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Differential Equations and Maximal Ideals on the Weyl Algebra $A_2(\mathbb{C})$.

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ABSTRACT - We characterize the differential operators $S = \partial/\partial x + \beta\, \partial/\partial y + \gamma$ such that the ideal $A_2(\mathbb{C})\, S$ is maximal in $A_2(\mathbb{C})$.

1. Introduction.

Let $A_n = \mathbb{C}[x_1, \ldots, x_n]\langle \partial/\partial x_1, \ldots, \partial/\partial x_n \rangle$ be the Weyl algebra, in n variables, over the complex field $\mathbb{C}$. In [3], the author proves, among other things, that the differential operators

$$\partial_1 + \sum_{i=2}^{n} (x_i a_i(x_1) + b_i(x_1))\, \partial_i + \sum_{i=2}^{n} h_i(x_1)\, x_i \in A_n \tag{1.1}$$

where: $\partial_i = \partial/\partial x_i$; the polynomials a_i, b_i and h_i belong to $\mathbb{C}[x_1]$; the a_i's are linearly independent on the field of rational numbers $\mathbb{Q}$, and moreover we have

$$\deg(a_i) > \max\{\deg(b_i),\ \deg(h_i)\} \geq 0\,,$$

generate left maximal ideals in A_n (Th. 3.6, page 412.)

The operators of type (1.1) generalize the following operator, of [5]:

$$P = x_1 + \partial_1 \left(\sum_{i=2}^{n} \lambda_i x_i \partial_i \right) + \sum_{i=2}^{n} (x_i - \partial_i) \in A_n\,, \tag{1.2}$$

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where the λ_i's in $\mathbb{C}$ are linearly independent over $\mathbb{Q}$, which generates a right maximal ideal in A_n [1].

Always in [5], page 627, it is proved that: also the operator,

$$x_2 + x_1 \partial_1 \partial_2 + \lambda(\partial_1^2 x_1 - \mu\partial_1) + x_1 \in A_2, \tag{1.3}$$

with $\lambda \in \mathbb{C} \setminus \mathbb{Q}$ and $\mu \notin \mathbb{Z}$, generates a right maximal ideal in A_2; and in [3], page 416, it is asked if it is possible to extend Theorem 3.6 in order to include the example 1.3.

This paper studies the operators of type (1.1) and (1.3) in A_2. The main result (Theorem 2.2 in the following section) is as follows:

Let $S = \partial_x + \beta \cdot \partial_y + \gamma \in A_2 = \mathbb{C}[x, y]\langle \partial x, \partial y\rangle$, (*or* $S = \partial_y + \beta \cdot \partial_x + \gamma \in A_2$). *Then, we have that $A_2 S$ is maximal in A_2 if and only if:* $\forall R \in \mathbb{C}[x, y]\langle \partial_y\rangle$ (*or* $\forall R \in \mathbb{C}[x, y]\langle \partial_x\rangle$), *it follows that*

$$[S, R] = SR - RS \notin \mathbb{C}[x, y]\, R\,. \tag{1.4}$$

In the case that $\beta \in \mathbb{C}[x, y]$ with $\deg_y \beta = 1$, namely, in the case of operators of type (1.1), one can easily rephrase the maximality of $A_2 S$ as follows (Theorem 2.4 of the following section):

Let $S = \partial_x + \beta \cdot \partial_y + \gamma \in A_2 = \mathbb{C}[x, y]\langle \partial x, \partial y\rangle$, *and let* $\beta = \beta_0 + \beta_1 y$, $\beta_1 \in \mathbb{C}[x] \setminus \{0\}$. *Then, $A_2 S$ is maximal in A_2 if and only if: the following equations*

$$\partial_x\left(\frac{p}{q}\right) + \beta\, \partial_y\left(\frac{p}{q}\right) + (\partial_y \beta)\left(\frac{p}{q}\right) = \partial_y \gamma\ ;$$

$$\partial_x\left(\frac{r}{s}\right) - \beta_1\left(\frac{r}{s}\right) = -\beta_0$$

don't have any solution respectively in $\mathbb{C}(x, y)$ and in $\mathbb{C}(x)$.

