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Polarized Surfaces (X, L) with $g(L) = q(X) + m$ and $h^0(L) \geqslant m + 2$.

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ABSTRACT - Let (X, L) be a polarized surface. In this paper, we study (X, L) with $g(L) = q(X) + m$ and $h^0(L) \geqslant m + 2$. In particular, we get that $\kappa(X) = -\infty$ if $h^0(L) > 0$. Furthermore we obtain that if X is minimal with $\kappa(X) = -\infty$ and $q(X) = 0$, then $h^0(L) > m + 2$ for any ample line bundle L of X. We also study some special cases with $\kappa(X) = -\infty$ and $q(X) \geqslant 1$.

0. Introduction.

Let X be a smooth projective variety over the complex number field $\mathbb{C}$ of $\dim X = n$, and let L be an ample (resp. a nef and big) line bundle on X. Then we call the pair (X, L) a *polarized* (resp. *quasi-polarized*) *manifold*. In order to study polarized manifolds (X, L) deeply, we often use some invariants of (X, L) (for example, the degree L^n, the delta genus $\Delta(L)$, etc.). The *sectional genus* $g(L)$ is one of famous invariants of (X, L) and it is defined as follows:

$$g(L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1},$$

where K_X is the canonical line bundle of X. If L is very ample, then the sectional genus is exactly the genus of a curve which is obtained by intersecting general $n-1$ elements of the complete linear system $|L|$.

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A classification of (X, L) with small value of sectional genus was obtained by several authors. On the other hand, Fujita proved the following theorem (see Theorem (II.13.1) in [Fj3]).

THEOREM. *Let (X, L) be a polarized manifold. Then for fixed $g(L)$ and $n = \dim X$, there are only finitely many deformation types of (X, L) unless (X, L) is a scroll over a smooth curve.*

(For a definition of the deformation type of (X, L), see § 13 of Chapter II in [Fj3].) By this theorem, Fujita proposed the following conjecture:

CONJECTURE (FUJITA). *Let (X, L) be a polarized manifold. Then $g(L) \geqslant q(X)$, where $q(X) = h^1(\mathcal{O}_X)$: the irregularity of X.*

(See (13.7) in [Fj3]. See also Question 7.2.11 in [BeSo].) Here we state the known facts about this conjecture.

First we consider the case where $\dim X = 2$. Then the conjecture is true if (X, L) is one of the following cases (see [Fk1]):

(a-1) $\kappa(X) \leqslant 1$,

(a-2) L is an ample line bundle on X with $h^0(L) := \dim H^0(L) > 0$.

By these facts, the remaining case for $\dim X = 2$ is the following:

(a-?) X is of general type and L is an ample line bundle with $h^0(L) = 0$.

We can also prove that the conjecture is true for some special cases of (a-?) (see [Fk2]), but in general it is unknown whether the conjecture is true or not for the case (a-?).

Next we consider the case where $\dim X = n \geqslant 3$. Then the conjecture is true if (X, L) is one of the following:

(b-1) $0 \leqslant \kappa(X) \leqslant 1$ and $L^n \geqslant 2$ (see [Fk3]),

(b-2) $\dim \mathrm{Bs}|L| \leqslant 0$, where $\mathrm{Bs}|L|$ denotes the base locus of $|L|$ (see [Fk5]),

(b-3) $\dim X = 3$ and $h^0(L) \geqslant 2$ (see [Fk7]).

In particular, we note that if L is very ample, then $g(L) \geqslant q(X)$ holds by virtue of Lefschetz' theorems on hyperplane sections. Here we also note that even if $\dim X \geqslant 3$ and $h^0(L) > 0$, it is unknown whether the conjecture is true or not. The above are representative cases, and we can also prove that the conjecture is true for many special polarized varieties. But in general it is also unknown whether the conjecture is true or not.

Here we consider the case where $\dim X = 3$ and $h^0(L) \geqslant 2$. Then we get that $g(L) \geqslant q(X)$ by (b-3) above. So it is natural to try to classify polarized manifolds (X, L) with $\dim X = 3$ and $h^0(L) \geqslant 2$ by the value of $g(L) - q(X)$. By this motivation, the author studied polarized 3-folds (X, L) with $h^0(L) \geqslant 2$, and we obtained the classification of polarized 3-folds (X, L) with the following types:

(b-3-1) $g(L) = q(X)$ and $h^0(L) \geqslant 3$ (see [Fk7]),
(b-3-2) $g(L) = q(X) + 1$ and $h^0(L) \geqslant 4$ (see [Fk4]),
(b-3-3) $g(L) = q(X) + 2$ and $h^0(L) \geqslant 5$ (see [Fk8]).

By considering this result of 3-dimensional case, it is natural to consider the following problem:

Problem. *Let* (X, L) *be a polarized manifold with* $\dim X = n$ *and* $g(L) = q(X) + m$, *where* m *is a nonnegative integer. Assume that* $h^0(L) \geqslant n + m$. *Then classify* (X, L) *with these properties.*

In [Fk9], we get a classification of polarized manifolds (X, L) with $n := \dim X \geqslant 3$, $g(L) = q(X) + m$, $\dim \mathrm{Bs}|L| \leqslant 0$, and $h^0(L) \geqslant m + n$.

In this paper, we consider the case in which $n = 2$, $g(L) = q(X) + m$ and $h^0(L) \geqslant m + 2$. The contents of this paper are the following:

(I) We will prove that if $h^0(L) > 0$, then $\kappa(X) = -\infty$ (see Theorem 2.1 and Theorem 2.2 below).

(II) We consider the case in which $\kappa(X) = -\infty$.

(II-1) If X is minimal with $q(X) = 0$, then for any ample line bundle L, we get that $h^0(L) > m + 2$ (see Theorem 2.5 below). We also study the case where X is not minimal with $q(X) = 0$ (see Remark 2.6).

(II-2) We consider some special cases with $q(X) \geqslant 1$ (see Proposition 2.7 and Theorem 2.9 below).

By considering Theorem 2.5 and Remark 2.6, the author thinks that there are many examples of polarized surfaces (X, L) with $g(L) = q(X) + m$ and $h^0(L) \geqslant m + 2$. So it is difficult to classify all (X, L) with $g(L) = q(X) + m$ and $h^0(L) \geqslant m + 2$.

Last of all, we note that if $n \geqslant 3$, then we can use the adjunction theory for $K_X + (n-2)L$. But if $n = 2$, then we cannot use the theory. So we need to study (X, L) by the value of the Kodaira dimension.

We use the customary notation in algebraic geometry.

