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Pointwise Decay for Solutions of the 2D Neumann exterior problem for the wave equation - II.

PAOLO SECCHI (*)

ABSTRACT - We consider the exterior problem in the plane for the wave equation with a Neumann boundary condition. We are interested to the asymptotic behavior for large times for the solution, and in particular to the dependence on the norms of the initial data in the estimate for the pointwise decay rate. In the paper we improve an estimate of this kind, proved by the author in a previous paper, by a cut-off technique which use in particular a new estimate of the local energy decay of the free space solution.

1. Introduction.

Let Ω be an exterior domain in $\boldsymbol{R}^2$; the boundary $\partial\Omega$ is a smooth, convex and compact hypersurface. Given $r>0$, we denote $\Omega_r = \Omega \cap B_r$, where $B_r = \{x \in \boldsymbol{R}^2 \mid |x| < r\}$. Below, $r_0 > 0$ is a fixed constant such that $\Omega^c \subset B_{r_0}$ (Ω^c is the complement of Ω). We set $Q = [0, \infty) \times \Omega$, $\Sigma = [0, \infty) \times \partial\Omega$.

We study the decay property of solutions to the mixed problem for

(*) Indirizzo dell'A.: Dipartimento di Matematica - via Valotti 9, 25133 Brescia, Italy.

the wave equation with Neumann boundary condition

$$
(1)\qquad \begin{aligned}
&(\partial_{tt}^2 - \Delta)\, u = 0 && \text{in } Q\,,\\
&\partial_\nu u = 0 && \text{on } \Sigma\,,\\
&u(0,\, x) = f(x)\,,\\
&\partial_t u(0,\, x) = g(x) && \text{in } \Omega\,.
\end{aligned}
$$

In the previous paper [6] we showed the decay rate $(1+t)^{-1/2}\log^2(e+t)$, slightly slower than the optimal rate $(1+t)^{-1/2}$ of the free space solution. The aim of the present paper is to improve the dependence on the data, in a form suitable for applications. As in [6], our proof is a combination by a cut-off argument of the estimate of the local energy decay following from the analysis of Kleinman and Vainberg [1], Morawetz [2], Vainberg [7] and decay estimates for the free space solution. Differently from [6], we use a new estimate of the local decay of the free space solution. In order to get a decay rate of local energy in the presence of an obstacle, some assumption on its shape should be taken, in order to exclude the existence of closed ray solutions. In fact, for the Dirichlet problem, Ralston [5] has shown that if there is a closed ray solution, there is no rate of decay. For the Dirichlet problem the obstacle should be *non-trapping*, see [3]; for the Neumann problem, the analysis of Kleinman and Vainberg [1], Morawetz [2], Vainberg [7] gives the decay rate for convex bodies. This is the reason why in this paper we take the boundary convex. The result of this paper will be applied to the study of the Euler compressible flow in a forecoming paper.

Let us introduce some notation. For a multi-index $\alpha = (\alpha_1,\, \alpha_2)$ we set $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}$, $|\alpha| = \alpha_1 + \alpha_2$, where $\partial_1 = \partial/\partial x_1$, $\partial_2 = \partial/\partial x_2$. Let $W^{m,p}(\Omega)$ be the usual Sobolev space of order m, $m = 1, 2, \ldots$ and order of integrability $p \geq 1$, and let $\|\cdot\|_{W^{m,p}}$ denote its norm. If $p = 2$ we set $W^{m,2}(\Omega) = H^m(\Omega)$ with norm $\|\cdot\|_{H^m}$. The norm of $L^2(\Omega)$ is denoted by $\|\cdot\|$, the norm of $L^p(\Omega)$, $1 \leq p \leq \infty$, by $|\cdot|_p$. For simplicity we use the abbreviated notation $W^{m,p}$, H^m, L^p. Let us define the weighted Sobolev space

$$
\widetilde{W}^{m,p} = \widetilde{W}^{m,p}(\Omega) := \{ f \in L^p \colon \|f\|_{\widetilde{W}^{m,p}} < \infty \}
$$

where

$$
\|f\|_{\widetilde{W}^{m,p}} := \Big(\sum_{|\alpha| \leq m} \|(1 + |\cdot|)\, \partial^\alpha f(\cdot)\|_{L^p}^p \Big)^{1/p}.
$$