Finally, in the third section, utilizing Corollary 3.1, a corollary of Theorem 2.2, we will give a somewhat simplified proof of the following result [5], Prop. 2.2 page 627:

[1] If $\mathcal{F}\colon A_n \to A_n$ is Fourier transformation, then we have $\mathcal{F}(x_i) = -\partial_i$ and $\mathcal{F}(\partial_i) = x_i$. Moreover, if $\mathcal{C} : A_n \to A_n$ is the standard transposition, $\mathcal{C}(x^\alpha \partial^\beta) = (-1)^{|\beta|} \partial^\beta x^\alpha$, then we obtain, in A_2, $\mathcal{F}(1.2)^{\mathcal{C}} = \partial_1 + (1 + \lambda x_1 x_2)\, \partial_2 + x_2$, which is of type (1.1).

The operators

$$\mathcal{F}(1.3)^{\mathcal{C}} = \partial_y + (1 + xy + \lambda x^2)\, \partial_x + \lambda(\mu + 2)\, x \in A_2, \qquad (\lambda, \mu) \in \mathbb{C}^2,$$

with $\lambda \notin \mathbb{Z}$ *and* $\mu \notin \mathbb{Z}$, *generate, in* A_2, *left maximal ideals.*

In the end, we would like to add some comments. The operator (1.2) of [5], solves, for the first time, the conjecture of [2], page 31, which asks:

Is it true that, for each finitely generated A_n*-module* M, *we have*

$$GK\dim(M) = Kr\dim(M) + n? \tag{1.5}$$

Here, $GK\dim$ indicates Gelfand-Kirillov dimension, and $Kr\dim$ means Krull dimension. If $n \geqslant 2$, since $(P)\, A_n$ is maximal in A_n, we have

$$GK\dim(A_n/(P)\, A_n) = 2n - 1 > Kr\dim(A_n/(P)\, A_n) + n \tag{1.6}$$

because $A_n/(P)\, A_n$ is simple.

In [1], they give many families of simple A_n-modules M which are not holonomic (namely, $GK\dim(M) > n$); anyway, as is said in [3], page 405, the examples 1.1 and 1.3 originally given in [5] are not members of any these families.

2. – Let $S = \partial_y + \beta\, \partial_x + \gamma \in A_2 = \mathbb{C}[x, y]\langle \partial_x, \partial_y \rangle$. If $P \in A_2$, then we have

$$P = QS + R, \quad \text{where } R \in \mathbb{C}[x, y]\langle \partial_x \rangle,$$

and moreover, $[S, R] = SR - RS \in \mathbb{C}[x, y]\langle \partial_x \rangle$. Therefore, in order to prove that $A_2 S$ is a maximal ideal in A_2, it is enough to prove that

$$A_2 S + A_2 R = A_2, \qquad \forall R \in \mathbb{C}[x, y]\langle \partial_x \rangle$$

Lemma 2.1. *Let* $A_2 S$ *be a maximal ideal in* A_2. *Then:* β *is not divisible by* x; *and moreover* $(\partial_x \beta)(\partial_x \gamma) \neq 0$.

Proof. Let $\beta = x\, \tilde{\beta}$, and let f be a holomorphic function in a neighborhood of zero in $\mathbb{C}^2$, such that

$$(\partial_y + x\, \tilde{\beta} + \gamma - \tilde{\beta})\, f = 0$$

and $f(0, 0) = 1$: such an f exists by Theorem of Cauchy-Kowalewsky, [4], page 119. We have:

$$S\left(\frac{f}{x}\right)=\frac{f_y}{x}+\frac{x\tilde{\beta}(xf_x-f)}{x^2}+\frac{\gamma f}{x}=0\,,$$

where f_x and f_y mean $\partial_x(f)$ and $\partial_y(f)$ respectively, and moreover, $\lambda S+\mu x=1$; hence we get $x(\lambda S(f/x)+\mu(f))=f$, and therefore,

$$x\mu(f)(x,\,0)=f(x,\,0):$$

which is impossible.

If $\partial_x\beta=0$, then let $p(x,\,y)=-\int_0^y\beta(t)\,dt+x$. We have $\partial_y p+\beta\partial_x p=0$. Now, if

$$(\partial_y+\beta\partial_x)\,f=-\gamma$$

where f is holomorphic in some neighborhood of zero in $\mathbb{C}^2$, then we have $S(e^f/p)=0$, and from the equation $\lambda S+\mu p=1$, we obtain

$$p\mu(e^f)=e^f:$$

which is a contradiction.

