REND. SEM. MAT. UNIV. PADOVA, Vol. 107 (2002)

Counting Lamé Differential Operators.

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ABSTRACT - Using some properties of Belyi functions (covers of $\mathbb{P}^1$ with at most three branching points) and the combinatorics of the associated «dessins d'enfants» we obtain some effective results on Lamé operators L_n for $n = 1$.

0. Introduction.

In this note we reconsider the problem of Lamé differential operators

$$L_n = D^2 + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-\lambda}\right) D - \frac{n(n+1)\,x + B}{4x(x-1)(x-\lambda)}$$

which have algebraic solutions, and more precisely the number of such operators in the case $n = 1$.

This problem was already considered by Chiarellotto [5]. The results and the estimates we obtain in what follows are not new, but they are obtained with different techniques. In a forthcoming article we shall apply these methods to prove some general results, and to calculate such estimates in some other cases (for example the case $n = 2$).

Our approach, as Chiarellotto's, is based on the fact that (cf. Theorem 2.1 below) such a Lamé operator is a pullback by a rational map of a

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Supported by the Research Training Network «Arithmetic Algebraic Geometry», (Fifth Framework Programme, contract number HPRN-CT-2000-00120).

hypergeometric operator in «the basic Schwarz list». The key point in our method is that such a rational function is a Belyi cover, and we use the Grothendieck correspondence with the «dessins d'enfants» in order to count, in fact, graphs on the Riemann sphere with specified combinatorial data.

In the first two sections we recall some facts concerning Belyi functions and second degree differential operators with algebraic solutions, essentially described in [7] and in [1], [2]. We investigate, in section 3, the rational maps which transform, by pull-back, a hypergeometric operator in another one, in the case when the monodromy groups are dihedral. The last section contains some general results on Lamé operators admitting a full set of algebraic solutions, and the estimates concerning the case $n=1$.

1. Belyi functions and «dessins d'enfants».

DEFINITION 1.1. *Let C be a curve defined over $\mathbb{C}$. A Belyi function is a cover $\beta : C \to \mathbb{P}^1$, unramified above $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.*

Belyi's theorem states that the existence of a Belyi function $\beta : C \to \mathbb{P}^1$ is equivalent to the possibility of defining C over a number field.

A Belyi function $\beta : C \to \mathbb{P}^1$ will be called clean if $\forall P \in \beta^{-1}(1)$ the multiplicity $e_P = 2$, and pre-clean if $\forall P \in \beta^{-1}(1)$, $e_P \leq 2$.

If we consider the real segment $[0, 1] \subset \mathbb{P}^1$, then $\beta^{-1}([0, 1])$ can be viewed as a bicoloured graph on the topological model of C, connected, with no mutual intersections of the open edges or cells, such that the vertices are endowed with two colors, corresponding to the points of $\beta^{-1}(0)$ («●») and $\beta^{-1}(1)$ (« $*$ ») respectively. Every edge is bounded by vertices of different colors. Such a graph is called «dessin d'enfants», the one associated to β being denoted $\mathcal{D}_\beta$.

There exists a «dictionary» between the ramification data of β and the combinatorial data of $\mathcal{D}_\beta$. For example, the multiplicity of an element of $\beta^{-1}(\{0, 1\})$ coincides with the number of edges incident to the corresponding vertex of $\mathcal{D}_\beta$; in every cell of the graph there is an inverse image of ∞, its multiplicity being half of the number of edges bounding the cell. In particular, to a morphism totally ramified over ∞ corresponds a tree, the degree being the number of edges. If β is clean, then every « $*$ » vertex has the valency 2; we call this a clean dessin. We can view a couple of edges ● $- - * - -$● as a single edge, so regard a clean dessin as an unmarked graph, the vertices being the elements of $\beta^{-1}(0)$. In this case the degree is twice the number of edges and the valency of every inverse image of ∞ is the number of edges bounding the corresponding cell.

Grothendieck correspondence. *There is a bijective correspondence between the set of clean Belyi couples* $(C, \beta : C \to \mathbb{P}^1)$ *modulo isomorphisms and the set of abstract clean dessins* (*i.e. isomorphism classes of dessins*).

The equivalence relations are the natural ones. The correspondence can be extended to pre-clean objects, and even to arbitrary ones (in the sense that to isomorphic couples correspond isomorphic dessins).

