int_\Omega (|\nabla e_\varepsilon|^2 - \lambda e_\varepsilon^2)\, dx - \frac{s^{2^*}}{2^*} \int_\Omega Q(x)\, e_\varepsilon^{2^*}\, dx + Cs^{2^*} \varepsilon^{(N-2)/2}$$

for some constant $C>0$. Hence

$$J(w+se_\varepsilon)\leqslant \frac{1}{N}\frac{(\int_\Omega(|\nabla e_\varepsilon|^2-\lambda e_\varepsilon^2)\,dx)^{N/2}}{(\int_\Omega Q(x)\,|e_\varepsilon|^{2^*}\,dx)^{(N-2)/2}}+O(\varepsilon^{(N-2)/2}).$$

Since

$$\int_\Omega |P^+U_\varepsilon|^{2^*}\,dx=\int_\Omega U_\varepsilon^{2^*}\,dx+O(\varepsilon^{N-2})$$

and

$$\int_\Omega |\nabla P^+U_\varepsilon|^2\,dx=\int_\Omega |\nabla U_\varepsilon|^2\,dx+O(\varepsilon^{N-2}),$$

we obtain

$$J(w+se_\varepsilon)\leqslant \frac{1}{N}\frac{(\int_\Omega(|\nabla U_\varepsilon|^2-\lambda U_\varepsilon^2)\,dx)^{N/2}}{(\int_\Omega Q(x)\,U_\varepsilon^{2^*}\,dx)^{(N-2)/2}}+O(\varepsilon^{(N-2)/2}).$$

We need the following asymptotic formulas (see [17])

$$\int_\Omega |\nabla U_\varepsilon|^2\,dx=\frac{K_1}{2}-I(\varepsilon)+o(\varepsilon),$$

$$\int_\Omega U_\varepsilon^{2^*}\,dx=\frac{K_2}{2}-\Pi(\varepsilon)+o(\varepsilon),$$

where $K_1=\int_{\mathbb{R}^N}|\nabla U|^2\,dx$, $K_2=\int_{\mathbb{R}^N}U^{2^*}\,dx$, $S=K_1/K_2^{(N-2)/N}$, $I(\varepsilon)=O(\varepsilon)$ and $\Pi(\varepsilon)=O(\varepsilon)$. Moreover, we have (see (3.17) in [17])

(4.2) $$\lim_{\varepsilon\to 0}\frac{I(\varepsilon)}{\Pi(\varepsilon)}>\frac{N-2}{N}\frac{K_2}{K_1}.$$

By assumption (A) we have

$$\int_\Omega Q(x)\,U_\varepsilon^{2^*}\,dx=\frac{Q_m K_2}{2}+o(\varepsilon).$$

Thus

$$(4.3)\qquad J(w+se_\varepsilon) \leqslant \frac{1}{N}\,\frac{\left(\frac{K_1}{2}-I(\varepsilon)+o(\varepsilon)\right)^{N/2}}{\left(\frac{Q_m K_2}{2}-\Pi(\varepsilon)\,Q_m+o(\varepsilon)\right)^{(N-2)/2}}+O(\varepsilon^{(N-2)/2}).$$

According to (4.2) we can find an $\varepsilon_0 > 0$ and a $\varrho > 0$ such that

$$(4.4)\qquad I(\varepsilon) > \frac{N-2}{N}\,\frac{K_1}{K_2}\,\Pi(\varepsilon)+\varrho$$

for $0 < \varepsilon \leqslant \varepsilon_0$. It then folllows from (4.3) and (4.4) that

$$\begin{aligned} J(w+se_\varepsilon) &\leqslant \left(\left(\frac{K_1}{2}\right)^{N/2}-\frac{N}{2}\left(\frac{K_1}{2}\right)^{(N-2)/2} I(\varepsilon)+o(\varepsilon)\right) \\ &\quad\times\left(\left(\frac{1}{2}K_2 Q_m\right)^{-(N-2)/2}+\frac{N-2}{2}Q_m\,\Pi(\varepsilon)\left(\frac{1}{2}K_2 Q_m\right)^{-N/2}+o(\varepsilon)\right) \\ &< \frac{S^{N/2}}{2NQ_m^{(N-2)/2}}-C\varrho \end{aligned}$$

for some constant $C > 0$ and the result follows. ■

We are now in a position to formulate the following result

THEOREM 4.2. *Suppose that the assumptions of Proposition* 4.1 *hold. Then problem* (4.1) *has at least two solutions.*