Finally, if $\partial_x\gamma=0$, then let $u(y)=e^{-\int_0^y\gamma(t)\,dt}$. Then, $S(u)=\partial_x(u)=$ $=0$, which contradicts to the equation $\lambda S+\mu\partial_x=1$. ■

THEOREM 2.2. *The following statements are equivalent.*

P_1): *A_2S is a maximal ideal in A_2.*

P_2): $\forall R\in\mathbb{C}[x,\,y]\langle\partial_x\rangle$, *where R is not a constant, we have* $[S,\,R]\notin$ $\notin\mathbb{C}[x,\,y]\,R$.

PROOF. First, we prove that P_1 implies P_2. Let $\lambda S+\mu R=1$; if $\deg_{\partial_y}\lambda=m$, then $\deg_{\partial_y}\mu=m+1$. Dividing λ and μ by S, we obtain

$$\lambda S+\mu R=\sum_{k=0}^{m}B_kS^{k+1}+\sum_{k=0}^{m+1}C_kS^kR=1\,,$$

for some B_k and C_k in $\mathbb{C}[x,\,y]\langle\partial_y\rangle$; hence $[y,\,\lambda S+\mu R]=-\sum_{k=0}^{m}(k+1)\,B_kS^k-$ $-\sum_{k=1}^{m+1}kC_kS^{k-1}R=0$; repeating in this manner m more times, we get

$$B_m+C_{m+1}R=0\,.$$

From here, if $[S, R] = \alpha R$, then we would have

$$\lambda S + \mu R = \sum_{k=0}^{m-1} B_k S^{k+1} + \sum_{k=0}^{m} C_k S^k R - C_{m+1}[R, S^{m+1}] =$$

$$= \sum_{k=0}^{m-1} E_k S^{k+1} + \sum_{k=0}^{m} D_k S^k R = 1 ,$$

for some E_k and D_k in $\mathbb{C}[x, y]\langle \partial_x \rangle$. Proceeding in this way, we obtain

$$ES + D_0 R + D_1 SR = 1 ;$$

where $E = -D_1 R$, and hence we get the following contradiction:

$$D_0 R + D_1 [R, S] = D_0 R - D_1 \alpha R = 1 .$$

We, now, prove that P_2) implies P_1). Let $R = \sum_{k=0}^{N} p_k \partial_x^k$; if $[S, R] = \sum_{k=0}^{N} q_k \partial_x^k$, then since $[S, R] \notin \mathbb{C}[x, y] R$, we have

$$0 \leqslant \deg_{\partial_x}(p_N [S, R] - q_N R) \leqslant N - 1 .$$

This inequality implies that the ideal $A_2 S + A_2 R$ contains a polynomial $p = \sum_{k=0}^{N} r_k x^k$ which is not zero.

Let N be the least degree in x among all the polynomials contained in $A_2 S + A_2 R$. If N were strictly greater than zero, then we would have

$$t = \deg_x [S, p] \geqslant N,$$

and therefore, $r_N^t [S, p] = \alpha p$: if r_N does not divide p, then we would have $[S, p] = \alpha_1 p$, which is impossible from the hypothesis. If, instead, r_N divides p, let us put $p = \alpha_0 p_0$, where α_0 is the greatest common divisor of the elements r_0, , …, r_N. Then, we would have

$$[S, \alpha_0 p_0] = [S, \alpha_0] p_0 + \alpha_0 [S, p_0] = \left(\frac{\alpha}{r_N^t} \right) \alpha_0 p_0,$$

which again contradicts the hypothesis. ■

Observation 1. The theorem is valid also for operators of type

$$\partial_x + \beta \partial_y + \gamma .$$

In this case, if $R = \sum_{k=0}^{n} p_k \partial_y^k$, where $n \geqslant 1$, the routine calculations give the following expression of $[S, R]$:

$$[S, R] = \ldots + \left[\partial_x(p_{n-1}) + \beta\partial_y(p_{n-1}) - \binom{n-1}{n-2} p_{n-1}\,\partial_y(\beta) \right.$$