1. Preliminaries.

THEOREM 1.1. *Let (X, L) be a polarized manifold with $n = \dim X \geqslant 2$. Assume that $|L|$ has a ladder and $g(L) \geqslant \Delta(L)$, where $\Delta(L)$ is the delta genus of (X, L).*

(1) *If $L^n \geqslant 2\Delta(L) + 1$, then $g(L) = \Delta(L)$ and $q(X) = 0$.*
(2) *If $L^n \geqslant 2\Delta(L)$, then $\mathrm{Bs}|L| = \emptyset$.*
(3) *If $L^n \geqslant 2\Delta(L) - 1$, then $|L|$ has a regular ladder.*

PROOF. See (I.3.5) in [Fj3]. ■

THEOREM 1.2. *Let (X, L) be a polarized manifold with $n = \dim X \geqslant 2$. If $\dim \mathrm{Bs}|L| \leqslant 0$ and $L^n \geqslant 2\Delta(L) - 1$, then $|L|$ has a ladder.*

PROOF. See (I.4.15) in [Fj3]. ■

DEFINITION 1.3. (See Definition 1.1 in [Fj1].) Let (X, L) be a polarized surface. Then (X, L) is called a *hyperelliptic polarized surface* if $\mathrm{Bs}|L| = \emptyset$, the morphism defined by $|L|$ is of degree two onto its image W, and if $\Delta(W, H) = 0$ for the hyperplane section H on W.

THEOREM 1.4. *Let (X, L) be a polarized manifold such that $\mathrm{Bs}|L| = \emptyset$, $L^n = 2\Delta(L)$, and $g(L) > \Delta(L)$. Then (X, L) is hyperelliptic unless L is simplely generated and (X, L) is a Fano-K3 variety.*

PROOF. See Theorem 1.4 in [Fj1]. ■

THEOREM 1.5. *Let (X, L) be a hyperelliptic polarized surface. Then (X, L) is one of the following types:*

Type	L^2	$g(L)$	$q(X)$
(I_a)	2	a	0
(IV_a)	8	$2a+1$	0
$({}^*II_a)$	4	$2a$	0
$(\sum(\delta_1, \delta_2)^+_{a,b})$	$2\|\delta\|$	$a\|\delta\| + b - 1$	0
$(\sum(\delta_1, \delta_2)^0_b)$	$2\|\delta\|$	$b-1$	$b-1$
$(\sum(\mu, \mu)^=_a)$	4μ	$a\mu - 1$	$a-1$
$(\sum(\mu + 2\gamma, \mu)^-_a)$	$4(\mu+\gamma)$	$a\mu + 2a\gamma - \gamma - 1$	0

Furthermore the Kodaira dimension of X is the following:

Value of $\kappa(X)$	2	1
(I_a)	$a > 2$	—
(IV_a)	$a > 2$	—
$({}^*II_a)$	$a > 1$	—
$(\sum(\delta_1, \delta_2)^+_{a,b})$	case (5)	case (4)
$(\sum(\delta_1, \delta_2)^0_b)$	—	—
$(\sum(\mu, \mu)^=_a)$	—	—
$(\sum(\mu + 2\gamma, \mu)^-_a)$	$a > 2$	$a = 2$ *and* $\gamma > 2$

Value of $\kappa(X)$	0	$-\infty$
(I_a)	$a = 2$	$a < 2$
(IV_a)	$a = 2$	—
$({}^*II_a)$	—	$a = 1$
$(\sum(\delta_1, \delta_2)^+_{a,b})$	case (3) and (6a)	case (1) and (2)
$(\sum(\delta_1, \delta_2)^0_b)$	—	*any* b
$(\sum(\mu, \mu)^=_a)$	—	*any* a
$(\sum(\mu + 2\gamma, \mu)^-_a)$	$a = \gamma = 2$	$a = 2$ *and* $\gamma > 1$

For the definition of the above types, see [Fj1]. *In particular for the cases of the type* $(\sum(\delta_1, \delta_2)^+_{a,b})$, *see* (5.20) *in* [Fj1].

PROOF. See [Fj1]. (Here we remark that the case (6b) of type $(\sum(\delta_1, \delta_2)^+_{a,b})$ is impossible because $\dim X = 2$.) ■

DEFINITION 1.6 (See Definition 1.9 in [Fk1].)

(1) Let (X, L) be a quasi-polarized surface. Then (X, L) is called *L-minimal* if $LE > 0$ for any (-1)-curve E on X.

(2) Let (X, L) and (Y, A) be quasi-polarized surfaces. Then (Y, A) is called an *L-minimalization of* (X, L) if there exists a birational mor-

phism $\mu : X \to Y$ such that $L = \mu^*(A)$ and (Y, A) is A-minimal. (We remark that an L-minimalization of (X, L) always exists.)

(3) Let (X, L) and (X', L') be polarized surfaces. Then (X, L) is called a *simple blowing up of* (X', L') if X is a blowing up of X' at $x \in X'$ and $(E, L_E) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ for the exceptional divisor E.

(4) Let (X, L) be a polarized surface. Then (X, L) is called the *reduction model* if (X, L) is not obtained by finite times of a simple blowing up of another polarized surface.

Remark 1.6.1. (1) Let X be a smooth projective surface and let L be an ample line bundle on X. Then (X, L) is L-minimal.

(2) Let (X, L) be a polarized surface. Then there exist a polarized surface (Y, A) and a birational morphism $\pi : X \to Y$ such that (X, L) is finite times of a simple blowing up of (Y, A), and (Y, A) is the reduction model. In this case we obtain that $g(L) = g(A)$, $q(X) = q(Y)$, and $h^0(L) \leq h^0(A)$.

Theorem 1.7. *Let* (X, L) *be a quasi-polarized surface with* $h^0(L) \geq 2$ *and* $\kappa(X) = 2$. *Assume that* $g(L) = q(X) + m$. *Then* $L^2 \leq 2m$. *Moreover if* $L^2 = 2m$ *and* (X, L) *is L-minimal, then* $X \cong C_1 \times C_2$ *and* $L \equiv C_1 + 2C_2$, *where* C_1 *and* C_2 *are smooth curves with* $g(C_1) \geq 2$ *and* $g(C_2) = 2$.

Proof. See Theorem 3.1 in [Fk6]. ■

Remark 1.7.1. Let (X, L) be as in Theorem 1.7. Then $L^2 \leq 2m$ is equivalent to $K_X L \geq 2q(X) - 2$.

Theorem 1.8. *Let* (X, L) *be a quasi-polarized surface with* $\kappa(X) = 0$ *or* 1. *Assume that* $g(L) = q(X) + m$. *Then* $L^2 \leq 2m + 2$.

If this equality holds and (X, L) *is L-minimal, then* (X, L) *is one of the following;*

(1) $\kappa(X) = 0$ *case.*

X is an Abelian surface and L is any nef and big divisor.

(2) $\kappa(X) = 1$ *case.*

$X \cong F \times C$ *and* $L \equiv C + (m + 1)F$, *where* F *and* C *are smooth curves with* $g(C) \geq 2$ *and* $g(F) = 1$. *If* $h^0(L) > 0$, *then* $L = C + \sum_{x \in I} m_x F_x$, *where* F_x *is a fiber of the second projection over* $x \in C$, I *is a set of a finite point of* C, *and* m_x *is a positive integer with* $\sum_{x \in I} m_x = m + 1$.

PROOF. See Theorem 2.1 in [Fk6]. ■

REMARK 1.8.1. Let (X, L) be as in Theorem 1.8. Then $L^2 \leqslant 2m + 2$ is equivalent to $K_X L \geqslant 2q(X) - 4$.