DEFINITION 1.2. *A Belyi function* $\beta : C \to \mathbb{P}^1$ *is called* $*$*-morphism if one of the following conditions is satisfied:*

(i) $g(C) = 0$ (*i.e.* $C = \mathbb{P}^1$) *and* $\{0, 1, \infty\} \subset \beta^{-1}(\{0, 1, \infty\})$;

(ii) $g(C) = 1$ (*i.e.* $C = (E, O)$ *elliptic curve*) *and*

$$O \in \beta^{-1}(\{0, 1, \infty\});$$

(iii) $g(C) \geqslant 2$.

In [7] we give some properties of the degree of Belyi functions, among which the following one will be used in this paper:

THEOREM 1.1. *Let* C *be a curve defined over a number field and* $M > 1$ *a fixed real number. The set of* $*$*-morphisms* $C \to \mathbb{P}^1$ *of degree at most* M *is finite.*

The cardinality of the set in the above theorem can be estimated in terms of the number of graphs on a topological surface of given genus, with a given number of vertices and edges.

2. Second degree differential operators with algebraic solutions.

Let C be a compact Riemann surface with function field $K(C)$ and D be a nontrivial derivation of $K(C)/\mathbb{C}$. We are dealing with the second degree differential operators on C:

$$L = D^2 + AD + B\,, \qquad A, B \in K(C)$$

If $P \in C$ and t_P is a local parameter at P, then locally

$$L = R\left(\left(\frac{d}{dt_P}\right)^2 + A_1 \frac{d}{dt_P} + B_1\right).$$

Then P is singular for L if A_1 and B_1 are not both holomorphic at P. We suppose that at each singular point P, L has two independent solutions:

$$x_{i,P} = t_P^{\gamma_{i,P}} \cdot u_{i,P}, \qquad i = 1, 2, \qquad \gamma_{i,P} \in \mathbb{Q}$$

$u_{i,P}$ being invertible power series in t_P (we say that P is a regular singular point). Let

$$\Delta_{P,L} = |\gamma_{1,P} - \gamma_{2,P}|, \qquad \forall P \in C \tag{2.1}$$

the exponential difference in P. If $\Delta_{P,L} \in \mathbb{N} \setminus \{0, 1\}$ we say that P is an apparent singularity.

Obviously $\Delta_{P,L} = 1$ if P is not singular. Let

$$\Delta_L = \sum_{P \in C} (\Delta_{P,L} - 1). \tag{2.2}$$

We say that two differential operators as above are projectively equivalent if they have a common ratio of algebraic solutions. Any such operator L is projectively equivalent to one in normalized form:

$$L = D^2 + B.$$

Projectively equivalent operators have isomorphic projective monodromy groups.

If (C, L) and (C', L') are two curves endowed with differential operators as above and $f : C \to C'$ is a meromorphic function, we say that L is a (weak) pull-back of L' on C if there are ratios of independent solutions of L and L', τ and τ', such that $\tau = \tau' \circ f$. In this case, for all $P \in C$

$$\Delta_{P,L} = e_{P,f} \cdot \Delta_{f(P),L'} \tag{2.3}$$

where $e_{P,f}$ is the multiplicity of P by respect to f. So we get ([1])

$$\Delta_L - 2(g(C) - 1) = \deg f(\Delta_{L'} - 2(g(C') - 1)). \tag{2.4}$$

The aim of this note is to «count», in some special case(s), such differential operators with a full set of algebraic solutions. Schwarz [8] determined all such operators on the projective line with three singular points

— the hypergeometric differential operators:

$$L_{\lambda,\mu,\nu} = D^2 + \frac{1-\lambda^2}{4x^2} + \frac{1-\mu^2}{4(x-1)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{4x(x-1)}, \qquad \lambda+\mu+\nu>1$$

which have exponential differences $\Delta_{P,L_{\lambda,\mu,\nu}}$ equal to λ, μ, ν at $0, 1, \infty$ respectively. We have the following possible cases («Schwarz list»):

(λ, μ, ν)	$G_{L_{\lambda,\mu,\nu}}$
$(1/n, 1, 1/n)$	cyclic of order n
$(1/2, 1/n, 1/2)$	dihedral of order $2n$
$(1/2, 1/3, 1/3)$	tetrahedral
$(1/2, 1/3, 1/4)$	octahedral
$(1/2, 1/3, 1/5)$	icosahedral

In general we have

THEOREM 2.1 (Klein, [6]; see also [1], [2]). *A second order differential operator L in normalized form on a Riemann surface C has finite projective monodromy group if and only if it is a pull-back of a hypergeometric one belonging to the Schwarz list, via a meromorphic function $f: C \to \mathbb{P}^1$.*

As the Wronskian of a differential operator in normalized form belongs to $K(C)$, it follows that it's realization as such a pull-back is equivalent to the fact that it admits a full set of algebraic solutions.

