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Fundamental Solutions of Pseudo-Differential Operators over p-Adic Fields.

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ABSTRACT - We show the existence of fundamental solutions for p-adic pseudo-differential operators with polynomial symbols.

1. Introduction.

Let K be a p-adic field, i.e. a finite extension of $\mathbb{Q}_p$ the field of p-adic numbers. Let R_K be the valuation ring of K, P_K the maximal ideal of R_K, and $\overline{K} = R_K/P_K$ the residue field of K. The cardinality of $\overline{K}$ is denoted by q. For $z \in K$, $v(z) \in \mathbb{Z} \cup \{+\infty\}$ denotes the valuation of z, $|z|_K = q^{-v(z)}$ and $ac(z) = z\pi^{-v(z)}$ where π is a fixed uniformizing parameter for R_K. Let Ψ denote an additive character of K trivial on R_K but not on P_K^{-1}.

A function $\Phi : K^n \to \mathbb{C}$ is called a Schwartz-Bruhat function if it is locally constant with compact support. We denote by $\mathcal{S}(K^n)$ the $\mathbb{C}$-vector space of Schwartz-Bruhat functions over K^n.The dual space $\mathcal{S}'(K^n)$ is the space of distributions over K^n. Let $f = f(x) \in K[x]$, $x = (x_1, \ldots, x_n)$, be a non-zero polynomial, and β a complex number satisfying $\mathrm{Re}(\beta) > 0$. If $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in K^n$, we set $[x, y] := \sum_{i=1}^{n} x_i y_i$.

A p-adic pseudo-differential operator $f(\partial, \beta)$, with symbol $|f|_K^\beta$, is an

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operator of the form

$$(1.1)\qquad \begin{array}{rccc} f(\partial,\beta): & \mathcal{S}(K^n) & \to & \mathcal{S}(K^n) \\ & \Phi & \to & \mathcal{F}^{-1}(|f|_K^{\beta}\mathcal{F}(\Phi)), \end{array}$$

where

$$(1.2)\qquad \begin{array}{rccc} \mathcal{F}: & \mathcal{S}(K^n) & \to & \mathcal{S}(K^n) \\ & \Phi & \to & \int_{K^n} \Psi(-[x,y])\,\Phi(x)\,dx \end{array}$$

is the Fourier transform. The operator $f(\partial, \beta)$has self-adjoint extension with dense domain in $L^2(K^n)$. We associate to $f(\partial, \beta)$ the following p-adic pseudo-differential equation:

$$(1.3)\qquad f(\partial,\beta)\,u = g\,, \qquad g \in \mathcal{S}(K^n).$$

A fundamental solution for (1.3) is a distribution E such that $u = E * g$ is a solution.

The main result of this paper is the following.

THEOREM 1.1. *Every p-adic pseudo-differential equation $f(\partial, \beta)\, u = g$, with $f(x) \in K[x_1, \ldots, x_n] \setminus K$, $g \in \mathcal{S}(K^n)$, and $\beta \in \mathbb{C}$, $\mathrm{Re}(\beta) > 0$, has a fundamental solution $E \in \mathcal{S}'(K^n)$.*

The p-adic pseudo-differential operators occur naturally in p-adic quantum field theory [11], [6]. Vladimirov showed the existence of a fundamental solutions for symbols of the form $|\xi|_K^{\alpha}$, $\alpha > 0$ [10], [11]. In [7], [6] Kochubei showed explicitly the existence of fundamental solutions for operators with symbols of the form $|f(\xi_1, \ldots, \xi_n)|_K^{\alpha}$, $\alpha > 0$, where $f(\xi_1, \ldots, \xi_n)$ is a quadratic form satisfying $f(\xi_1, \ldots, \xi_n) \neq 0$ if $|\xi_1|_K + \ldots + |\xi_n|_K \neq 0$. In [8] Khrennikov considered spaces of functions and distributions defined outside the singularities of a symbol, in this situation he showed the existence of a fundamental solution for a p-adic pseudo-differential equation with symbol $|f|_K \neq 0$. The main result of this paper shows the existence of fundamental solutions for operators with polynomial symbols. Our proof is based on a solution of the division problem for p-adic distributions. This problem is solved by adapting the ideas developed by Atiyah for the archimedean case [1], and Igusa's theorem on the meromorphic continuation of local zeta functions [3], [4]. The connection between local zeta functions (also called Igusa's local zeta functions) and fundamental solutions of p-adic pseudo-differential operators has been explicitly showed in particular cases by Jang and Sato [5], [9]. In [9] Sato studies the asymptotics of the Green function G of the following pseudo-differential equation

$$(f(\partial, 1) + m^2)\, u = g, \qquad m > 0. \tag{1.4}$$

The main result in [9, theorem 2.3] describes the asymptotics of $G(x)$ when the polynomial f is a relative invariant of some prehomogeneous vector spaces (see e.g. [3, Chapter 6]). The key step is to establish a connection between the Green function $G(x)$ and the local zeta function attached to f.

All the above mentioned results suggest a deep connection between Igusa's work on local zeta functions (see e.g. [3]) and p-adic pseudo-differential equations.