5. Case $Q_M \geqslant 2^{2/(N-2)} Q_m$.

If $Q_M \geqslant 2^{(2/N)-2} Q_m$, then $S_\infty = \dfrac{S^{N/2}}{NQ_M^{(N-2)/2}}$. We assume that

$$(5.1)\qquad |Q(x)-Q(y)| = o(|x-y|^2)$$

for some $y \in \Omega$ with $Q(y) = Q_M$ and x close to y.

Assuming that $y = 0$, we have

$$\int_\Omega |\nabla U_\varepsilon|^2\,dx = K_1 + O(\varepsilon^{N-2}),$$

$$\int_\Omega Q(x)\,U_\varepsilon^{2^*}\,dx = K_2 Q_M + o(\varepsilon^2)$$

and

$$\int_\Omega U_\varepsilon^2\,dx \geqslant c_1\,\varepsilon^2$$

for some constant $c_1 > 0$ independent of ε. As in the proof of Proposition 4.1 we have

$$\begin{aligned} J(w+se_\varepsilon) &\leqslant \frac{(\int_\Omega(|\nabla U_\varepsilon|^2-\lambda U_\varepsilon^2)\,dx)^{N/2}}{N(\int_\Omega Q(x)\,U_\varepsilon^{2^*}\,dx)^{(N-2)/2}} + O(\varepsilon^{(N-2)/2}) \\ &\leqslant \frac{(K_1+O(\varepsilon^{N-2})-c_1\,\varepsilon^2)^{N/2}}{N(K_2\,Q_M+o(\varepsilon^2))^{(N-2)/2}} + O(\varepsilon^{(N-2)/2}). \end{aligned}$$

If $N \geqslant 7$, taking $\varepsilon > 0$ sufficiently small, we can check that

$$\max_{v\in D} J(v) S^{N/2}/Q_M^{(N-2)/2}.$$

THEOREM 5.1. *Let* $N \geqslant 7$. *Suppose that* $Q_M \geqslant 2^{2/(N-2)}Q_m$ *and that* (5.1) *holds. Then problem* (1.1) *has two solutions.*

6. Existence of solutions in the case $\lambda = 0$.

In this case problem (1.1) takes the form

$$\text{(6.1)}\qquad \begin{cases} -\Delta u = Q(x)\,u_+^{2^*-1} + f(x) & \text{in } \Omega \\ \dfrac{\partial}{\partial\nu}u(x) = 0 \quad \text{on } \partial\Omega, \end{cases}$$

Obviously a necessary condition for the existence of a solution of problem (6.1) is the condition

$$\text{(6.2)}\qquad \int_\Omega f(x)\,dx < 0.$$

Since the eigenfunctions corresponding to $\lambda = 0$ are constant, we decompose $H^1(\Omega)$ as $H^1(\Omega) = \text{span}\,\{1\} \oplus E^+$, where

$$E^+ = \left\{ v \in H^1(\Omega);\ \int_\Omega v\,dx = 0 \right\}.$$

Thus for every function $u \in H^1(\Omega)$ we have $u = t + v$, where $t \in \mathbb{R}$ and

$\int_\Omega v\,dx = 0$. The variational functional J for (6.1) is given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \frac{1}{2^*}\int_\Omega Q(x)(t+v)_+^{2^*}\,dx - \int_\Omega f(x)(t+v)\,dx.$$