Clearly $\|f\|_{W^{m,p}} \leqslant \|f\|_{\widetilde{W}^{m,p}}$. If $p=2$ we set $\widetilde{W}^{m,2} = \widetilde{H}^m$ with norm $\|\cdot\|_{\widetilde{H}^m}$. If $m=0$ we set $\widetilde{W}^{0,p} = \widetilde{L}^p$ with norm $\|\cdot\|_{\widetilde{L}^p}$. Similarly we introduce the spaces $W^{m,p}(\boldsymbol{R}^2)$, $H^m(\boldsymbol{R}^2)$, $L^p(\boldsymbol{R}^2)$, $\widetilde{W}^{m,p}(\boldsymbol{R}^2)$, $\widetilde{H}^m(\boldsymbol{R}^2)$ and $\widetilde{L}^p(\boldsymbol{R}^2)$ with norms denoted by $\|\cdot\|_{W^{m,p}(R^2)}$, $\|\cdot\|_{H^m(R^2)}$, $\|\cdot\|_{L^p(R^2)}$, $\|\cdot\|_{\widetilde{W}^{m,p}(R^2)}$, $\|\cdot\|_{\widetilde{H}^m(R^2)}$ and $\|\cdot\|_{\widetilde{L}^p(R^2)}$ respectively. We will also use the same symbol for spaces of vector valued functions.

THEOREM 1.1. *Suppose u is a solution of the exterior problem* (1.1). *Assume the initial data satisfy $f \in \widetilde{W}^{3,1} \cap \widetilde{H}^3$, $g \in \widetilde{W}^{2,1} \cap \widetilde{H}^2$. Then there exists a constant $C>0$ such that, for every $t \geqslant 0$,*

$$\text{(2)} \qquad |\partial_t u(t,\cdot)|_\infty + |\nabla u(t,\cdot)|_\infty \leqslant C(1+t)^{-1/2}\log^2(e+t) \times$$
$$\times\left[\|f\|_{\widetilde{H}^3}^{1/2}(\|f\|_{\widetilde{W}^{3,1}} + \|f\|_{\widetilde{H}^3})^{1/2} + \|g\|_{\widetilde{H}^2}^{1/2}(\|g\|_{\widetilde{W}^{2,1}} + \|g\|_{\widetilde{H}^2})^{1/2}\right].$$

The decay rate obtained in (2) is slightly slower than the optimal rate decay $t^{-1/2}$ of the free space solution.

2. Proof of Theorem 1.1.

Let us take functions $\tilde{f}, \tilde{g}: \boldsymbol{R}^2 \to \boldsymbol{R}$ such that $\tilde{f} = f$, $\tilde{g} = g$ on Ω, and such that $\tilde{f} \in \widetilde{W}^{3,1}(\boldsymbol{R}^2) \cap \widetilde{H}^3(\boldsymbol{R}^2)$, $\tilde{g} \in \widetilde{W}^{2,1}(\boldsymbol{R}^2) \cap \widetilde{H}^2(\boldsymbol{R}^2)$,

$$\text{(3)} \qquad \begin{aligned} &\|\tilde{f}\|_{\widetilde{H}^3(R^2)} + \|\tilde{g}\|_{\widetilde{H}^2(R^2)} \leqslant C(\|f\|_{\widetilde{H}^3} + \|g\|_{\widetilde{H}^2}), \\ &\|\tilde{f}\|_{\widetilde{W}^{3,1}(R^2)} + \|\tilde{g}\|_{\widetilde{W}^{2,1}(R^2)} \leqslant C(\|f\|_{\widetilde{W}^{3,1}} + \|g\|_{\widetilde{W}^{2,1}}). \end{aligned}$$

For this, observe that it's enough to take extensions over the bounded set Ω^c with the regularity $\tilde{f} \in H^3(\boldsymbol{R}^2)$, $\tilde{g} \in H^2(\boldsymbol{R}^2)$, since the required behavior at infinity is already furnished by f, g.