If L and $L_{\lambda,\mu,\nu}$ in the theorem have isomorphic projective monodromy groups, then the function f is unique, under the additional assumption that $\lambda \neq \mu$. In general the group of L is a subgroup of that of $L_{\lambda,\mu,\nu}$, and the realization of L as in the theorem is not unique.

The following proposition is an easy consequence of (2.3) and theorem 2.1:

PROPOSITION 2.2. *Let L be a second order differential operator in normalized form, with finite monodromy group, on a Riemann surface C. If L has no apparent singularity, then the function f in theorem* 2.1 *is a Belyi function.*

REMARK 2.1. *We are interested in counting operators modulo automorphisms of C, so we can suppose, in the hypothesis of the above proposition, that f is a $*$-morphism.*

COROLLARY 2.3. *In the hypothesis of the above proposition, the curve C and the function f can be defined over a number field. Moreover, in this case the singular points are defined over $\overline{\mathbb{Q}}$.*

The cases considered in the next sections will concern only the situation $C=\mathbb{P}^1$.

3. The case of three singular points.

We consider rational functions $f:\mathbb{P}^1\to\mathbb{P}^1$ such that the pull-back of a hypergeometric operator in the Schwarz list is again such an operator. According to the proposition 2.2 and remark 2.1, f is a $*$-morphism.

PROPOSITION 3.1. *Let $L:=L_{\frac{1}{2},\frac{1}{N},\frac{1}{2}}$, $L':=L_{\frac{1}{2},\frac{1}{N'},\frac{1}{2}}$ two hypergeometric operators with projective monodromy groups the dihedral groups of order $2N$, $2N'$ respectively, $N, N'>2$. Then there exists at most a $*$-morphism $f:\mathbb{P}^1\to\mathbb{P}^1$, modulo homographies, such that $f^*L=L'$. More precisely, this morphism exists if and only if $N'|N$.*

PROOF. The equation (2.4) implies that the degree of an eventual f should be $\deg f=N/N'$. Thus, such an f does not exist if N' does not divide N.

On the other hand, the Hurwitz formula implies that

$$\#f^{-1}(\{0,1,\infty\})=N/N'+2$$

and as $\{0,1,\infty\}\subset f^{-1}(\{0,1,\infty\})$, let $a_1,\ldots,a_{N/N'-1}$ be the other points in $f^{-1}(\{0,1,\infty\})$. Their distribution in the ramified fibers, with their possible multiplicities, is according to the following table ([5]):

	0	1	∞	a_i
0	$0,1$	$0,2/N'$	$0,1$	$0,2$
1	$0,N/2$	$0,N/N'$	$0,N/2$	$0,N$
∞	$0,1$	$0,2/N'$	$0,1$	$0,2$

The sum of the multiplicities on each row must be N/N', and in every column there is only one element $\neq 0$. If N/N' is even, the analysis of this table leads to the only possibility as follows:

	0	1	∞	
0	1		1	$+1/2(N'/N'-2)$ points with multiplicity 2
1		N/N'		
∞				$+N/2N'$ points with multiplicity 2

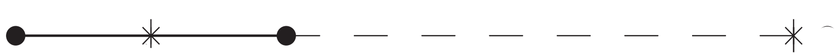
$\Big($and the one with the first and the last line switched, which is obtained from this one applying the homography $x \mapsto \frac{1}{x}\Big)$.