$$\left. - \binom{n}{n-2} p_n\,\partial_y^2(\beta) \right] \partial_y^{n-1} + \left[\partial_x(p_n) + \partial_y(p_n) - \binom{n}{n-1} (\partial_y(\beta))\, p_n \right] \partial_y^n,$$

where we follow the convention that $\binom{n}{0} = 1$, and that, if $n = 1$, $\binom{n-1}{n-2} = \binom{n}{n-2} = 0$. Therefore, in the expression

$$p_n[S, R] - (\partial_x(p_n) + \beta\partial_y(p_n) - n\partial_y(\beta)\, p_n)\, R\ ,$$

the coefficient, $c_n \in \mathbb{C}[x, y]$, of ∂_y^{n-1} is the following:

$$c_n = p_n^2 \left[\partial_x\left(\frac{p_{n-1}}{p_n}\right) + \beta\partial_y\left(\frac{p_{n-1}}{p_n}\right) + \right.$$
$$\left. + \partial_y(\beta)\left(\frac{p_{n-1}}{p_n}\right) - n\partial_y(\gamma) - \left(\frac{n(n-1)}{2}\right) \partial_y^2\beta \right]. \tag{2.3}$$

In the case of operators, in A_2, as the operator (1.1) of [3], we have the following theorem:

THEOREM 2.4. *Let* $S = \partial_x + \beta\,\partial_y + \gamma \in A_2$, *where* $\beta(x, y) = \beta_0(x) + \beta_1(x)\, y$. *Then the following statements are equivalent:*

P_1): $A_2 S$ *is a maximal ideal in* A_2.

P_2): *The equations*

$$\partial_x\left(\frac{p}{q}\right) + \beta\,\partial_y\left(\frac{p}{q}\right) + (\partial_y\beta)\,\frac{p}{q} = \partial_y\gamma \tag{2.5}$$

and

$$\partial_x\left(\frac{r}{s}\right) - \beta_1\left(\frac{r}{s}\right) = -\beta_0, \tag{2.6}$$

don't have any solutions respectively in $\mathbb{C}(x, y)$ *and in* $\mathbb{C}(x)$.

PROOF. First, we prove that P_1) implies P_2). If there were a solution p/q of (2.5), then letting $R = p + q\partial_y$, we would have

$$q[S, R] - \lambda R = 0 ,$$

where λ is the coefficient of ∂_y in $[S, R]$, which is contrary to Theorem 2.2.

Now, if r/s were a solution of (2.6), let us put $p = r(x) + s(x)\, y$, and again we would obtain

$$s[S, p] - (s' + \beta_1 s)\, p = 0 ,$$

which contradicts Theorem 2.2.

Conversely, let us assume P_2). Please observe that, in the case that $\beta = \beta_0 + \beta_1 y$, the equation (2.3) is of the following form:

$$c_n = np_n^2 \left[\partial_x \left(\frac{p_{n-1}}{np_n} \right) + \beta \partial_y \left(\frac{p_{n-1}}{np_n} \right) + \beta_1 \left(\frac{p_{n-1}}{np_n} \right) - \partial_y \gamma \right];$$

hence, if $R = \sum_{k=0}^{n} p_k \partial_y^k$ and if the equation (2.5) does not have any solution, then the ideal $A_2 S + A_2 R$ contains an element of the form

$$R_1 = \sum_{k=0}^{n-1} q_k \partial_y^k, \quad \text{where} \ \deg_{\partial_y} R_1 = n - 1 ,$$

and therefore,

$$(A_2 S + A_2 R) \cap (\mathbb{C}[x, y] \setminus \{0\}) \neq \emptyset .$$

Similarly, since the equation (2.6) does not have any solution, we conclude that $(A_2 S + A_2 R) \cap (\mathbb{C}[x] \setminus \{0\}) \neq \emptyset$, and hence $A_2 S + A_2 R = A_2$. ■

OBSERVATION 2. This research was initiated by the direct verification (see Observation 3) that the equation (2.5), in the case that $\beta = 1 + xy$, does not have any solution in $\mathbb{C}(x, y)$.

OBSERVATION 3 [M. Takagi]. *The equation*

$$L(f) = (\partial_x + (1 + xy)\, \partial_y + x)\, f = 1$$

does not have any solution $f \in \mathbb{C}(x, y)$.