Let u_1 be the solution of the Cauchy problem

$$\text{(4)} \qquad \begin{aligned} &(\partial_{tt}^2 - \Delta)\, u_1 = 0 \qquad \text{in } [0, \infty) \times \boldsymbol{R}^2, \\ &u_1(0,x) = \tilde{f}(x), \\ &\partial_t u_1(0,x) = \tilde{g}(x) \quad \text{in } \boldsymbol{R}^2. \end{aligned}$$

From [4], Theorem 2.1, we have

$$\text{(5)} \qquad |\partial_t u_1(t)|_{L^\infty(R^2)} + |\nabla u_1(t)|_{L^\infty(R^2)} \leqslant C(1+t)^{-1/2}\|(\nabla\tilde{f}, \tilde{g})\|_{W^{2,1}(R^2)}.$$

Choosing $r > r_0$ and $\chi(x) \in C_0^\infty(\boldsymbol{R}^2)$ so that $\chi(x) = 1$ if $|x| \leqslant r$ and $= 0$ if $|x| \geqslant r+1$, we put

$$u_2 = u - (1-\chi)u_1, \qquad G = -u_1 \varDelta\chi - 2\nabla u_1 \cdot \nabla\chi$$

The function u_2 is the solution of the initial boundary value problem

$$(6)\qquad \begin{aligned} &(\partial_{tt}^2 - \varDelta)\, u_2 = G && \text{in } Q, \\ &\partial_\nu u_2 = 0 && \text{on } \Sigma, \\ &u_2(0, x) = \chi f(x), && \\ &\partial_t u_2(0, x) = \chi g(x) && \text{in } \Omega . \end{aligned}$$

Observe that $\operatorname{supp} G(t, \cdot) \subseteq \{x \mid r \leqslant |x| \leqslant r+1\}$ for all $t \geqslant 0$, and $\operatorname{supp} \chi f \subseteq \Omega_{r+1}$, $\operatorname{supp} \chi g \subseteq \Omega_{r+1}$. From (17) in the Appendix with $w = u_2$, $w_0 = \chi f$, $w_1 = \chi g$, we obtain

$$(7)\qquad |\partial_t u_2(t)|_{L^\infty(\Omega_{r+2})} + |\nabla u_2(t)|_{L^\infty(\Omega_{r+2})} \leqslant C_r(1+t)^{-1}(\|\nabla(\chi f)\|_{H^2} + \|\chi g\|_{H^2})$$

$$+ C_r \int_0^t (1+t-s)^{-1} \|G(s)\|_{H^2}\, ds .$$

We estimate $\|G(s)\|_{H^2}$. First of all we observe that

$$\|G(s)\|_{H^2} \leqslant C_r \|u_1(s)\|_{H^3(B_{r+1})}.$$

We apply to $u = u_1$ the local decay estimate given in Corollary 3.1 and use (3). This yields

$$(8)\qquad \|G(s)\|_{H^2} \leqslant C_r M_1 (1+s)^{-1}$$

where

$$M_1 = \|f\|_{\tilde{H}^3}^{1/2}(\|f\|_{\tilde{W}^{3,1}} + \|f\|_{\tilde{H}^3})^{1/2} + \|g\|_{\tilde{H}^2}^{1/2}(\|g\|_{\tilde{W}^{2,1}} + \|g\|_{\tilde{H}^2})^{1/2}.$$