It is easy to show that the function $t \to J(t+v)$ is bounded above. Let $v \in E^+$ and set

$$g(t) = J(t+v).$$

It is clear that for every $v \in E^+$ there exists $t(v) > 0$ such that

$$g(t(v)) = \max_{t\in\mathbb{R}} g(t),$$

that is, $J(t+v) \leqslant J(t(v)+v)$ for every $t \in \mathbb{R}$. Thus by the implicit function theorem we can define a continuously differentiable mapping

$$v \in E^+ \Rightarrow t(v) \in \mathbb{R},$$

such that $J(t+v) \leqslant J(t(v)+v)$ for every $t \neq t(v)$. Since

$$0 = g'(t(v)) = -\int_\Omega Q(x)(t(v)+v)^{2^*-1}\,dx - \int_\Omega f\,dx,$$

we see that

$$\int_\Omega Q(x)(t(v)+v)_+^{2^*-1}\,dx + \int_\Omega f(x)\,dx = 0 \tag{6.3}$$

for every $v \in E^+$. In particular, if $v = 0$, then

$$\int_\Omega Q(x)(t(0))_+^{2^*-1}\,dx = -\int_\Omega f(x)\,dx.$$

This combined with (6.2) yields $t(0) > 0$ and we have

$$t(0)^{2^*-1}\int_\Omega Q(x)\,dx = -\int_\Omega f(x)\,dx. \tag{6.4}$$

We now claim that the functional

$$F(v) = J(v + t(v))$$

attains its minimum on some ball $B_\varrho(0)$. We set

$$A = -\int_\Omega f(x)\,dx \quad \text{and} \quad B = \int_\Omega Q(x)\,dx.$$

By easy computations using (6.4), we verify that

$$F(0) = \frac{N+2}{2N} \frac{A^{2N/(N+2)}}{B^{(N-2)/(N+2)}}.$$

We now estimate $F(v)$ from below:

$$F(v) \geqslant J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \frac{1}{2^*}\int_\Omega Qv_+^{2^*}\,dx - \int_\Omega fv\,dx$$

$$\geqslant \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \frac{1}{2^*}\int_\Omega Qv_+^{2^*}\,dx - \|f\|_2\|v\|_2.$$

We now observe that

$$\int_\Omega |\nabla v|^2\,dx \geqslant \lambda_2 \int_\Omega v^2\,dx$$

for every $v \in E^+$. Since $\int_\Omega v\,dx = 0$, we can use the Sobolev inequality to obtain

$$F(v) \geqslant \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \frac{Q_M}{2^*} S^{-N/(N-2)} \left(\int_\Omega |\nabla v|^2\,dx\right)^{2^*/2}$$

$$-\|f\|_2 \lambda_2^{-1} \left(\int_\Omega |\nabla v|^2\,dx\right)^{1/2}.$$

Letting $\varrho = \|\nabla v\|_2$ we derive from the above estimate

$$F(v) \geqslant \frac{\varrho^2}{2} - \frac{Q_M}{2^*} S^{-N/N_2} \varrho^{2^*} - \|f\|_2 \lambda_2^{-1/2} \varrho$$

$$= \varrho z \left(\frac{\varrho}{2} - \frac{Q_M}{2^*} S^{-N/N-2} \varrho^{2^*-1} - \|f\|_2 \lambda_2^{-1/2}\right) = \varrho j(\varrho).$$

Since $j(\varrho)$ achieves its maximum at

$$\varrho_0 = \left(\frac{N}{(N+2)\,Q_M}\right)^{(N-2)/4} S^{N/4}$$

we see that

$$F(v) \geqslant \varrho_0\left(\varrho_0\left(\frac{1}{2} - \frac{N-2}{2(N+2)}\right) - \|f\|_2 \lambda_2^{-1/2}\right) \tag{6.6}$$

$$= \varrho_0\left(\frac{2\varrho_0}{N+2} - \|f\|_2 \lambda_2^{-1/2}\right).$$

We now assume that

$$\|f\|_2 \leqslant \frac{\lambda_2^{1/2}}{N+2}\left(\frac{N}{(N+2)\,Q_M}\right)^{(N-2)/4} S^{N/4} \tag{6.7}$$

and

$$-\int_\Omega f(x)\,dx \leqslant \tag{6.8}$$

$$\leqslant \left(\frac{2N}{N+2}\right)^{(N+2)/2N} N^{-(N+2)/2N} Q_M^{-(N^2-4)/4N} S^{(N+2)/4}\left(\int_\Omega Q(x)\,dx\right)^{1/2^*}.$$

As an immediate consequence of (6.5), (6.6), (6.7) and (6.8) we can state the following lemma:

LEMMA 6.1. *Suppose that* (6.7) *and* (6.8) *hold. Then* $F(v) > F(0)$ *for all* $v \in E^+$ *such that* $\|v\| = \varrho_0$, *and moreover*

$$F(0) < \frac{S^{N/2}}{NQ_M^{(N-2)/2}}.$$

We can now formulate the existence result in the case $\lambda = 0$.

