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Irrational Rapidly Convergent Series.

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ABSTRACT - The main result of this paper is a criterion for irrational series which consist of rational numbers and converge very quickly.

1. Introduction.

Mahler in [6] introduced the main method of proving the irrationality of sums of infinite series. This method has been extended several times and Nishioka's book [7] contains a survey of these results. Other methods are given in Sándor [8], Hančl [5] and Erdös [4].

In 1987 in [1] Badea proved the following theorem.

THEOREM 1. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two sequences of positive integers such that for every large* n, $a_{n+1} > \frac{b_{n+1}}{b_n} a_n^2 - \frac{b_{n+1}}{b_n} a_n + 1$. *Then the sum* $\alpha = \sum_{k=1}^{\infty} \frac{b_n}{a_n}$ *is an irrational number.*

Later in [2] he improved this result. Erdös in [4] introduced the notion of irrational sequences of positive integers and proved that the sequence $\{2^{2^n}\}_{n=1}^{\infty}$ is irrational. In [5] the present author extended this definition of irrational sequences to sequences of positive real numbers.

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DEFINITION 1. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of positive real numbers. If, for every sequence* $\{c_n\}_{n=1}^{\infty}$ *of positive integers, the sum* $\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$ *is an irrational number then the sequence* $\{a_n\}_{n=1}^{\infty}$ *is called irrational. If the sequence is not irrational then it is rational.*

The same paper also gives a criterion for the irrationality of infinite sequences and series.

In 1999 in [3] Duverney proved the following theorem.

THEOREM 2. *Let* β *be a positive rational number and* γ *be a non-negative real number with* $0 \leqslant \gamma < 2$. *Assume that* $\{u_n\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are sequences of nonzero integers such that*

$$\lim_{n\to\infty} u_n = \infty,$$

$$u_{n+1} = \beta u_n^2 + O(u_n^{\gamma}),$$

$$\log|a_n| = o(2^n)$$

and

$$\log|b_n| = o(2^n).$$

Then $\alpha = \sum_{n=1}^{\infty} \frac{a_n}{b_n u_n}$ *is a rational number if and only if there is* n_0 *such that for every* $n > n_0$

$$u_{n+1} = \beta u_n^2 - \frac{a_{n+1} b_n}{a_n b_{n+1}} u_n + \frac{a_{n+2} b_{n+1}}{\beta a_{n+1} b_{n+2}}.$$

The main result of this paper is Theorem 3. It deals with a criterion for the irrationality of sums of infinite series of rational numbers which depends on the speed and character of the convergence. In particular it does not depend on arithmetical properties like divisibility.

THEOREM 3. *Let* $A > 1$ *be a real number. Suppose that* $\{d_n\}_{n=1}^{\infty}$ *is a sequence of real numbers greater than one. Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two sequences of positive integers such that*

$$\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = A \tag{1}$$

and for all sufficiently large n

$$\frac{A}{a_n^{\frac{1}{2^n}}} > \prod_{j=n}^{\infty} d_j \tag{2}$$

and

$$\lim_{n\to\infty} \frac{d_n^{2^n}}{b_n} = \infty . \tag{3}$$

Then the series $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{a_n}$ *is an irrational number.*

COROLLARY 1. *Let* $A > 1$ *be a real number. Assume that* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are two sequences of positive integers such that* $\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = A$ *and for every sufficiently large positive integer n,* $a_n^{\frac{1}{2^n}}\left(1+\frac{1}{n}\right) \leqslant A$ *and* $b_n \leqslant 2^{\frac{1}{n^4}2^n}$. *Then the series* $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ *is irrational.*

This is an immediate consequence of Theorem 3. It is enough to put $d_n = 1 + \frac{1}{n^3}$.

COROLLARY 2. *Let* $A > 1$ *be a real number. Assume that* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are two sequences of positive integers such that* $\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = A$ *and for every sufficiently large positive integer n,* $a_n^{\frac{1}{2^n}}(1+4(2/3)^n) \leqslant A$ *and* $b_n \leqslant 2^{(4/3)^{n-1}}$. *Then the series* $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ *is irrational.*

This is an immediate consequence of Theorem 3. It is enough to put $d_n = 1 + (2/3)^n$.