We obtain from (7), (8) and (25) in the Appendix that

$$(9)\qquad |\partial_t u_2(t)|_{L^\infty(\Omega_{r+2})} + |\nabla u_2(t)|_{L^\infty(\Omega_{r+2})} \leqslant C_r(1+t)^{-1}(\|\nabla(\chi f)\|_{H^2} + \|\chi g\|_{H^2})$$

$$+ C_r M_1 \int_0^t (1+t-s)^{-1}(1+s)^{-1}\, ds$$

$$\leqslant C_r M_1 (1+t)^{-1} \log(e+t) \qquad \forall t \geqslant 0,$$

since $\|f\|_{H^m} \leqslant C\|f\|_{\tilde{H}^m}$. Choosing $\psi(x) \in C_0^\infty(\boldsymbol{R}^2)$ so that $\psi(x) = 1$ if $|x| \geqslant r+2$ and $=0$ if $|x| \leqslant r+1$, we observe that

$$\psi\chi f = 0, \qquad \psi\chi g = 0, \qquad \psi G = 0\,.$$

Let us define

$$H = -u_2 \Delta\psi - 2\nabla u_2 \cdot \nabla\psi\,.$$

The function ψu_2 solves the Cauchy problem

$$\begin{aligned} &(\partial_{tt}^2 - \Delta)(\psi u_2) = H \quad \text{in } [0,\infty)\times \boldsymbol{R}^2,\\ &\psi u_2(0,x) = 0,\\ &\partial_t(\psi u_2)(0,x) = 0 \quad \text{in } \boldsymbol{R}^2. \end{aligned} \tag{10}$$

From (5) and the Duhamel's principle we get

$$\begin{aligned} &|\partial_t(\psi u_2)(t)|_{L^\infty(R^2)} + |\nabla(\psi u_2)(t)|_{L^\infty(R^2)}\\ &\leqslant C\int_0^t (1+t-s)^{-1/2}\|H(s,\cdot)\|_{W^{2,1}(R^2)}\,ds\\ &\leqslant C_r\int_0^t (1+t-s)^{-1/2}\|u_2(s)\|_{H^3(\Omega_{r+2})}\,ds\,. \end{aligned} \tag{11}$$

On the other hand, applying (18) in the Appendix to the solution u_2 of (6) yields

$$\begin{aligned} &\|u_2(t)\|_{H^3(\Omega_{r+2})} \leqslant C_r(1+t)^{-1}(\|\chi f\|_{H^3} + \|\chi g\|_{H^2})\\ &+C_r\int_0^t (1+t-s)^{-1}\|G(s)\|_{H^2}\,ds\\ &\leqslant C_r M_1(1+t)^{-1} + C_r M_1\int_0^t (1+t-s)^{-1}(1+s)^{-1}\,ds\\ &\leqslant C_r M_1(1+t)^{-1}\log(e+t)\,. \end{aligned} \tag{12}$$

Then from (11), (12) and (27) in the Appendix one has

$$
\begin{aligned}
&|\partial_t(\psi u_2)(t)|_{L^\infty(R^2)} + |\nabla(\psi u_2)(t)|_{L^\infty(R^2)} \\
&\leqslant C_r M_1 \int_0^t (1+t-s)^{-1/2}(1+s)^{-1}\log(e+s)\,ds \qquad (13)\\
&\leqslant C_r M_1 (1+t)^{-1/2}\log^2(1+t).
\end{aligned}
$$

Moreover, from (12) and a Sobolev imbedding we have

$$
|u_2(t)|_{L^\infty(\Omega_{r+2})} \leqslant C_r M_1 (1+t)^{-1}\log(e+t). \qquad (14)
$$

Since $u = (1-\chi)u_1 + u_2$, we have

$$
\begin{aligned}
&|\partial_t u(t)|_\infty + |\nabla u(t)|_\infty \\
&\leqslant |(1-\chi)\,\partial_t u_1(t)|_\infty + |\nabla((1-\chi)\,u_1(t))|_\infty + |\partial_t u_2(t)|_\infty + |\nabla u_2(t)|_\infty \\
&\leqslant |\partial_t u_1(t)|_{L^\infty(R^2)} + |\nabla u_1(t)|_{L^\infty(R^2)} + C|u_1(t)|_{L^\infty(B_{r+1}\setminus B_r)} \\
&+ |\partial_t(\psi u_2(t))|_{L^\infty(R^2)} + |\nabla(\psi u_2(t))|_{L^\infty(R^2)} \\
&+ |\partial_t u_2(t)|_{L^\infty(\Omega_{r+2})} + |\nabla u_2(t)|_{L^\infty(\Omega_{r+2})} + C|u_2(t)|_{L^\infty(\Omega_{r+2})}.
\end{aligned}
$$