THEOREM 1.9. *Let (X, L) be a quasi-polarized surface. Assume that $\Delta(L) = 0$. Then $\kappa(X) = -\infty$.*

PROOF. See Corollary 1.7 in [Fj2] and Theorem 3.1 in [Fk1]. ■

2. Main Theorem.

THEOREM 2.1. *Let X be a smooth projective surface defined over the complex number field $\mathbb{C}$, and let L be an ample line bundle on X. Assume that* $\dim \mathrm{Bs}|L| \leqslant 0$ *and* $h^0(L) \geqslant m + 2$, *where* $m = g(L) - q(X)$. *Then* $\kappa(X) = -\infty$.

PROOF. We assume that $\kappa(X) \geqslant 0$. By taking the reduction model, if necessary, we may assume that (X, L) is the reduction model by Remark 1.6.1 (2).

(1) The case in which $\kappa(X) = 2$.

Then by Theorem 1.7, we get that $L^2 \leqslant 2m$. We put $t = 2m - L^2$. Then we get that

$$\begin{aligned}
\Delta(L) &= 2 + L^2 - h^0(L) \\
&= 2 + 2m - t - h^0(L) \\
&\leqslant 2 + 2m - t - (m + 2) \\
&= m - t \\
&= m - (2m - L^2) \\
&= L^2 - m \\
&\leqslant L^2 - \frac{1}{2} L^2 \\
&= \frac{1}{2} L^2.
\end{aligned}$$

Since $g(L)=1+(1/2)(K_X+L)L>(1/2)L^2\geqslant\Delta(L)$ and $\dim \mathrm{Bs}|L|\leqslant 0$, we get that $|L|$ has a ladder and by Theorem 1.1 we get that $\mathrm{Bs}|L|=\emptyset$.

If $L^2\geqslant 2\Delta(L)+1$, then we get that L is very ample, $g(L)=\Delta(L)$, and $q(X)=0$, that is, $g(L)=m$. By the definition of sectional genus, we get that

$$1+\frac{1}{2}(K_X+L)L=m,$$

that is,

$$2m-2=(K_X+L)L\geqslant L^2.$$

On the other hand, we get that

$$\begin{aligned}\Delta(L)&=2+L^2-h^0(L)\\&\leqslant 2+2m-2-(m+2)\\&=m-2<g(L).\end{aligned}$$

But this is impossible. Therefore we may assume that $L^2=2\Delta(L)$.

Then $h^0(L)=m+2$ and $L^2=2m$. Here we remark that $g(L)>$ $>(1/2)L^2=\Delta(L)$. By Theorem 1.4, we get that (X, L) is a hyperelliptic polarized surface. So we can use the list of the classification of hyperelliptic polarized surfaces. (See Theorem 1.5.) Since $\kappa(X)=2$, we obtain that $q(X)=0$ and $g(L)=m$. Therefore

$$\begin{aligned}m=g(L)&=1+(1/2)(K_X+L)L\\&=1+(1/2)K_XL+m\\&>m.\end{aligned}$$

But this is impossible.

(2) The case in which $\kappa(X)=0$ or 1.

Then by Theorem 1.8, we get that $L^2\leqslant 2m+2$.

(2.A) The case where $L^2=2m+2$.

Then $K_XL=2q(X)-4$. If $\kappa(X)=1$, then $\dim \mathrm{Bs}|L|=1$ by Theorem 1.8 (2), and this is impossible. If $\kappa(X)=0$, then by Theorem 1.8 (1) X is an abelian surface and $h^0(L) = L^2/2 \leqslant m+1$ and this is a contradiction by hypothesis. So we assume that $L^2 \leqslant 2m+1$.

(2.B) The case where $L^2 = 2m+1$.

Then

$$\begin{aligned}\Delta(L) = 2 + L^2 - h^0(L) &\leqslant 2m+3-(m+2)\\ &= m+1\\ &= \frac{1}{2}(2m)+1\\ &= \frac{1}{2}(L^2-1)+1\\ &= \frac{1}{2}L^2+\frac{1}{2}.\end{aligned}$$

Hence $L^2 \geqslant 2\Delta(L)-1$ and $|L|$ has a regular ladder by Theorem 1.1 (3).

(2.B.1) The case in which $\kappa(X) = 0$.

Then $g(L) = q(X) + m \leqslant 2 + m$ by the classification of surfaces. So we get that

$$\begin{aligned}2+m \geqslant g(L) &= 1 + \frac{1}{2}(K_X L + 2m + 1)\\ &= \frac{1}{2}K_X L + m + \frac{3}{2}.\end{aligned}$$

Hence $K_X L \leqslant 1$. Since $L^2 = 2m+1$ is odd, we get that $K_X L = 1$ and $q(X) = 2$. Therefore X is a one point blowing up of an abelian surface S, $\pi: X \to S$ and $L = \pi^*(A) - E$ for an ample line bundle A on S. Since $A^2 = 2m+2$, we get that $h^0(A) = A^2/2 = m+1$. But since $m+2 \leqslant$ $\leqslant h^0(L) = h^0(\pi^*A - E) \leqslant h^0(\pi^*A) = h^0(A) = m+1$, we get a contradiction.

(2.B.2) The case where $\kappa(X) = 1$.

Then there exists an elliptic fibration $f: X \to C$, where C is a smooth projective curve. Let $\mu: X \to S$ be the relatively minimal model of $f: X \to C$ and let $h: S \to C$ be a relatively minimal elliptic fibration such that $f = h \circ \mu$. We put $A = \mu_*(L)$. Then A is ample. Here we remark that $q(X) \leqslant g(C) + 1$.

(2.B.2.1) The case in which $g(C) \leqslant 1$.

Then $q(X) \leqslant 2$ and $g(L) \leqslant m + 2$. So we get that

$$2 + m \geqslant g(L) = 1 + \frac{1}{2}(K_X L + 2m + 1)$$

$$= \frac{1}{2} K_X L + m + \frac{3}{2}.$$

Therefore $K_X L = 1$ because $\kappa(X) = 1$ and L is ample. If X is not relatively minimal, then $K_X L > K_S A$ and this is impossible because $K_X L = 1$. So X is relatively minimal. By the canonical bundle formula, we get that

$$K_X \equiv (2g(C) - 2 + \chi(\mathcal{O}_X))\, F + \sum_i (m_i - 1)\, F_i,$$

where F is a general fiber of f and $m_i F_i$ is the multiple fiber of f.

If $g(C) = 0$, then $q(X) \leqslant 1$, and $g(L) \leqslant 1 + m$. Then

$$1 + m \geqslant g(L) = 1 + \frac{1}{2}(K_X L + 2m + 1)$$

$$= \frac{1}{2} K_X L + m + \frac{3}{2}$$

$$> m + \frac{3}{2},$$

and this is a contradiction.

If $g(C) = 1$ and $q(X) = 2$, then $\chi(\mathcal{O}_X) = 0$ and $K_X \equiv \sum_i (m_i - 1)\, F_i$. Since $\kappa(X) = 1$, there exists at least two multiple fibers (see Proposition 1.3 in [Se2]). So we get that $K_X L \geqslant 2$ and this is a contradiciton.