If we permute $\{0, 1, \infty\}$ in the image such that $0 \mapsto 0$, $\infty \mapsto 1$, $1 \mapsto \infty$ $\Big($apply the homography $x \mapsto \frac{x}{x-1}\Big)$, the Belyi function we obtain corresponds to a clean dessin, which is a tree with 2 vertices of valency 1 and $\frac{N}{2N'} - 1$ vertices of valency 2. There is only one such possibility:

If N/N' is odd, we have, again, essentially one possibility:

	0	1	∞	
0	1			$+1/2(N'/N'-1)$ points with multiplicity 2
1		N/N'		
∞			1	$+1/2(N/N'-1)$ points with multiplicity 2

and the dessin (which is pre-clean this time) is



So the proposition is proven. ■

This is the useful case in what follows. We can treat in the same manner the other possible cases of projective monodromy groups in Schwarz's list.

4. The case of Lamé operators.

We consider Lamé operators on the projective line:

$$L_n = D^2 + \frac{f'}{2f} D - \frac{n(n+1)\,x + B}{f}, \qquad n \in \mathbb{N}^*$$

where $f = 4(x-e_1)(x-e_2)(x-e_3)$, $e_1 \neq e_2 \neq e_3 \neq e_1$, $e_i, B \in \mathbb{C}$.

The singular points are e_i and ∞, with exponential differences

$$\Delta_{L_n, e_i} = \frac{1}{2}, \qquad \Delta_{L_n, \infty} = n + \frac{1}{2}.$$

As we are interested in such operators modulo homographies, we can suppose that $e_1 = 0$, $e_2 = 1$, $e_3 = \lambda \in \mathbb{C}$. The only possible finite monodromy group is the dihedral one of order $2N$ (see [3]). In this case, there exists a $*$-morphism F and a hypergeometric operator with the same monodromy group H_N such that $L_n = F^* H_N$ ([1]). The degree of F is nN, by 2.4. The theorem 1.1 implies

Theorem 4.1. *For any fixed N and n, there are only finitely many Lamé operators L_n with dihedral monodromy group of order $2N$.*

The Hurwitz formula implies that

$$\# F^{-1}(\{0, 1, \infty\}) = nN + 2.$$

Let $a_1, \ldots, a_{nN-2}$ be the points of $F^{-1}(\{0, 1, \infty\}) \backslash \{0, 1, \infty, \lambda\}$. Their distribution in the ramified fibers, with their possible multiplicities, is according to the following table ([5]):

	0	1	λ	∞	a_i
0	$0, 1$	$0, 1$	$0, 1$	$0, 2n+1$	$0, 2$
1	$0, N/2$	$0, N/2$	$0, N/2$	$0, nN+N/2$	$0, N$
∞	$0, 1$	$0, 1$	$0, 1$	$0, 2n+1$	$0, 2$

The sum of the multiplicities on each row must be nN, and in every column there is only one element $\neq 0$. The analysis of this table leads to the possible cases as follows (modulo homographies):

Ia

	0	1	λ	∞	
0					$+nN/2$ points with multiplicity 2
1					$+n$ points with multiplicity N
∞	1	1	1	$2n+1$	$+1/2(nN-2n-4)$ points with multiplicity 2

Ib

	0	1	λ	∞	
0			1		$+1/2(nN-1)$ points with multiplicity 2
1					$+n$ points with multiplicity N
∞	1	1		$2n+1$	$+1/2(nN-2n-3)$ points with multiplicity 2

Ic

	0	1	λ	∞	
0		1	1		$+1/2(nN-2)$ points with multiplicity 2
1					$+n$ points with multiplicity N
∞	1			$2n+1$	$+1/2(nN-2n-2)$ points with multiplicity 2

Id

	0	1	λ	∞	
0	1	1	1		$+1/2(nN-3)$ points with multiplicity 2
1					$+n$ points with multiplicity N
∞				$2n+1$	$+1/2(nN-2n-1)$ points with multiplicity 2

II

	0	1	λ	∞	
0					$+nN/2$ points with multiplicity 2
1		$N/2$	$N/2$		$+n-1$ points with multiplicity N
∞	1			$2n+1$	$+1/2(nN-2n-2)$ points with multiplicity 2

We have:

Proposition 4.2. *There is no Lamé operator having dihedral monodromy group of order* 4.

PROOF. Suppose there exists such an operator. Then $N = 2$ and let F be the rational map which realizes it as a pull-back of a hypergeometric operator. In each of the five possible cases, the multiplicity of $\infty \in \in F^{-1}(\infty)$ is $2n + 1 > 2n = \deg F$, which is impossible. ■

Using the Grothendieck correspondence, we shall count in what follows these operators for $n = 1$.