REMARK 1. *Theorem* 3 *does not hold if we omit condition* (2). *To see this let* a_0 *be a positive integer greater than* 1 *and for every positive integer* n, $a_n = a_{n-1}^2 - a_{n-1} + 1$. *Then*

$$a_n^{\frac{1}{2^{n+1}}} = a_n^{\frac{1}{2^n}}\left(1 - \frac{1}{a_{n-1}} + \frac{1}{a_{n-1}^2}\right)^{\frac{1}{2^{n+1}}} \leqslant a_n^{\frac{1}{2^n}}$$

and

$$a_n^{\frac{1}{2^{n+1}}} \geqslant \left(\frac{3}{4}\right)^{\frac{1}{2^{n+1}}} a_n^{\frac{1}{2^n}} \geqslant \left(\frac{3}{4}\right)^{1-\frac{1}{2^n}} a_0 \geqslant \frac{3}{5} a_0 > 1 \, .$$

Hence $\lim_{n\to\infty} a_n^{\frac{1}{2^n}} > 1$ *and*

$$\sum_{n=0}^{\infty} \frac{1}{a_n} = \frac{1}{a_0} + \sum_{n=1}^{\infty} \frac{1}{a_n} =$$

$$= \frac{1}{a_0} + \sum_{n=1}^{\infty} \frac{a_n - 1}{a_n(a_n-1)} = \frac{1}{a_0} + \sum_{n=1}^{\infty}\left(\frac{1}{a_n-1} - \frac{1}{a_n(a_n-1)}\right) =$$

$$= \frac{1}{a_0} + \sum_{n=1}^{\infty}\left(\frac{1}{(a_0-1)\prod_{j=0}^{n-1} a_j} - \frac{1}{(a_0-1)\prod_{j=0}^{n} a_j}\right) = \frac{1}{a_0} + \frac{1}{a_0(a_0-1)} = \frac{1}{a_0-1}$$

is a rational number.

EXAMPLE 1. *Let* $[x]$ *be the greatest integer less then or equal to* x, $\pi(n)$ *be the number of primes less then or equal to* n *and* $d(n)$ *be the number of positive divisors of the number* n. *As an immediate consequence of Corollary* 1 *we obtain that the series*

$$\sum_{n=2}^{\infty} \frac{[d(n)^{(3/2)^n}] + 2n}{\left[\left(\sqrt{2} - \frac{1}{\pi(n)}\right)^{2^n}\right] - n!} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{[d(n)^{2(4/3)^n}] + n^2}{\left[\left(\sqrt{2} - \frac{1}{\pi(n)}\right)^{2^n}\right] - n^n}$$

are irrational numbers.

EXAMPLE 2. *Let* $[x]$, $\pi(n)$ *and* $d(n)$ *be defined as in Example* 1. *As an immediate consequence of Corollary* 2 *we obtain that the series*

$$\sum_{n=2}^{\infty} \frac{[d(n)^{(5/4)^n}] + n}{\left[\left(\sqrt{2} - \frac{1}{2^{\pi(n)}}\right)^{2^n}\right] - n^n} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{[d(n)^{(6/5)^n}] + n^2}{\left[\left(\sqrt{2} - \frac{1}{2^{\pi(n)}}\right)^{2^n}\right] - n!}$$

are irrational numbers.

OPEN PROBLEM 1. *It is an open problem if for every sequence* $\{c_n\}_{n=1}^{\infty}$ *of positive integers the sum* $\sum_{n=1}^{\infty} \frac{1}{c_n 2^{2^n} + 2^n + 2}$ *is irrational or not.*

2. Proof.

Proof (of Theorem 3). From (1) and (2) we obtain that $\lim_{n\to\infty} a_n^{\frac{1}{2^{n+1}}} = \sqrt{A}$ and $\lim_{n\to\infty} d_n = 1$. This, (1) and (3) imply $\lim_{n\to\infty} (d_{n+1} a_n^{\frac{1}{2^{n+1}}} a_{n+1}^{-\frac{1}{2^{n+1}}}) = \frac{1}{\sqrt{A}} < 1$. From this and (3) we obtain that

$$\lim_{n\to\infty} \frac{\frac{b_{n+1}}{a_{n+1}}}{\frac{b_n}{a_n}} = \lim_{n\to\infty} \left(\frac{b_{n+1}}{a_{n+1}} \frac{a_n}{b_n} \right) \leqslant \lim_{n\to\infty} d_{n+1}^{2^{n+1}} \frac{a_n}{a_{n+1}} = \tag{4}$$

$$= \lim_{n\to\infty} (d_{n+1} a_n^{\frac{1}{2^{n+1}}} a_{n+1}^{-\frac{1}{2^{n+1}}})^{2^{n+1}} = 0.$$