If $g(C) = 1$ and $q(X) = 1$, then $g(L) = m + 1$ and this is impossible by the same argument as above.

(2.B.2.2) The case in which $g(C) \geqslant 2$.

Then $K_X L \geqslant K_S A \geqslant (2g(C) - 2 + \chi(\mathcal{O}_X))AF$. Since $L^2 = 2m + 1$, we get that

$$g(L) = 1 + \frac{1}{2}(K_X + L)\,L$$
$$\geqslant 1 + \frac{1}{2}(2g(C) - 2 + \chi(\mathcal{O}_X))\,AF + m + \frac{1}{2}\,.$$

If $q(X) = g(C)$, then this is impossible.

If $q(X) = g(C) + 1$, then $\chi(\mathcal{O}_X) = 0$ (see Lemma 1.6 in [Se1]) and so we have

$$g(L) \geqslant 1 + (g(C) - 1)\,AF + m + \frac{1}{2}$$
$$= m + \frac{3}{2} + (g(C) - 1)\,AF.$$

If $AF \geqslant 2$, then

$$g(L) \geqslant m + \frac{3}{2} + 2g(C) - 2$$
$$= g(C) + m + g(C) - \frac{1}{2}$$
$$= g(C) + 1 + m + g(C) - \frac{3}{2}$$
$$> q(X) + m.$$

So we get that $AF = 1$. Since $q(X) = g(C) + 1$, by Lemma 1.13 in [Fk1], we get that $S \cong C \times F$ and $A \equiv C + aF$ for an integer a. Since $LF = AF = 1$, we get that X is minimal. Here we remark that f has no multiple fiber because $LF = 1$. Hence by the canonical bundle formula we get that

$$g(L) = 1 + \frac{1}{2}(K_X + L)\,L$$
$$= 1 + (g(C) - 1) + \frac{1}{2}L^2$$
$$= q(X) - 1 + \frac{1}{2}L^2.$$

But this is a contradiction because $L^2 = 2m + 1$ by assumption.

(2.C) The case in which $L^2 \leqslant 2m$.

Here we do the above argument. We put $t := 2m - L^2$. Then $t \geqslant 0$ and we get that

$$\begin{aligned} \varDelta(L) &= 2 + L^2 - h^0(L) \\ &= 2 + 2m - t - h^0(L) \\ &\leqslant 2 + 2m - t - (m+2) \\ &= m - t \\ &= m - (2m - L^2) \\ &= L^2 - m \\ &\leqslant L^2 - \frac{1}{2}L^2 \\ &= \frac{1}{2}L^2. \end{aligned}$$

Therefore we get $L^2 \geqslant 2\varDelta(L)$. Here we remark that

$$\begin{aligned} g(L) &= 1 + \frac{1}{2}(K_X + L)L \\ &\geqslant 1 + \frac{1}{2}L^2 \\ &> \varDelta(L). \end{aligned}$$

Since $\dim Bs|L| \leqslant 0$, we get that $|L|$ has a ladder by Theorem 1.2. If $L^2 \geqslant 2\varDelta(L) + 1$, then $g(L) = \varDelta(L)$ and $q(X) = 0$ by Theorem 1.1 (1). Hence $m = g(L) = \varDelta(L) \leqslant (L^2 - 1)/2$, that is, $L^2 \geqslant 2m + 1$ and this is impossible. Therefore we may assume that $L^2 = 2\varDelta(L)$. In this case $L^2 = 2m$, $\varDelta(L) = m$, and $h^0(L) = m + 2$ by the above inequalities. By Theorem 1.1 (2), we get that $\mathrm{Bs}|L| = \emptyset$. So by Theorem 1.4, we get that (X, L) is either a hyperelliptic polarized surface or a polarized K3-surface.

If X is a K3-surface, then $q(X) = 0$ and $g(L) = m$. So we get that

$$m = g(L) = 1 + \frac{1}{2}(K_X + L)L = 1 + \frac{1}{2}L^2.$$

Hence $L^2 = 2m - 2$ and $\varDelta(L) = m - 1$. But then $h^0(L) = 2 + L^2 - (m-1) = m + 1$ and this is impossible.

If (X, L) is a hyperelliptic polarized surface, then since $\kappa(X) = 0$ or 1

by the list of hyperelliptic polarized surface (see Theorem 1.5) we get that (X, L) is one of the following types:

Type	L^2	$g(L)$	$q(X)$
(I_2)	2	2	0
$(\sum(\delta_1, \delta_2)^+_{a,b})$	$2\|\delta\|$	$a\|\delta\| + b - 1$	0
$(\sum(\mu + 2\gamma, \mu)^-_2)$	$4(\mu + \gamma)$	$2\mu + 3\gamma - 1$	0

Here we remark that if (X, L) is the type $(\sum(\mu + 2\gamma, \mu)^-_2)$, then $\gamma \geqslant 2$.

If (X, L) is the type (I_2) (resp. (IV_2), $(\sum(\mu + 2\gamma, \mu)^-_2)$), then $h^0(L) = \Delta(L) + 2 = \dfrac{L^2}{2} + 2 = 3$ (resp. 6, $2(\mu + \gamma + 1)$).

If (X, L) is the type (I_2) or (IV_2), then $h^0(L) = g(L) + 1 = m + 1 < m + 2$.

If (X, L) is the type $(\sum(\mu + 2\gamma, \mu)^-_2)$, then $h^0(L) = g(L) - \gamma = m - \gamma < m + 2$. In each case we get a contradiction.

If (X, L) is the type $(\sum(\delta_1, \delta_2)^+_{a,b})$, then (X, L) is the case (3), the case (4) or the case (6a) in (5.20) in [Fj1]. Here we use the notation in (5.20) in [Fj1].

Assume that (X, L) is the case (3) in (5.20) in [Fj1]. Then $q(X) = 0$, $h^0(K_X) = 1$, and $K_X = 0$. Hence $m = g(L) = 1 + (L^2/2)$, that is, $L^2 = 2m - 2$. By the Riemann-Roch theorem and the Kodaira vanishing theorem, we get that

$$\begin{aligned} h^0(L) = \chi(L) &= \chi(\mathcal{O}_X) + \frac{1}{2}(L^2 - K_X L) \\ &= 2 + m - 1 \\ &= m + 1 \end{aligned}$$

and this is a contradiction.

Assume that (X, L) is the case (4) in (5.20) in [Fj1]. In this case $a = \dim X = 2$ and $g(L) = 2|\delta| + b - 1$. Here we calculate $h^0(L)$.

$$\begin{aligned} h^0(L) &= 2 + L^2 - \Delta(L) \\ &= 2 + 2\Delta(L) - \Delta(L) \\ &= \Delta(L) + 2 \\ &= \frac{L^2}{2} + 2 \\ &= |\delta| + 2. \end{aligned}$$

Here we remark that $L^2 = 2m$ and $\Delta(L) = m$. So in particular $h^0(L) = m + 2$. Hence $|\delta| = m$. On the other hand since $q(X) = 0$, we get that $m = g(L) = 2|\delta| + b - 1 = 2m + b - 1$. Therefore $|\delta| + b - 1 = 0$. But since $b' = |\delta| + b - 2 > 0$ by (5.20) in [Fj1]. This is a contradiction.