PROPOSITION 4.3. *The number of* $*$*-morphisms compatible with the table* Ia *is*

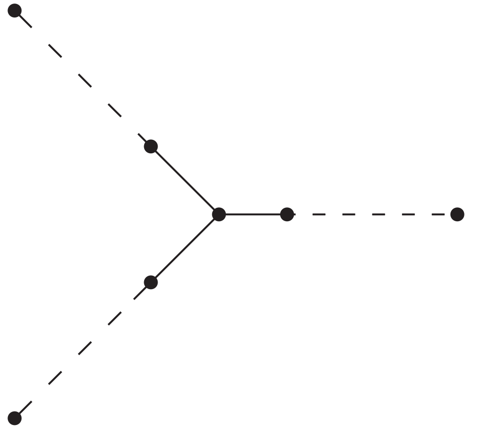
$$\left[\frac{N}{6}\right] \cdot \frac{N - 3[N/6] - 3}{2} + \varepsilon$$

if $N \geqslant 6$ *and is even, and* 0 *if not, where* $\varepsilon = 1$ *if* $3 \mid N$ *and* $\varepsilon = 0$ *if not.*

PROOF. Permuting in a convenient way the points $0, 1, \infty$, the number of such covers is the same with the one of the covers with the following ramification data:

	0	1	λ	∞	
0	1	1	1	3	$+1/2(N-6)$ points with multiplicity 2
1					$+N/2$ points with multiplicity 2
∞					$+1$ point with multiplicity N

But these are clean Belyi functions, which correspond to dessins with the following topological and combinatorial properties:



– they are trees with $\frac{N}{2}$ edges (as we have one inverse image of ∞ with multiplicity N);

– there is only one vertex with valency 3;

– there are three vertices with valency 1, all the others having valency 2.

The three branches have respectively A, B, C edges. Then

$$A + B + C = \frac{N}{2}$$

$A, B, C > 0$ and, for avoiding symmetries, $A \leqslant B$, $A \leqslant C$. It follows that

$$A \in \left\{1, 2, \ldots, \left[\frac{N}{6}\right]\right\}$$

and for each A

$$B \in \left\{A, A+1, \ldots, \frac{N}{2} - 2A - 1\right\}.$$

We also have an additional case if $3 \mid N \left(A = B = C = \frac{N}{6}\right)$. A simple calculation says that the numbers of couples (A, B) satisfying these conditions is the one in the statement. ■

PROPOSITION 4.4. *The number of* $*$*-morphisms compatible with the table Ib is*

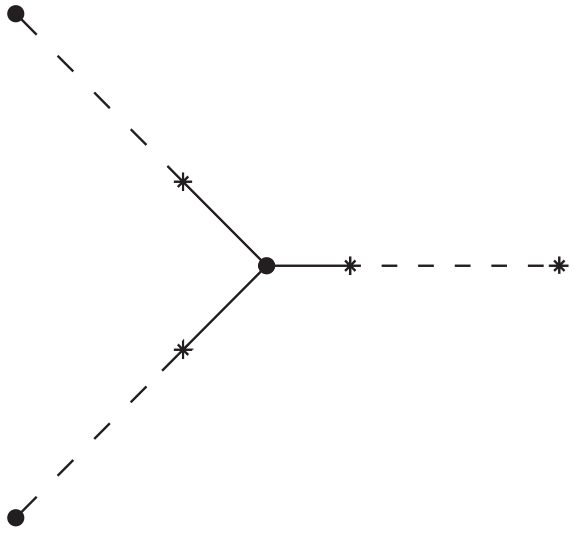
$$\frac{N-3}{4} \cdot \frac{N-1}{2}$$

if $N \geqslant 5$ *and is odd, and* 0 *if not.*

PROOF. Permuting in a convenient way the points $0, 1, \infty$, the number of such covers is the same with the one of the covers with the following ramification data:

	0	1	λ	∞	
0	1	1		3	$+1/2(N-5)$ points with multiplicity 2
1			1		$+1/2(N-1)$ points with multiplicity 2
∞					$+1$ point with multiplicity N

But these are Belyi functions, which correspond to pre-clean dessins with the following topological and combinatorial properties:



– they are trees with N edges (as we have one inverse image of ∞ with multiplicity N);

– there is only one «●» vertex with valency 3;

– there are three vertices with valency 1, all the others having valency 2; two of the three are «●» vertices, the third one is a « $*$ » vertex.