2. Local zeta functions and division of distributions.

The local zeta function associated to f is the distribution

$$\langle |f|_K^s, \Phi \rangle = \int_{K^n} \Phi(x)\, |f(x)|_K^s\, dx, \tag{2.1}$$

where $\Phi \in \mathcal{S}(K^n)$, $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$, and $dx$is the Haar of K^n normalized so that $vol(R_K^n) = 1$. The local zeta functions were introduced by Weil [12] and their basic properties for general f were first studied by Igusa [3], [4]. A central result in the theory of local zeta functions is the following.

THEOREM 2.1 (Igusa, [3, Theorem 8.2.1]). *The distribution* $|f|_K^s$ *admits a meromorphic continuation to the complex plane such that* $\langle |f|_K^s, \Phi \rangle$ *is a rational function of* q^{-s} *for each* $\Phi \in \mathcal{S}(K^n)$. *In addition the real parts of the poles of* $|f|_K^s$ *are negative rational numbers.*

The archimedean counterpart of the previous theorem was obtained jointly by Bernstein and Gelfand [2], independently by Atiyah [1]. The following lemma is a consequence of the previous theorem.

LEMMA 2.1. *Let* $f(x) \in K[x_1, \ldots, x_n]$ *be a non-constant polynomial, and* β *a complex number satisfying* $\mathrm{Re}(\beta) > 0$. *Then there exists a distribution* $T \in \mathcal{S}'(K^n)$ *satisfying* $|f|_K^{\beta}\, T = 1$.

Proof. By theorem 2.1 $|f|_K^s$ has a meromorphic continuation to $\mathbb{C}$ such that $\langle |f|_K^s, \Phi\rangle$ is a rational function of q^{-s} for each $\Phi \in \mathcal{S}(K^n)$. Let

$$|f|_K^s = \sum_{m \in \mathbb{Z}} c_m (s+\beta)^m \tag{2.2}$$

be the Laurent expansion at $-\beta$ with $c_m \in \mathcal{S}'(K^n)$ for all m. Since the real parts of the poles of $|f|_K^s$ are negative rational numbers by theorem 2.1, it holds that $|f|_K^{s+\beta} = |f|_K^\beta \, |f|_K^s$ is holomorphic at $s = -\beta$. Therefore $|f|_K^\beta c_m = 0$ for all $m < 0$ and

$$|f|_K^{s+\beta} = c_0 \, |f|_K^\beta + \sum_{m=1}^{\infty} c_m \, |f|_K^\beta (s+\beta)^m . \tag{2.3}$$

By using the Lebesgue lemma and (2.3) it holds that

$$\lim_{s \to -\beta} \langle |f|_K^{s+\beta}, \Phi\rangle = \int_{K^n} \Phi(x)\, dx = \langle 1, \Phi \rangle \tag{2.4}$$

$$= c_0 \, |f|_K^\beta .$$

Therefore we can take $T = c_0$. ∎

If $T \in \mathcal{S}'(K^n)$ we denote by $\mathcal{F}T \in \mathcal{S}'(K^n)$ the Fourier transform of the distribution T, i.e. $\langle \mathcal{F}T, \Phi\rangle = \langle S, \mathcal{F}(\Phi)\rangle$, $\Phi \in \mathcal{S}(K^n)$.

3. Proof of the main result.

By lemma 2.1 there exists a $T \in \mathcal{S}'(K^n)$ such that $|f|_K^\beta T = 1$. We set $E = \mathcal{F}^{-1} T \in \mathcal{S}'(K^n)$ and assert that E is a fundamental solution for (1.3). This last statement is equivalent to assert that $\mathcal{F}(\Phi) = (\mathcal{F}E)\, \mathcal{F}(g)$ satisfies $|f|_K^\beta \mathcal{F}(\Phi) = \mathcal{F}(g)$. Since $|f|_K^\beta \mathcal{F}(\Phi) = |f|_K^\beta (\mathcal{F}E)\, \mathcal{F}(g) = |f|_K^\beta T\, \mathcal{F}(g) = \mathcal{F}(g)$, we have that E is a fundamental solution for (1.3).

4. Operators with twisted symbols.

Let $\chi : R_K^\times \to \mathbb{C}$ be a non-trivial multiplicative character, i.e. a homomorphism with finite image, where $R_K^\times$ is the groupof units of R_K. We put formally $\chi(0) = 0$. If $f(x) \in K[x_1, \dots, x_n] \setminus K$, we say that $\chi(ac(f))\, |f|_K^\beta$, with $\beta \in \mathbb{C}$, $\mathrm{Re}(\beta) > 0$, is a *twisted symbol*, and call the pseudo-differential operator

$$(4.1)\quad \Phi \to f(\partial, \beta, \chi)\, \Phi = \mathcal{F}^{-1}(\chi(ac(f))\, |f|_K^{\beta} \mathcal{F}(\Phi)), \qquad \Phi \in \mathcal{S}(K^n),$$

a *twisted operator*. Since the distribution $\chi(ac(f))\,|f|_K^{\beta}$ satisfies all the properties stated in theorem 2.1 (cf. [3, Theorem 8.2.1]), theorem 1.1 generalizes literally to the case of twisted operators. In [6, chapter 2] Kochubei showed explicitly the existence of fundamental solutions for twisted operators in some particular cases.

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