Assume that (X, L) is the case (6a) in (5.20) in [Fj1]. Then $\delta_1 = \delta_2 + 1$ and $|\delta| = \delta_1 + \delta_2 = 2\delta_2 + 1$. Furthermore $a = \dim X + 1 = 3$ and $b = -(\dim X + 1)\delta_2 = -3\delta_2$. Therefore

$$\begin{aligned} g(L) &= 3(2\delta_2 + 1) - 3\delta_2 - 1 \\ &= 3\delta_2 + 2, \end{aligned}$$

and

$$\begin{aligned} L^2 &= 2(2\delta_2 + 1) \\ &= 4\delta_2 + 2. \end{aligned}$$

On the other hand

$$h^0(L) = \Delta(L) + 2 = \frac{L^2}{2} + 2 = 2\delta_2 + 3.$$

Since $h^0(L) \geqslant m + 2$ and $g(L) = q(X) + m = m$, we get that

$$2\delta_2 + 3 = h^0(L) \geqslant m + 2 = g(L) + 2 = 3\delta_2 + 4,$$

that is, $\delta_2 \leqslant -1$. But then $g(L) = 3\delta_2 + 2 \leqslant -1$ and this is impossible. These complete the proof of Theorem 2.1. ■

Theorem 2.2. *Let (X, L) be a polarized surface with* $\dim \mathrm{Bs}|L| = 1$ *and $h^0(L) \geqslant m + 2$ for $m = g(L) - q(X)$. Then $\kappa(X) = -\infty$.*

Proof. Assume that $\kappa(X) \geqslant 0$. By the Kodaira vanishing theorem and the Riemann-Roch theorem we get that

$$h^0(K_X + L) - h^0(K_X) = g(L) - q(X).$$

If $h^0(K_X) > 0$, then since $h^0(L) \geqslant m + 2$ we get that

$$m + 1 \leqslant h^0(L) - 1 \leqslant h^0(K_X + L) - h^0(K_X) = g(L) - q(X) = m$$

and this is impossible. Hence $h^0(K_X) = 0$. By the classification theory of surfaces we get that $q(X) \leqslant 1$. Let (Y, A) be the reduction model of (X, L). Then $K_Y + A$ is nef, $g(A) = q(Y) + m$, $q(Y) \leqslant 1$, and $h^0(A) \geqslant$

$\geqslant m+2$. Let M_A be the movable part of $|A|$ and let Z_A be the fixed part of $|A|$.

(I) The case where M_A is nef and big.

Then we get that

$$
\begin{aligned}
1+m \geqslant q(Y)+m = g(A) & \\
&= 1+\frac{1}{2}(K_Y+A)A \\
&\geqslant 1+\frac{1}{2}(K_Y+A)M_A \\
&\geqslant 1+\frac{1}{2}(K_Y+M_A)M_A.
\end{aligned}
$$

Hence $(K_Y+M_A)M_A \leqslant 2m$. Here we also remark that $g(M_A) \leqslant q(Y)+m$ by the above inequalities.

If $M_A^2 \leqslant m$, then

$$
\begin{aligned}
\Delta(M_A) &= 2+M_A^2-h^0(M_A) \\
&\leqslant 2+m-(m+2)=0.
\end{aligned}
$$

So we get that $\Delta(M_A)=0$ and by Theorem 1.9 this is impossible because $\kappa(Y) \geqslant 0$. Hence $M_A^2 \geqslant m+1$ and $K_Y M_A \leqslant m-1$, that is, $M_A^2 > K_Y M_A$. In particular $h^2(M_A) = h^0(K_Y - M_A) = 0$ because M_A is nef. So by the Riemann-Roch theorem, we get that

$$h^0(M_A) = h^1(M_A) + \chi(\mathcal{O}_Y) + \frac{1}{2}(M_A^2 - K_Y M_A).$$

(I.1) The case in which $K_Y M_A > 0$.

(I.1.1) The case in which a general member of $|M_A|$ is irreducible.

Let C be a general member of $|M_A|$. Then C is an irreducible curve. So by the following exact sequence

$$0 \to K_Y - M_A \to K_Y \to K_Y|_C \to 0,$$

we get the following exact sequence

$$0 = H^0(K_Y) \to H^0(K_Y|_C) \to H^1(K_Y - M_A) \to H^1(K_Y).$$

Since $g(C)=g(M_A)\geqslant 2$, we get that $h^0(K_Y|_C)\leqslant \deg K_Y|_C = K_YM_A$.

If $q(Y)=0$, then $h^1(K_Y-M_A)=h^0(K_Y|_C)\leqslant K_YM_A$ and $\chi(\mathcal{O}_Y)=1$. So we get that

$$\begin{aligned}h^0(M_A)&=h^1(M_A)+\chi(\mathcal{O}_Y)+\frac{1}{2}(M_A^2-K_YM_A)\\&\leqslant K_YM_A+1+\frac{1}{2}(M_A^2-K_YM_A)\\&=g(M_A)\leqslant m\end{aligned}$$

and this is impossible.

If $q(Y)=1$, then $h^1(K_Y-M_A)=h^0(K_Y|_C)+1\leqslant K_YM_A+1$ and $\chi(\mathcal{O}_Y)=0$. So we get that

$$\begin{aligned}h^0(M_A)&=h^1(M_A)+\chi(\mathcal{O}_Y)+\frac{1}{2}(M_A^2-K_YM_A)\\&\leqslant K_YM_A+1+\frac{1}{2}(M_A^2-K_YM_A)\\&=g(M_A)\leqslant m+1\end{aligned}$$

and this is also impossible.

(I.1.2) The case in which a general member of M_A is not irreducible.

Let $D=\sum_{i=1}^{a}C_i$ be a general member of M_A, where $a\geqslant 2$. Then D is reduced since dim Bs$|M_A|\leqslant 0$. In this case we get that $h^0(K_Y|_{C_i})\leqslant$ $\leqslant \deg(K_Y|_{C_i})+1$ and $h^0(K_Y|_D)\leqslant h^0(K_Y|_{C_i})$ for any i. Hence

$$\begin{aligned}ah^0(K_Y|_D)&\leqslant\left(\sum_{i=1}^{a}\deg(K_Y|_{C_i})\right)+a\\&=K_YD+a\\&=K_YM_A+a.\end{aligned}$$

Therefore

$$h^0(K_Y|_D)\leqslant\frac{1}{a}K_YM_A+1.$$

Claim 2.2.1. *$h^0(K_Y|_D)\leqslant K_YM_A$ if $K_YM_A\geqslant 1$.*

PROOF. If $K_Y M_A = 1$, then $h^0(K_Y|_D) \leqslant (1/a) + 1$. But since $a \geqslant 2$ and $h^0(K_Y|_D)$ is integer, we get that $h^0(K_Y|_D) \leqslant 1 = K_Y M_A$.