The three branches have respectively A, B, C edges. Two of the branches end with «●» vertices, the third one with a « $*$ » vertex. We have

$$A + B + C = N$$

N odd, $A, B, C > 0$, A odd and B, C even. It follows that

$$A \in \{1, 3, \dots, N-4\}$$

and for each A

$$B \in \{2, 4, \dots, N-A-2\}$$

so $\frac{N-A-2}{2}$ possibilities. The number of these trees is

$$\frac{N-3}{2} + \frac{N-5}{2} + \ldots + \frac{2}{2} = 1 + 2 + \ldots + \frac{N-3}{2} = \frac{N-3}{4} \cdot \frac{N-1}{2}. \qquad \blacksquare$$

PROPOSITION 4.5. *The number of* $*$*-morphisms compatible with the table Ic is*

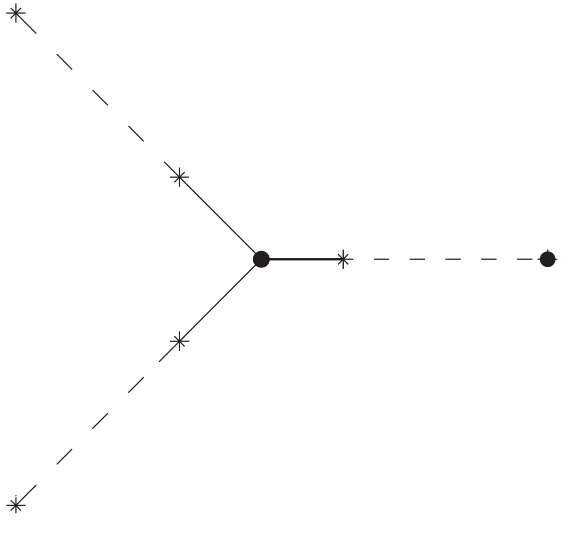
$$\frac{N-2}{4} \cdot \frac{N}{2}$$

if $N \geqslant 5$ *and is even, and* 0 *if not.*

PROOF. Permuting in a convenient way the points $0, 1, \infty$, the number of such covers is the same with the one of the covers with the following ramification data:

	0	1	λ	∞	
0	1			3	$+1/2(N-4)$ points with multiplicity 2
1		1	1		$+1/2(N-2)$ points with multiplicity 2
∞					$+1$ point with multiplicity N

As in the previous proposition, these are Belyi functions, which correspond to pre-clean dessins as follows:



– they are trees with N edges (as we have one inverse image of ∞ with multiplicity N);

– there is only one «●» vertex with valency 3;

– there are three vertices with valency 1, all the others having valency 2; two of the three are « $*$ » vertices, the third one is a «●» vertex.

The three branches have respectively A, B, C edges. Two of the branches end with « $*$ » vertices, the third one with a «●» vertex. We have

$$A + B + C = N$$

N even, $A, B, C > 0$, A even and B, C odd. It follows that

$$A \in \{2, 4, \ldots, N-2\}, \qquad B \in \{1, 3, \ldots, N-A-1\}$$

and, as in the previous proof, the number of these trees is

$$\frac{N-2}{2} + \frac{N-4}{2} + \ldots + \frac{2}{2} = 1 + 2 + \ldots + \frac{N-2}{2} = \frac{N-2}{4} \cdot \frac{N}{2}. \qquad \blacksquare$$

Proposition 4.6. *The number of $*$-morphisms compatible with the table Id is*

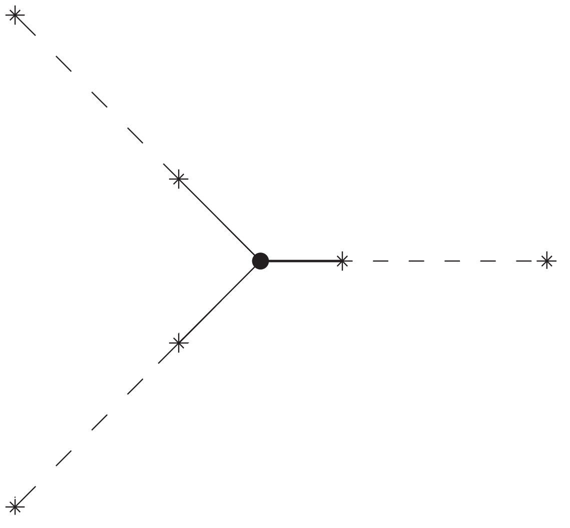
$$\left[\frac{N+3}{6}\right] \cdot \frac{N - 3\left[\frac{N+3}{6}\right]}{2} + \varepsilon$$

if $N \geqslant 3$ and is odd, and 0 if not, where $\varepsilon = 1$ if $3 \mid N$ and $\varepsilon = 0$ if not.