From (3), (5), Corollary 3.1 and a Sobolev imbedding, (9), (13), (14) and Lemma 3.3 we finally obtain

$$
\begin{aligned}
&|\partial_t u(t)|_\infty + |\nabla u(t)|_\infty \\
&\leqslant C(1+t)^{-1/2}\|(\nabla\tilde{f},\tilde{g})\|_{W^{2,1}(R^2)} + C_r M_1(1+t)^{-1} \\
&+ C_r M_1 (1+t)^{-1/2}\log^2(1+t) \qquad (15)\\
&+ C_r M_1 (1+t)^{-1}\log(e+t) \\
&\leqslant C_r M_1 (1+t)^{-1/2}\log^2(e+t) \qquad \forall t \geqslant 0.
\end{aligned}
$$

3. Appendix.

Let us consider the initial boundary value problem

$$(16)\qquad \begin{aligned} &(\partial_{tt}^2 - \Delta)\, w = G && \text{in } Q, \\ &\partial_\nu w = 0 && \text{on } \Sigma, \\ &w(0,x) = w_0, && \\ &\partial_t w(0,x) = w_1 && \text{in } \Omega . \end{aligned}$$

From [6] we have

LEMMA 3.1. *Let* (w_0, w_1) *and* $G(t,\cdot)$ *have compact support for each* $t > 0$. *Assume* $(\nabla w_0, w_1) \in H^2$, $G(t,\cdot) \in H^2$ *for each* $t > 0$. *Then the solution* w *of* (16) *satisfies the estimate*

$$(17)\qquad \begin{aligned} |\partial_t w(t)|_{L^\infty(\Omega_R)} + |\nabla w(t)|_{L^\infty(\Omega_R)} &\leqslant C_R(1+t)^{-1}(\|\nabla w_0\|_{H^2} + \|w_1\|_{H^2}) \\ &+ C_R \int_0^t (1+t-s)^{-1} \|G(s,\cdot)\|_{H^2}\, ds . \end{aligned}$$

If $(w_0, w_1) \in H^3 \times H^2$, $G(t,\cdot) \in H^2$ *for each* $t > 0$, *then*

$$(18)\qquad \begin{aligned} \|w(t)\|_{H^3(\Omega_R)} &\leqslant C_R(1+t)^{-1}(\|w_0\|_{H^3} + \|w_1\|_{H^2}) \\ &+ C_R \int_0^t (1+t-s)^{-1} \|G(s,\cdot)\|_{H^2}\, ds . \end{aligned}$$

The above inequalities hold for every $R > r_0$ *and* $t \geqslant 0$; C_R *depends on* R, *the support of the data and the geometry of* $\partial\Omega$.

Let us consider the initial value problem

$$(19)\qquad \begin{aligned} &(\partial_{tt}^2 - \Delta)\, u = 0 && \text{in } [0, \infty) \times \boldsymbol{R}^2, \\ &u(0,x) = f(x), && \\ &\partial_t u(0,x) = g(x) && \text{in } \boldsymbol{R}^2. \end{aligned}$$

In the following lemma all norms, if not explicitly indicated otherwise, are evaluated over $\boldsymbol{R}^2$.

LEMMA 3.2. *The solution u of the Cauchy problem* (19) *satisfies the estimate*

$$\|u(t)\|_{H^1(B_R)} \leqslant C_R(1+t)^{-1}\big[\|f\|_{\widetilde{H}^1}^{1/2}(\|f\|_{\widetilde{W}^{1,1}}+\|f\|_{\widetilde{H}^1})^{1/2}+\|g\|_{\widetilde{L}^2}^{1/2}(\|g\|_{\widetilde{L}^1}+\|g\|_{\widetilde{L}^2})^{1/2}\big].$$

for every $R>0$ and $t \geqslant 0$, where C_R depends on R.