PROOF. Assume that there were a solution of $L(f)=1$, where $f=p/q\in\mathbb{C}(x,y)$ such that p and q are mutually prime. Then, we have

$$q(\lambda-q)=p\mu\ ,$$

where $\lambda=\partial_x p+(1+xy)\,\partial_y p+xp$ and $\mu=\partial_x q+(1+xy)\,\partial_y q$. Therefore, there exists $r\in\mathbb{C}[x,\,y]$, such that

$$\begin{cases}\lambda-q=pr\\ \mu=qr\ .\end{cases}$$

We, now, show that r is of type: $r(x,\ y)=ax+b$, where $(a,\,b)\in\mathbb{C}^2$. In fact, if $\deg_x q=k$ and $\deg_y q=h$, then we have

$$\deg_x\mu\leqslant k+1,\quad\text{and}\quad\deg_y\mu\leqslant h\ .$$

Therefore, from the equation $\mu=qr$, we obtain $\deg_x r\leqslant 1$ and $\deg_y r=0$.

For k and h, we see easily that $h\geqslant 1$ because $q=f(x)$, where f is a polynomial, does not satisfy the equation $\mu=q(ax+b)$. It is also easy to verify that $k\geqslant 2$ because if

$$q=xg(y)+h(y)\,,$$

where g and h are polynomials, then q does not satisfy the equation $\mu=q(ax+b)$.

Finally, we show that any of

$$q=\sum_{i=0}^{k}\sum_{j=0}^{h}a_{i,j}x^iy^j,\quad\text{where } k\geqslant 2\ \text{ and }\ h\geqslant 1\ ,$$

cannot satisfy the equation $\mu=q(ax+b)$. In fact, we obtain

$$\mu=\sum_{i=0}^{k}\sum_{j=0}^{h}ia_{i,j}x^{i-1}y^j+\sum_{i=0}^{k}\sum_{j=1}^{h}ja_{i,j}x^iy^{j-1}+\sum_{i=0}^{k}\sum_{j=1}^{h}ja_{i,j}x^{i+1}y^j$$

and

$$q(ax+b)=\sum_{i=0}^{k}\sum_{j=0}^{h}aa_{i,j}x^{i+1}y^j+\sum_{i=0}^{k}\sum_{j=0}^{h}ba_{i,j}x^iy^j.$$

Since only qax contains x^{k+1}, it must be that $aa_{k,0}=0$. Now, if $a=0$ we must have

$$ja_{k,j}=aa_{k,j},$$

and hence, $a_{k,j}=0$ for all $j=1,\,\dots,\,h$. Since $(a_{k,h},\,\dots,\,a_{k,0})\neq 0$. Comparing the coefficients of x^k in $\mu=q(ax+b)$, we have

$$a_{k,1}=aa_{k-1,0}+ba_{k,0},$$

which implies that $b = 0$. Therefore, $(\partial_a + (1 + xy)\,\partial_y)\,q = 0$, that is, $q = 0$.

If, instead, $a_{k,0} = 0$, then, comparing, always in $\mu = q(ax + b)$, the coefficients of $x^{k+1}y^j$, where $1 \leqslant j \leqslant h$, we must have

$$ja_{k,j} = aa_{k,j},$$

from which we obtain that $a = l$ for some $l \in \{1, \ldots, h\}$, $a_{k,l} \neq 0$, and $a_{k,j} = 0$ if $j \neq l$.

If $l = h$, then, confronting the coefficients of $x^k y^h$, we have

$$ha_{k-1,h} = ha_{k-1,h} + ba_{k,h}$$

with $a_{k,h} \neq 0$ and therefore, $b = 0$. Moreover, from the coefficients of $x^{k-1}y^h$, we also have

$$ka_{k,h} + ha_{k-2,h} = ha_{k-2,h}$$

which gives the contradiction: $a_{k,h} = 0$.