If $K_Y M_A \geqslant 2$, then $K_Y M_A \geqslant 2 \geqslant a/(a-1)$. Hence

$$\frac{a-1}{a} K_Y M_A \geqslant 1$$

and so we get that

$$K_Y M_A = \frac{1}{a} K_Y M_A + \frac{a-1}{a} K_Y M_A \geqslant \frac{1}{a} K_Y D + 1 \geqslant h^0(K_Y|_D).$$

This completes the proof of Claim 2.2.1. ■

If $q(Y) = 0$, then $h^1(K_Y - M_A) = h^0(K_Y|_D) \leqslant K_Y M_A$ and $\chi(\mathcal{O}_Y) = 1$. So we get that

$$\begin{aligned} h^0(M_A) &= h^1(M_A) + \chi(\mathcal{O}_Y) + \frac{1}{2}(M_A^2 - K_Y M_A) \\ &\leqslant K_Y M_A + 1 + \frac{1}{2}(M_A^2 - K_Y M_A) \\ &= g(M_A) \leqslant m \end{aligned}$$

and this is impossible.

If $q(Y) = 1$, then $h^1(K_Y - M_A) = h^0(K_Y|_D) + 1 \leqslant K_Y M_A + 1$ and $\chi(\mathcal{O}_Y) = 0$. So we get that

$$\begin{aligned} h^0(M_A) &= h^1(M_A) + \chi(\mathcal{O}_Y) + \frac{1}{2}(M_A^2 - K_Y M_A) \\ &\leqslant K_Y M_A + 1 + \frac{1}{2}(M_A^2 - K_Y M_A) \\ &= g(M_A) \leqslant m + 1 \end{aligned}$$

and this is also impossible.

(I.2) The case in which $K_Y M_A = 0$.

Let (S, H) be an M_A-minimal model of (Y, M_A). Then (S, H) is a quasi-polarized surface. Since $K_Y M_A = 0$ and $\kappa(Y) \geqslant 0$, we get that S is minimal with $K_S H = 0$. Since $H^2 > 0$ and $K_S^2 \geqslant 0$, we get that $K_S \equiv 0$ and so we obtain that $\kappa(S) = 0$. Here we remark that $g(M_A) = g(H)$, $M_A^2 = H^2$, and $q(Y) = q(S) \leqslant 1$. In particular $H^2 > K_S H$ and by the Riemann-

Roch theorem we get that

$$h^0(H) = h^1(H) + \chi(\mathcal{O}_Y) + \frac{1}{2}(H^2 - K_Y H).$$

If $q(S) = 0$, then $H^2 = 2m - 2$, $h^1(K_S - H) \leqslant 1$, and $\chi(\mathcal{O}_S) = 1$. Hence

$$\begin{aligned} h^0(M_A) = h^0(H) &= h^1(H) + \chi(\mathcal{O}_S) + \frac{1}{2}(H^2 - K_S H) \\ &\leqslant 2 + m - 1 \\ &= m + 1 \end{aligned}$$

and this is impossible.

If $q(S) = 1$, then S is a bielliptic surface and $H^2 = 2m$. So we get that $h^0(L) = h^0(A) = h^0(M_A) = h^0(H) = H^2/2 = m$ and this is impossible.

(II) The case where M_A is nef but not big.

Then $\mathrm{Bs}|M_A| = \emptyset$. By using this linear system $|M_A|$, we get that there exists a fiber space $f: Y \to C$ over a smooth projective curve C. Moreover we get that $M_A \equiv aF$, where F is a general fiber of f. So we get that

$$\begin{aligned} g(A) &= 1 + \frac{1}{2}(K_Y + A)\,A \\ &\geqslant 1 + \frac{1}{2}(K_Y + A)\,M_A \\ &= 1 + \frac{a}{2}(K_Y + A)\,F \\ &= 1 + \frac{a}{2}(2g(F) - 2 + AF) \\ &= 1 + a(g(F) - 1) + \frac{a}{2}AF. \end{aligned}$$

Here we remark that $g(F) \geqslant 1$ because $\kappa(Y) \geqslant 0$.

If $g(F) \geqslant 2$, then $1 + m \geqslant g(A) > a + 1$, that is, $a < m$. So we get that $h^0(L) \leqslant h^0(A) = h^0(M_A) \leqslant a < m$ and this is a contradiction.

If $AF \geqslant 2$, then $1 + m \geqslant g(A) \geqslant a + 1$, that is, $a \leqslant m$. So we get that $h^0(L) \leqslant h^0(A) = h^0(M_A) \leqslant a \leqslant m$ and this is impossible.

So we may assume that $g(F) = 1$ and $AF = 1$. Since A is ample with

$AF = 1$, we get that f is relatively minimal, and f has no multiple fiber. Therefore by the canonical bundle formula, we get that the case where $g(C) = 0$ is impossible because $\chi(\mathcal{O}_Y) \leqslant 1$. So we obtain that $g(C) = 1$. Since $1 \geqslant q(Y) \geqslant g(C) = 1$, we get that $q(Y) = 1$ and $\chi(\mathcal{O}_Y) = 0$. By the canonical bundle formula, we obtain that $K_Y \equiv 0$ and since $q(Y) = 1$, we get that $A^2 = 2m$. Hence Y is a bielliptic surface because $q(Y) = 1$. By using the Kodaira vanishing theorem and the Riemann-Roch theorem, we have

$$h^0(A) = \frac{A^2}{2} = \frac{2m}{2} = m.$$

But then $h^0(L) \leqslant h^0(A) = m$ and this is impossible. These complete the proof of Theorem 2.2. ■

REMARK 2.2.1. By the same argument as the proof of Theorem 2.2, we can prove that $\kappa(X) = -\infty$ if (X, L) is a polarized surface with $\dim \mathrm{Bs}|L| \leqslant 0$, $h^0(L) \geqslant m + 2$, and $m = g(L) - q(X)$.

Next we consider the case in which $\kappa(X) = -\infty$. First we fix the notation which is used later.

NOTATION 2.3 (See also Chapter V in [Ha].) Let $X = \mathbb{P}(\mathcal{E})$ be a $\mathbb{P}^1$-bundle over a smooth projective curve C and let $\pi : X \to C$ be its projection, where $\mathcal{E}$ is a vector bundle of rank two on C. Assume that $\mathcal{E}$ is normalized. Let C_0 be a minimal section of π and let F be a fiber of π. We put $e = -C_0^2$. Then $e \geqslant -g(C)$ by Nagata's theorem.

REMARK 2.4. Here we use Notation 2.3. We put $L = aC_0 + bF$ for some integer a and b. Then

$$g(L) = q(X) + (a-1)\left(q(X) - 1 + b - \frac{1}{2}ae\right)$$

and

$$L^2 = 2ab - a^2 e.$$

Here we put $m = (a-1)\left(q(X) - 1 + b - \frac{1}{2}ae\right)$.

First we study the case in which $q(X) = 0$.