THEOREM 6.2. *Suppose that* (6.7) *and* (6.8) *hold. Then problem* (6.1) *has a solution.*

PROOF. It follows from Lemma 6.1 that

$$m = \inf_{v \in B_{\varrho_0}(0)} F(v) < \frac{S^{N/2}}{NQ_M^{(N-2)/2}}. \tag{6.9}$$

Let $\{v_n\}$ be a minimizing sequence for (6.9). Since $\{v_n\}$ is bounded in $H^1(\Omega)$, we may assume that $v_n \rightharpoonup v_0$ in $H^1(\Omega)$ and $v_n \rightharpoonup v_0$ in $L^q(\Omega)$ for every $2 \leqslant q < 2^*$. By the low semicontinuity of norm with respect to weak convergence we have

$$\|v_0\| \leqslant \liminf_{n\to\infty} \|v_n\| \leqslant \varrho_0 .$$

Estimate (6.6) shows that F is bounded away from 0 near the boundary of $B_{\varrho_0}(0)$ for f small enough. On the other hand $m \leqslant F(0)$ and $F(0)$ is close to 0 for small f. Therefore we can always assume that the minimizing sequence $\{v_n\}$ is contained in the interior of the ball $B_{\varrho_0}(0)$, say $\{v_n\} \subset \subset B_{\varrho_0/2}(0)$. It then follows from the Ekeland variational principle that

$$F(v_n) \to m \quad \text{and} \quad F'(v_n) \to 0.$$

Since $F'(v_n) \to 0$ means that $J'(v_n + t(v_n)) \to 0$ we obtain

$$\frac{1}{2}\int_\Omega |\nabla v_n|^2\,dx - \frac{1}{2^*}\int_\Omega Q(x)(v_n + t(v_n))_+^{2^*}\,dx - \int_\Omega f(v_n + t(v_n))\,dx = m + o(1)$$

and also by (6.3) we have

$$\int_\Omega |\nabla v_n|^2\,dx - \int_\Omega Q(x)(v_n + t(v_n))_+^{2^*-1} v_n\,dx - \int_\Omega f v_n\,dx = o(1).$$

Since v_0 is a weak solution of (6.1) we have

$$\int_\Omega \left(|\nabla v_0|^2 - Q(x)(v_0 + t(v_0))_+^{2^*-1} v_0 - f v_0\right) dx = 0 \tag{6.10}$$

and

$$\int_\Omega \left(Q(x)(v_0 + t(v_0))_+^{2^*-1} + f\right) dx = 0. \tag{6.11}$$

We need to show that $v_n \to v_0$ in $H^1(\Omega)$. As in [12] we show that $\lim_{n\to\infty} t(v_n) = t(v_0)$. We set $w_n = v_n - v_0$. By the Brézis-Lieb Lemma, we have

$$F(v_0) + \frac{1}{2}\int_\Omega |\nabla w_n|^2\,dx - \frac{1}{2^*}\int_\Omega Q(x)(w_n)_+^{2^*}\,dx = m + o(1) \tag{6.12}$$

and

$$\int_\Omega |\nabla w_n|^2\,dx - \int_\Omega Q(x)(w_n)_+^{2^*}\,dx - \int_\Omega Q(x)(v_0 + t(v_0))_+^{2^*}\,dx$$
$$+ \int_\Omega |\nabla v_0|^2\,dx - \int_\Omega f(v_0 + t(v_0))\,dx = o(1).$$

It then follows from (6.10) and (6.11) that

$$\int_\Omega |\nabla w_n|^2\,dx - \int_\Omega Q(x)(w_n)_+^{2^*}\,dx = o(1)\,.$$

Hence by (6.12) we get

$$F(v_0) + \frac{1}{N}\int_\Omega |\nabla w_n|^2\,dx = m + o(1)\,.$$

Since $F(v_0) \geqslant m$, this implies that $\int_\Omega |\nabla w_n|^2\,dx = o(1)$ and consequently $v_n \to v_0$ in $H^1(\Omega)$. ■

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