Let us, now, suppose that $l < h$. Then, $a_{k,h} = 0 = a_{k,0}$. Equating the coefficients of $x^k y^h$, we obtain

$$ha_{k-1,h} = aa_{k-1,h} + ba_{k,h} \qquad (a = l < h),$$

and therefore, $a_{k-1,h} = 0$; while confronting the coefficients of $x^i y^k$ $\forall i$, we have

$$(i+1)\,a_{i+1,h} = ha_{i-1,h} + ba_{i,h}, \qquad 0 \leqslant i \leqslant k \quad (a_{k-1,h} = 0),$$

which imply that $a_{i,h} = 0$, and this contradicts to the hypothesis that $\deg_y q = h$. ■

3. – If $S = \partial_y + \beta\,\partial_x + \gamma$, then the equation (2.3) has the following form:

$$c_n = p_n^2\left[\partial_y\left(\frac{p_{n-1}}{p_n}\right) + \beta\partial_x\left(\frac{p_{n-1}}{p_n}\right) + \right.$$
$$\left. + \partial_x(\beta)\left(\frac{p_{n-1}}{p_n}\right) - n\partial_x(\gamma) - \left(\frac{n(n-1)}{2}\right)\partial_x^2(\beta)\right].$$

The following corollary of Theorem 2.2 is immediate.

COROLLARY 3.1. *If the equations*

$$c_n = 0 \qquad \forall n \geqslant 1 \tag{3.2}$$

do not have solutions in $\mathbb{C}(x, y)$; *and if* $\forall p = \sum_{k=0}^{n} p_k(y)\, y^k$, *we have*

$$[S, p] \notin \mathbb{C}[x, y]\, p\,, \tag{3.3}$$

then, $A_2 S$ *is a maximal ideal.*

PROOF. The equation (3.2) says that, $\forall R = \sum_{k=0}^{n} r_k \partial_x^k$, $n \geqslant 1$, the ideal $A_2 S + A_2 R$ contains an element $p \in \mathbb{C}[x, y] \setminus \{0\}$.

The hypothesis (3.3) says that $A_2 S + A_2 p$ contains some element $q \in \mathbb{C}[y] \setminus \{0\}$. ■

OBSERVATION 4. *If* $\lambda \notin \mathbb{Z}$ *and* $\mu \notin \mathbb{Z}$, *then, the differential operator*

$$S = (\mathcal{F}(y + x\partial_x \partial_y + \lambda(\partial_x^2 x - \mu\partial_x) + x))^{\mathcal{C}} =$$

$$= \partial_y + (1 + xy + \lambda x^2)\, \partial_x + \lambda(\mu + 2)\, x\,,$$

satisfies the hypotheses of Corollary 3.1, *and therefore,* $A_2 S$ *is a maximal ideal in* A_2.

PROOF. Let us suppose that $c_n(p/q) = 0$ for some $n \in \mathbb{N}$, $n \geqslant 1$, with $p/q \in \mathbb{C}(x, y)$.

a) First, we show that q cannot be a constant. If it were, then we would have

$$p_y + (1 + xy + \lambda x^2)\, p_x + (y + 2\lambda x)\, p = n\lambda(\mu + n + 1) \neq 0\,,$$

which is impossible.

b) Now, let us assume that p and q are mutually prime. Then, the equation $c_n(p/q) = 0$ gives the following:

$$q_y + (1 + xy + \lambda x^2)\, q_x = rq \qquad (q_x \neq 0)\,,$$

for some $r \in \mathbb{C}[x, y]$. If v is an (non-constant) irreducible factor of q, we also have

$$v_y + (1 + xy + \lambda x^2)\, v_x = r_0 v\,, \qquad r_0 \in \mathbb{C}[x, y]\,.$$

Let $l(y)$ be a function defined implicitly by the equation $v(l(y), y) = 0$. Then we have

$$v_y + l'(y)\, v_x = 0\,,$$

and therefore, the function l is an algebraic solution (that is, l belongs to a finite extension of $\mathbb{C}(y)$) of the differential equation

$$l'(y) = 1 + yl(y) + \lambda l(y)^2,$$

which is also impossible [cf. Observation 5].

Up to this point, we have shown that the equations $c_n = 0$ do not possess any solutions in $\mathbb{C}(x, y)$, namely, that: $\forall R \in \mathbb{C}[x, y]\langle \partial_x \rangle$, $\deg_{\partial_x} R \geqslant 1$, we have

$$(A_2 S + A_2 R) \cap (\mathbb{C}[x, y] \backslash \{0\}) \neq \emptyset\,.$$

Similarly, we can show (3.3). ■

OBSERVATION 5 [M. Takagi]. 1) *The differential equation*

$$y' = 1 + xy + \lambda y^2 \tag{3.4}$$

does not have rational solutions provided that $\lambda \notin \mathbb{Z}$.