THEOREM 2.5. *Let X be a two-dimensional projective space $\mathbb{P}^2$ or a Hirzebruch surface, and let L be an ample line bundle on X. Then L is very ample and $h^0(L) > m + 2$, where $m = g(L) - q(X) = g(L)$.*

PROOF. Assume that $X = \mathbb{P}^2$. Let $L = \mathcal{O}(a)$. Then L is very ample,

$$h^0(L) = \frac{(a+2)(a+1)}{2},$$

and

$$g(L) = 1 + \frac{a(a-3)}{2}.$$

Then

$$m = g(L) = 1 + \frac{a(a-3)}{2}.$$

So we get that

$$\begin{aligned} h^0(L) - m &= \frac{(a+2)(a+1)}{2} - 1 - \frac{a(a-3)}{2} \\ &= 3a \geqslant 3. \end{aligned}$$

Hence $h^0(L) \geqslant m + 3$.

Assume that X is a Hirzebruch surface. Here we use Notation 2.3. Then by Remark 2.4, we get that

$$m = (a-1)\left(b - \frac{1}{2}ae - 1\right).$$

Furthermore by calculating $h^0(L)$, we get that

$$h^0(L) = (a+1)(b+1) - \frac{1}{2}a(a+1)e$$

because L is ample. Hence

$$\begin{aligned} h^0(L) - m &= (a+1)(b+1) - \frac{1}{2}a(a+1)e - (a-1)\left(b - \frac{1}{2}ae - 1\right) \\ &= 2a + 2b - ae \\ &= 2a + b + (b - ae). \end{aligned}$$

Since L is ample, we get that $a > 0$ and $b - ae > 0$ by Corollary 2.18 of Chapter V in [Ha]. Here we also remark that $e \geqslant 0$. Hence $b > 0$. Therefore

$$\begin{aligned} h^0(L) - m &= 2a + b + (b - ae) \\ &\geqslant 2 + 1 + 1 = 4. \end{aligned}$$

Hence $h^0(L) \geqslant m + 4$. By Corollary 2.18 of Chapter V in [Ha] we get that L is very ample. These complete the proof of Theorem 2.5. ■

REMARK 2.6. Let X be a smooth projective surface with $\kappa(X) = -\infty$. Let L be an ample line bundle on X. Assume that $q(X) = 0$ and X is not minimal.

Let $\varrho : X \to X'$ be a minimal model of X. We put $X_0 := X$ and $L_0 := L$. Let $\varrho_{i+1}: X_i \to X_{i+1}$ be a blowing down of (-1)-curve E_i on X_i such that $\varrho = \varrho_n \circ \ldots \circ \varrho_1: X_0 \to X_1 \to \ldots \to X_n$, where X_i is a smooth projective surface for $i = 1, \ldots, n$. We put $L_{i+1} = (\pi_{i+1})_*(L_i)$. Then $X' = X_n$ and $L' = L_n$. We put $L_i = (\pi_{n+1})^*(L_{i+1}) - m_i E_i$, where m_i is a positive integer. Then

$$g(L') = g(L) + \sum_{i=0}^{n-1} \frac{m_i(m_i - 1)}{2}.$$

We put $g(L) = q(X) + m$. Then

$$\begin{aligned} g(L') &= q(X) + m + \sum_{i=0}^{n-1} \frac{m_i(m_i - 1)}{2} \\ &= q(X') + m + \sum_{i=0}^{n-1} \frac{m_i(m_i - 1)}{2}. \end{aligned}$$

We put

$$m' = m + \sum_{i=0}^{n-1} \frac{m_i(m_i - 1)}{2}.$$

Here we calculate $h^0(L)$. Then

$$\begin{aligned} h^0(L) &= h^0(L_0) \\ &\geqslant h^0(L_1) - \frac{m_0(m_0 + 1)}{2} \\ &\geqslant h^0(L_2) - \sum_{i=0}^{1} \frac{m_i(m_i + 1)}{2} \\ &\vdots \\ &\geqslant h^0(L') - \sum_{i=0}^{n-1} \frac{m_i(m_i + 1)}{2}. \end{aligned}$$

So if $h^0(L') \geqslant m' + 2 + \sum_{i=1}^{n-1} m_i$, then

$$\begin{aligned} h^0(L) &\geqslant h^0(L') - \sum_{i=0}^{n-1} \frac{m_i(m_i+1)}{2} \\ &\geqslant m' + 2 + \sum_{i=1}^{n-1} m_i - \sum_{i=0}^{n-1} \frac{m_i(m_i+1)}{2} \\ &= m' - \sum_{i=0}^{n-1} \frac{m_i(m_i-1)}{2} + 2 \\ &= m + 2. \end{aligned}$$

Next we consider the case where $q(X) \geqslant 1$.

PROPOSITION 2.7. *Let (X, L) be a polarized surface with $\dim \mathrm{Bs}|L| \leqslant 0$ and $h^0(L) \geqslant m + 2$, where $m = g(L) - q(X)$. Assume that $\kappa(X) = -\infty$, $q(X) \geqslant 1$, and $L^2 \leqslant 2m$, then (X, L) is the following type;*

(X, L) is a hyperelliptic polarized surface of the type $(\sum(2, 2)_4^{=})$ with $L^2 = 8$, $g(L) = 7$, $q(X) = 3$, and $\kappa(X) = -\infty$.

PROOF. If $L^2 \leqslant 2m$, then by the same argument as Theorem 2.1 we get that

$$\begin{aligned} \varDelta(L) &= 2 + L^2 - h^0(L) \\ &= 2 + (2m - t) - h^0(L) \\ &\leqslant 2m + 2 - t - m - 2 \\ &= m - t \\ &= m - (2m - L^2) \\ &= L^2 - m \\ &\leqslant \frac{1}{2} L^2, \end{aligned}$$

where $t = 2m - L^2$. Therefore $L^2 \geqslant 2\varDelta(L)$. Here we remark that $g(L) = q(X) + m \geqslant (1/2) L^2 \geqslant \varDelta(L)$. Therefore $|L|$ has a ladder because $\dim \mathrm{Bs}|L| \leqslant 0$.

If $L^2 \geqslant 2\varDelta(L) + 1$, then $q(X) = 0$ by Theorem 1.1 (1). But this is a contradiction by hypothesis.

Hence we may assume that $L^2 = 2\varDelta(L)$. Then $L^2 = 2m$ and $m =$

$= \varDelta(L)$. Since $q(X) \geqslant 1$, we get that $g(L) = q(X) + m > m = \varDelta(L)$. So by Theorem 1.1 (2) we get that $\mathrm{Bs}|L| = \emptyset$ and (X, L) is a hyperelliptic polarized surface. Since $q(X) \geqslant 1$, we get that (X, L) is one of the types $(\sum^2(\delta_1, \delta_2)_b^0)$ and $(\sum(\mu, \mu)_a^=)$ by the classification of hyperelliptic polarized surfaces (see Theorem 1.5).

If (X, L) is the type $(\sum^2(\delta_1, \delta_2)_b^0)$, then $g(L) = q(X)$. So we get that $m = 0$. But since $L^2 \leqslant 2m$ by hypothesis, this is impossible.