PROOF. For the sake of brevity here we write $\|\cdot\|$ instead of $\|\cdot\|_{L^2(R^2)}$. First we assume $g=0$. Let us denote by $\widehat{u}(t, \xi)$ the Fourier transfom of u w.r.t. the x-variables:

$$\widehat{u}(t, \xi) = \int_{R^2} u(t, x)\, e^{-i\xi\cdot x}\, dx\, .$$

Moreover, $\widehat{f}$ denotes the Fourier transform of f. From the Cauchy problem it follows that $\widehat{u}(t, \xi) = \cos(t|\xi|)\, \widehat{f}(\xi)$. Then

$$u(t, x) = \frac{1}{(2\pi)^2}\int_{R^2} \cos(t|\xi|)\, \widehat{f}(\xi)\, e^{i\xi\cdot x}\, d\xi\, .$$

Let $0 \leqslant t \leqslant 1$. By the Plancherel theorem

$$\|u(t)\| = (2\pi)^{-1}\|\cos(t|\xi|)\,\widehat{f}\,\| \leqslant (2\pi)^{-1}\|\widehat{f}\,\| = \|f\| \leqslant \|f\|_{\widetilde{L}^2}. \tag{20}$$

Let $t \geqslant 1$. By integrating by parts we get

$$u(t, x) = -\frac{1}{(2\pi)^2 t}\int_{R^2} \sin(t|\xi|)\left\{\frac{\partial}{\partial|\xi|}\widehat{f}(\xi) + \frac{\widehat{f}(\xi)}{|\xi|} + i\frac{\xi\cdot x}{|\xi|}\widehat{f}(\xi)\right\} e^{i\xi\cdot x}\, d\xi\, .$$

We decompose the integral as the sum of integrals over $B_\alpha = \{|\xi| < \alpha\}$ and $\{|\xi| > \alpha\}$ respectively, where α will be chosen below. Accordingly u is written in the form $u = u_1 + u_2$. Then, passing to polar coordinates, we show that

$$|u_1(t,x)| \leqslant \frac{1}{(2\pi)^2 t}\int_{S^1}\int_0^\alpha\left\{\varrho\left\|\frac{\partial}{\partial|\xi|}\widehat{f}\right\|_{L^\infty(B_\alpha)} + (1+\varrho R)\|\widehat{f}\|_{L^\infty(B_\alpha)}\right\} d\varrho\, d\omega \tag{21}$$

$$\leqslant \frac{C_R}{t}\alpha(1+\alpha)\|(1+|x|)\, f\|_{L^1}$$

for every $|x| < R$, since

$$\left\| \frac{\partial}{\partial|\xi|} \widehat{f} \right\|_{L^\infty} \leqslant \|xf\|_{L^1}, \qquad \|\widehat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}.$$

Set $\langle x \rangle = (1 + |x|^2)^{1/2}$ and denote by $\chi_\alpha(\varrho)$ the characteristic function of $\{\varrho \geqslant \alpha\}$. Again by the Plancherel theorem and by $\left\|\frac{\partial}{\partial|\xi|}\widehat{f}\right\| \leqslant C\|xf\|$, we obtain

$$\begin{aligned}
\|u_2(t)\|_{L^2(B_R)} &\leqslant C_R \|\langle x\rangle^{-1} u_2(t, x)\| \\
&\leqslant \frac{C_R}{t} \left\| \langle x \rangle^{-1} \int_{R^2} \chi_\alpha(|\xi|) \sin(t|\xi|) \left\{ \frac{\partial}{\partial|\xi|} \widehat{f}(\xi) + \frac{\widehat{f}(\xi)}{|\xi|} \right\} e^{i\xi\cdot x} d\xi \right\| \\
&\quad + \frac{C_R}{t} \left\| \frac{ix}{\langle x \rangle} \cdot \int_{R^2} \chi_\alpha(|\xi|) \sin(t|\xi|) \frac{\xi}{|\xi|} \widehat{f}(\xi)\, e^{i\xi\cdot x} d\xi \right\| \\
&\leqslant \frac{C_R}{t} \left\| \chi_\alpha(|\xi|) \sin(t|\xi|) \left\{ \frac{\partial}{\partial|\xi|} \widehat{f}(\xi) + \frac{\widehat{f}(\xi)}{|\xi|} \right\} \right\| \\
&\quad + \frac{C_R}{t} \left\| \chi_\alpha(|\xi|) \sin(t|\xi|) \frac{\xi}{|\xi|} \widehat{f}(\xi) \right\| \\
&\leqslant \frac{C_R}{t} \left(\left\| \frac{\partial}{\partial|\xi|} \widehat{f}(\xi) \right\| + \left(\frac{1}{\alpha} + 1 \right) \|\widehat{f}(\xi)\| \right) \leqslant \frac{C_R}{t} \frac{1+\alpha}{\alpha} (\|xf\| + \|f\|).
\end{aligned} \tag{22}$$