PROOF. If $p(x)/q(x) (\in \mathbb{C}(x))$ were a solution of the equation (3.4), where p and q are mutually prime, then, we would obtain the following system

$$\begin{cases} q_x = rq - \lambda p \\ p_x = (x + r)\, p + q \end{cases}$$

for some $r \in \mathbb{C}[x]$. We, now, examine the three cases.

If $r \neq 0$ and $r + x \neq 0$, then we have that $\deg(rq) = \deg p$ from the first equation and that $\deg (x + r)\, p = \deg q$ from the second. Hence, it follows that $\deg r(x + r)\, p = \deg rq = \deg p$, which is impossible because $p \neq 0$.

If $r = 0$, then the system becomes as follows:

$$\begin{cases} q_x = q - \lambda p \\ p_x = xp + q\,. \end{cases}$$

Taking the derivative of the second equation, and substituting q_x with the first equation, we obtain

$$p_{xx} = (1-\lambda)\, p + xp_x .$$

Hence, we have $(1-\lambda)\, c_N x^N + Nc_N x^N = 0$, where c_N is the leading coefficient of p. However, this equation contradicts the assumption that λ is not an integer.

Finally, if $r + x = 0$, then the system is the following:

$$\begin{cases} q_x = -xq - \lambda p \\ p_x = q \,. \end{cases}$$

Differentiating the first equation, and substituting p_x with q, we have

$$q_{xx} = -(1+\lambda)\, q - xq_x .$$

Therefore, it follows that $(1+\lambda)\, d_M x^M + Md_M x^M$, where d_N is the leading coefficient of q and hence $1 + \lambda + M = 0$, which is impossible because λ is not an integer. ■

2) *Let $p \in \mathbb{N}$ such that $p \geqslant 2$. Suppose that*

$$v(x) = \sum_{k=-N}^{+\infty} c_k (x-\alpha)^{k/p}, \qquad \alpha \in \mathbb{C}, \qquad N \in \mathbb{Z}\,,$$

satisfies (2) *the equation* (3.4). *Then,* $c_k = 0$ *if* $\frac{k}{p} \notin \mathbb{Q} \backslash \mathbb{Z}$, *namely, each algebraic solution of* (3.4) *is rational.*

Proof. Let us prove, first, that $N = p$. Let

$$w(t) = \sum_{k=-N}^{+\infty} c_k t^k \qquad \text{and} \qquad t = t(x) = (x-\alpha)^{1/p} .$$

Then, $v(x) = w \circ t(x)$, and since $\dfrac{dt}{dx} = \dfrac{1}{p} t^{1-p}$, we obtain

$$\frac{dv}{dx} = \frac{dw}{dt} \cdot \frac{dt}{dx} = \frac{1}{p} \sum_{k=-N-p}^{+\infty} (k+p)\, c_{k+p} t^k .$$

(2) The series $v(x) = \sum_{k=-N}^{+\infty} c_k (x-\alpha)^{k/p}$ is called Puiseux series of v, $0 < |x - \alpha| \leqslant \varepsilon$.

Utilizing the equation $x = t^p + \alpha$, and substituting v', x, and v in the equation (3.4), we have the following equation of series:

$$(3.5)\qquad \frac{1}{p}\sum_{k=-N-p}^{+\infty}(k+p)\,c_{k+p}t^k =$$

$$= 1 + \sum_{k=-N+p}^{+\infty} c_{k-p}t^k + \alpha\sum_{k=-N}^{+\infty} c_k t^k + \lambda\sum_{k=-2N}^{+\infty}\sum_{n=-N}^{k+N} c_n c_{k-n} t^k.$$

Comparing the coefficient of t^{-2N}, we conclude that $c_k = 0$ if $k < -p$, and hence, we may write $w(t) = \sum_{k=-p}^{+\infty} c_k t^k$.