If (X, L) is the type $(\sum(\mu, \mu)_a^=)$, then we get that

$$(*) \qquad \begin{aligned} g(L) = a\mu - 1 &= (a-1) + a(\mu - 1) \\ &= q(X) + a(\mu - 1), \end{aligned}$$

and so we obtain that $\mu \neq 1$ because $g(L) - q(X) = m \neq 0$. Since $L^2 = 4\mu$ by Theorem 1.5, we get that $2m = L^2 = 4\mu$. Hence $m = 2\mu$. On the other hand $m = a(\mu - 1)$ by $(*)$. So we get that $a = (2\mu)/(\mu - 1) = 2 + (2/(\mu - 1))$. Since a is integer and $0 < L^2 = 4\mu$, we get that $\mu = 2$. Hence $L^2 = 8$, $\mu = 2$, $a = 4$, $q(X) = 3$, $m = 4$, and $g(L) = 7$. This completes the proof of Proposition 2.7. ■

REMARK 2.8. Assume that X is a $\mathbb{P}^1$-bundle with $q(X) \geqslant 1$. Here we use Notation 2.3. By Proposition 2.7, we may assume that $L^2 > 2m$ if $\dim \mathrm{Bs}|L| \leqslant 0$. Then by Remark 2.4 we get that

$$\begin{aligned} L^2 &= 2ab - a^2 e \\ &= 2a\left(b - \frac{1}{2}ae\right) \\ &= 2a\left(\frac{m}{a-1} - q(X) + 1\right) \\ &= \frac{2a}{a-1}m - 2a(q(X) - 1) \\ &= 2m + \frac{2}{a-1}m - 2a(q(X) - 1). \end{aligned}$$

Therefore

$$\frac{2}{a-1}m > 2a(q(X) - 1)$$

that is,

$$\frac{m}{a(a-1)}+1>q(X).$$

THEOREM 2.9. *Let X be a $\mathbb{P}^1$-bundle over a smooth projective curve C with $g(C)\geqslant 1$. Let L be an ample line bundle on X such that $L^2>2m$, where $m=g(L)-q(X)$. Assume that $m\geqslant 1$. If $LF\geqslant m$, then $g(C)=1$ and L is one of the following types:* (*Here we put $L=aC_0+bF$.*)

e	a	b	m
0	$m+1$	1	$\geqslant 1$
0	$\frac{m}{2}+1$	2	*even with* $m\geqslant 2$
1	2	3	2
-1	5	-2	2
-1	2	1	2
-1	7	-3	3
-1	4	-1	3
-1	3	0	3
-1	$2m+1$	$-m$	$\geqslant 1$
-1	$m+1$	$\frac{1-m}{2}$	*odd with* $m\geqslant 1$

PROOF. First we remark that (X, L) is not a scroll over a smooth curve because $m\geqslant 1$. In particular $a\geqslant 2$. By Remark 2.8 we get that $q(X)=1$ because $a=LF\geqslant m$.

(1) The case in which $m\geqslant 2$.

Since L is ample, we get that $b-(1/2)\,ae>0$. So since $q(X)=1$ and $a\geqslant m$, we get that

$$\begin{aligned} m &= (a-1)\left(q(X)-1+b-\frac{1}{2}ae\right) \\ &\geqslant (m-1)\left(q(X)-1+b-\frac{1}{2}ae\right) \\ &= (m-1)\left(b-\frac{1}{2}ae\right). \end{aligned}$$

Hence we get that

$$1 + \frac{1}{m-1} \geqslant b - \frac{1}{2}ae.$$

Since $m \geqslant 2$, we obtain that $b - (ae)/2 \leqslant 2$.

If $e \geqslant 0$, then $b - ae > 0$. Hence we get that

$$b - \frac{1}{2}ae > \frac{1}{2}ae.$$

If $b - (ae)/2 = 1/2$, then $ae < 1$. But this is impossible because $a \geqslant 2$, and a and b are integer.

If $b - (ae)/2 = 1$, then $ae < 2$. So we get that $e = 0$, $a = m + 1$ and $b = 1$ because a and b are integer.

If $b - (ae)/2 = 3/2$, then $ae < 3$. But this is impossible because $a \geqslant 2$, and a and b are integer.

If $b - (ae)/2 = 2$, then $ae < 4$. So we get that $e = 0$, $a = 1 + (m/2)$ is any, and $b = 2$ or $e = 1$, $a = 2$ and $b = 3$ because $a \geqslant 2$, and a and b are integer.

If $e < 0$, then $e = -1$ because $e \geqslant -g(C) = -1$.

If $m = 2$, then $0 < b - (ae)/2 \leqslant 2$ and $2 = (a-1)(b - (ae)/2)$. Hence we get the following type:

$b - (ae)/2$	a	b
$1/2$	5	-2
2	2	1

If $m = 3$, then $0 < b - (ae)/2 \leqslant 3/2$ and $3 = (a-1)(b - (ae)/2)$. Hence we get the following type:

$b - (ae)/2$	a	b
$1/2$	7	-3
1	4	-1
$3/2$	3	0

If $m \geqslant 4$, then $0 < b - (ae)/2 \leqslant 1$.

If $b - (ae)/2 = 1/2$, then $m = (a-1)/2$ and $b = -m$.

If $b - (ae)/2 = 1$, then $m = (a-1)$ and $b = (1-m)/2$. In particular m is odd in this case.

(2) The case in which $m=1$.

Since (X, L) is not scroll over a smooth curve, $LF \geqslant 2$. Since $b-(ae/2)>0$ and

$$1=m=(a-1)\left(q(X)-1+b-\frac{1}{2}ae\right)=(a-1)\left(b-\frac{1}{2}ae\right),$$

we get that $(a, b-(ae)/2)=(2, 1)$ or $(3, 1/2)$.

If $e \geqslant 0$, then $b-ae>0$. Hence if $b-(ae)/2=1$, then

$$\frac{1}{2}ae+1-ae>0.$$

Hence $ae \leqslant 1$. So $e=0$ and $b=1$.

If $b-(ae)/2=1/2$, then

$$\frac{1}{2}ae+\frac{1}{2}-ae>0.$$

Hence $ae \leqslant 0$. So $e=0$ and $b=1/2$. But this is impossible because b is integer.

If $e=-1$, $a=2$, and $b-(ae)/2=1$, then $b=0$.

If $e=-1$, $a=3$, and $b-(ae)/2=1/2$, then $b=-1$. These complete the proof of Theorem 2.9. ■

PROBLEM 2.10. *Let (X, L) be a polarized surface with $\kappa(X)=-\infty$ and $h^0(L) \geqslant m+2$ for $m=g(L)-q(X)$.*

(1) *Classify (X, L) such that X is not minimal with $q(X)=0$.*

(2) Classify (X, L) *such that X is a $\mathbb{P}^1$-bundle with $q(X) \geqslant 1$ and $LF \leqslant m-1$.*

(3) *Classify (X, L) such that X is not minimal with $L^2 \geqslant 2m+1$ and $q(X) \geqslant 1$.*

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