Inequality (4) implies that the series $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is convergent and for every sufficiently large positive integer n

$$\sum_{j=n+1}^{\infty} \frac{b_j}{a_j} \leqslant \frac{2b_{n+1}}{a_{n+1}}. \tag{5}$$

Let us suppose that the series $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is a rational number. Then there exist positive integers p and q such that $\alpha = \frac{p}{q}$. Thus we have

$$\alpha = \frac{p}{q} = \sum_{j=1}^{\infty} \frac{b_j}{a_j} = \sum_{j=1}^{n} \frac{b_j}{a_j} + \sum_{j=n+1}^{\infty} \frac{b_j}{a_j}.$$

It implies that

$$A_n = \left(p - q \sum_{j=1}^{n} \frac{b_j}{a_j} \right) \prod_{j=1}^{n} a_j = q \prod_{j=1}^{n} a_j \sum_{j=n+1}^{\infty} \frac{b_j}{a_j} \tag{6}$$

is a positive integer for every $n = 1, 2, \dots$. Thus

$$A_n \geqslant 1. \tag{7}$$

Now we find a positive integer n such that $A_n < 1$, a contradiction with (7). Let s be a positive integer such that for every $n \geqslant s$, (2) holds. This and (2) imply that for every n and k with $s \leqslant k \leqslant n$

$$\frac{A}{a_k^{\frac{1}{2^k}}} > \prod_{j=k}^{\infty} d_j \geqslant \prod_{j=n}^{\infty} d_j.$$

Hence

$$\frac{A}{\max_{j=s,\ldots,n} a_j^{\frac{1}{2^j}}} > \prod_{j=n}^{\infty} d_j. \tag{8}$$

From (1) and (8) we see that for infinitely many n

$$a_{n+1}^{\frac{1}{2^{n+1}}} \geqslant d_{n+1} \max_{j=s,\ldots,n} a_j^{\frac{1}{2^j}} \tag{9}$$

otherwise there exists n_0 with $n_0 \geqslant s$ such that for every $n \geqslant n_0$

$$a_{n+1}^{\frac{1}{2^{n+1}}} < d_{n+1} \max_{j=s,\ldots,n} a_j^{\frac{1}{2^j}} \leqslant \ldots \leqslant \prod_{j=n_0+1}^{n+1} d_j \max_{j=s,\ldots,n_0} a_j^{\frac{1}{2^j}},$$

contradicting (1) and (8) for sufficiently large n. Inequality (9) and the fact that $2^s + \ldots + 2^n < 2^{n+1}$ imply

$$a_{n+1} \geqslant d_{n+1}^{2^{n+1}} \Big(\max_{j=s,\ldots,n} a_j^{\frac{1}{2^j}} \Big)^{2^{n+1}} > d_{n+1}^{2^{n+1}} \Big(\max_{j=s,\ldots,n} a_j^{\frac{1}{2^j}} \Big)^{2^n + 2^{n-1} + \ldots + 2^s} =$$

$$= d_{n+1}^{2^{n+1}} \prod_{i=s}^{n} \Big(\max_{j=s,\ldots,n} a_j^{\frac{1}{2^j}} \Big)^{2^i} \geqslant d_{n+1}^{2^{n+1}} \prod_{j=s}^{n} a_j = d_{n+1}^{2^{n+1}} \Big(\prod_{j=1}^{n} a_j \Big) \Big(\prod_{j=1}^{s-1} a_j \Big)^{-1}.$$

From this, (3), (5) and (6) we obtain for infinitely many n

$$A_n = q \Big(\prod_{j=1}^{n} a_j \Big) \sum_{j=n+1}^{\infty} \frac{b_j}{a_j} \leqslant q \Big(\prod_{j=1}^{n} a_j \Big) \frac{2 b_{n+1}}{a_{n+1}} <$$

$$< 2q \Big(\prod_{j=1}^{n} a_j \Big) \frac{b_{n+1}}{d_{n+1}^{2^{n+1}} (\prod_{j=1}^{n} a_j)(\prod_{j=1}^{s-1} a_j)^{-1}} = 2q \Big(\prod_{j=1}^{s-1} a_j \Big) \frac{b_{n+1}}{d_{n+1}^{2^{n+1}}} < 1$$

and the proof of Theorem 3 is complete. ■

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