Now, let us prove that $c_{-p+1} = \dots = c_{-1} = 0$. In order to do so, we rewrite the equation (3.5):

$$\frac{1}{p}\sum_{k=-2p}^{+\infty}(k+p)\,c_{k+p}t^k =$$

$$= 1 + \sum_{k=0}^{+\infty} c_{k-p}t^k + \alpha\sum_{k=-p}^{+\infty} c_k t^k + \lambda\sum_{k=-2p}^{+\infty}\sum_{n=-p}^{k+p} c_n c_{k-n} t^k.$$

Confronting the coefficients of t^{-2p}, t^{-2p+1}, $\dots$, t^{-p-1}, we have the following system of equations:

$$\begin{cases} \dfrac{1}{p}(-p)\,c_{-p} = \lambda c_{-p}^2 \\ \dfrac{1}{p}(-p+1)\,c_{-p+1} = \lambda\left(\displaystyle\sum_{n=-p}^{-p+1} c_n c_{-2p+1-n}\right) \\ \vdots \\ \dfrac{1}{p}(-1)\,c_{-1} = \lambda\displaystyle\sum_{n=-p}^{-1} c_n c_{-p-1-n}. \end{cases}$$

From the first equation, we see immediately that $c_{-p} = 0$ or $\lambda = -1/c_{-p}$. Hence, we consider the two cases.

If $c_{-p} = 0$, then from the second equation, $c_{-p+1} = 2\lambda C_{-p}c_{-p+1} = 0$. Similarly, by induction, we conclude that $c_l = 0$, where $-p+1 \leqslant l \leqslant -1$.

If $c_{-p} \neq 0$, then $\lambda = -1/c_{-p}$, and from the second equation, we have

$\frac{p+1}{p} c_{-p+1} = 0$. Since $p+1 \neq 0$, we have $c_{-p+1} = 0$. Similarly, by induction, we obtain that $c_l = 0$, where $-p+1 \leqslant l \leqslant -1$. Therefore, in either case, we may write

$$w(t) = c_{-p} t^{-p} + \sum_{k=0}^{+\infty} c_k t^k,$$

where, c_{-p} may, or may not be zero.

Let us suppose that we have shown that $w(t) = \sum_{l=-1}^{N-1} c_{lp} t^{lp} + \sum_{k=Np}^{+\infty} c_k t^k$, and we will prove that $w(t) = \sum_{l=-1}^{N} c_{lp} t^{lp} + \sum_{k=(N+1)p}^{+\infty} c_k t^k$.

With this form of $w(t)$, the equation (3.4) becomes

$$-c_{-p} t^{-2p} + \sum_{l=0}^{N-2} (l+1)\, c_{(l+1)p} t^{lp} + \frac{1}{p} \sum_{k=(N-1)p}^{+\infty} (k+p)\, c_{k+p} t^k =$$

$$1 + \sum_{l=0}^{N+2} c_{(l-1)p} t^{lp} + \alpha \sum_{l=-1}^{N-1} c_{lp} t^{lp} + \sum_{k=(N+1)p}^{+\infty} c_{k-p} t^k + \alpha \sum_{k=Np}^{+\infty} c_k t^k +$$

$$\lambda \left(\sum_{l=-1}^{N-1} c_{lp} t^{lp} \right)^2 + 2\lambda \left(\sum_{l=-1}^{N-1} c_{lp} t^{lp} \right) \left(\sum_{k=Np}^{+\infty} c_k t^k \right) + \lambda \left(\sum_{k=Np}^{+\infty} c_k t^k \right)^2,$$

where we agree that $\sum_{l=0}^{N-2} (l+1)\, c_{(l+1)p} t^{lp} = 0$ if $N = 0$. Comparing the coefficients of the terms, $t^{(N-1)p+1}, \ldots, t^{Np-1}$, we obtain the following equations if $N \neq 0$:

$$\begin{cases} \frac{1}{p}((N-1)\,p + 1 + p)\, c_{Np+1} = 0 \\ \vdots \\ \frac{1}{p}(Np - 1 + p)\, c_{Np-1+p} = 0\,. \end{cases}$$

Therefore, we conclude that $c_{Np+1} = \ldots = c_{(N+1)p-1} = 0$.

If, instead, $N = 0$, we have the following:

$$\begin{cases} \frac{1}{p} c_1 = 2\lambda c_{-p} c_1 \\ \vdots \\ \frac{1}{p} c_{p-1} = 2\lambda c_{-p} c_{p-1} . \end{cases}$$

Since $1/p \neq 2\lambda c_{-p}$, we conclude that $c_1 = \ldots = c_{p-1} = 0$. ∎

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