We choose $\alpha = \|f\|_{\tilde{L}^1}^{-1/2} \|f\|_{\tilde{L}^2}^{1/2}$. Adding (20)-(22) after the substitution of α gives

$$\|u(t)\|_{L^2(B_R)} \leqslant C_R (1+t)^{-1} \|f\|_{\tilde{L}^2}^{1/2} (\|f\|_{\tilde{L}^1} + \|f\|_{\tilde{L}^2})^{1/2}. \tag{23}$$

Then, we take the spatial derivatives of (19) and apply the estimate just shown, obtaining the estimate of ∇u in terms of ∇f. Finally, we assume $f = 0$. Since the proof is similar to the first case we omit the details. ■

COROLLARY 3.1. *The solution u of the Cauchy problem* (19) *satisfies the estimate*

$$\|u(t)\|_{H^3(B_R)} \leqslant C_R (1+t)^{-1} \left[\|f\|_{\widetilde{H}^3}^{1/2} (\|f\|_{\widetilde{W}^{3,1}} + \|f\|_{\widetilde{H}^3})^{1/2} + \|g\|_{\widetilde{H}^2}^{1/2} (\|g\|_{\widetilde{W}^{2,1}} + \|g\|_{\widetilde{H}^2})^{1/2} \right].$$

for every $R > 0$ and $t \geqslant 0$, where C_R depends on R.

LEMMA 3.3. *If* $f \in \tilde{L}^1 \cap \tilde{L}^2$, *then*

$$\|f\|_{L^1} \leq C\|f\|_{\tilde{L}^1}^{1/2} \|f\|_{\tilde{L}^2}^{1/2}.$$

If $f \in \widetilde{W}^{m,1} \cap \widetilde{H}^m$, *then*

$$\|f\|_{W^{m,1}} \leq C\|f\|_{\widetilde{W}^{m,1}}^{1/2} \|f\|_{\widetilde{H}^m}^{1/2}.$$

PROOF. Applying the Hölder inequality gives

$$\|f\|_{L^1(R^2)} = \int_{R^2} [(1+|x|)|f|]^{1/2} [(1+|x|)|f|]^{1/2} (1+|x|)^{-1} dx$$

$$\leq \left(\int_{R^2} (1+|x|)|f|\,dx \right)^{1/2} \left(\int_{R^2} (1+|x|)^2 |f|^2 dx \right)^{1/4} \left(\int_{R^2} (1+|x|)^{-4} dx \right)^{1/4} \tag{24}$$

$$\leq C\|f\|_{\tilde{L}^1}^{1/2} \|f\|_{\tilde{L}^2}^{1/2}.$$

This inequality immediately yields the second thesis. ■

We finally report some elementary estimates used above. For the proof see [6].

LEMMA 3.4. *There exists a constant* $C > 0$ *such that for all* $t \geq 0$

$$\int_0^t (1+t-s)^{-1} (1+s)^{-1} ds \leq C(1+t)^{-1} \log(1+t), \tag{25}$$

$$\int_0^t (1+t-s)^{-1} (1+s)^{-1/2} ds \leq C(1+t)^{-1/2} \log(1+t). \tag{26}$$

$$\int_0^t (1+t-s)^{-1/2} (1+s)^{-1} \log(e+s)\, ds \leq C(1+t)^{-1/2} \log^2(1+t). \tag{27}$$

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