Proof. Permuting in a convenient way the points $0, 1, \infty$, the number of such covers is the same with the one of those with the ramification data:

	0	1	λ	∞	
0				3	$+1/2(N-3)$ points with multiplicity 2
1	1	1	1		$+1/2(N-3)$ points with multiplicity 2
∞					$+1$ point with multiplicity N

As in the proposition 4.3, these are Belyi functions, which correspond to dessins with the following topological and combinatorial properties:



– they are trees with N edges (as we have one inverse image of ∞ with multiplicity N;

– there is only one «●» vertex with valency 3;

– there are three vertices with valency 1, all the others having valency 2; all three are « $*$ » vertices.

The three branches have respectively A, B, C edges. Then

$$A + B + C = \frac{N+3}{2}$$

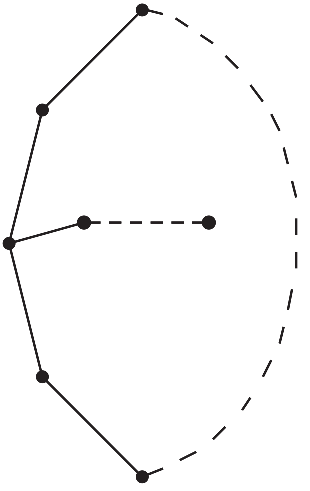
A, B, $C > 0$ odd and, for avoiding symmetries, $A \leqslant B$, $A \leqslant C$. It follows, by a calculation similar to the one in the proof of the proposition 4.3 (realized with $N+3$ instead of N), that the numbers of couples (A, B) satisfying these conditions is the one in the statement. ■

PROPOSITION 4.7. *There is no* $*$*-morphism compatible with the table* II.

PROOF. Permuting in a convenient way the points $0, 1, \infty$, the number of the covers compatible with the table II is the same with the one of the covers with the following ramification data:

	0	1	λ	∞	
0	1			3	$+1/2(N-4)$ points with multiplicity 2
1					$+N/2$ points with multiplicity 2
∞		$N/2$	$N/2$		

It is a clean Belyi cover, and the corresponding dessin has two cells.



The cells have the same valency, but on the other hand there is a unique vertex with valency three, so there is an edge starting in this vertex, inside one of the cells. But in such a situation, the two cells cannot have the same valency, so there is no cover with such a ramification data. ■

Taking into account that N must be even in the cases Ia and Ic, and odd in the cases Ib and Id, we obtain (as in [5]):

THEOREM 4.8. *The number $\mathcal{C}(1,N)$ of non-homographic covers $\mathbb{P}^1 \to \mathbb{P}^1$ which transform by pull-back a hypergeometric operator H_N*

into a Lamé operator L_1 *is*

$$\mathcal{C}(1, N) = \frac{(N-1)(N-2)}{6} + \frac{2\varepsilon}{3} \tag{4.5}$$

where $\varepsilon = 1$ *if* $3 \mid N$ *and* $\varepsilon = 0$ *if not.*

Finally, we have

THEOREM 4.9. *Let* $\mathcal{L}(1, N)$ *be the number of non-homographic Lamé operators* L_1 *with finite dihedral monodromy group of order* $2N$. *Then*

$$\mathcal{C}(1, N) = \sum_{N' \mid N,\, N' \neq 2} \mathcal{L}(1, N'). \tag{4.6}$$

PROOF. The proposition 4.2 implies that $N' \neq 2$. Let N be a fixed integer and $N' \mid N$, $N' \neq 2$. As remarked at the beginning of this section, for any Lamé operator L_1 with finite dihedral monodromy group of order $2N'$ there exists an unique $*$-morphism F and a hypergeometric operator with the same monodromy group, $H_{N'}$, such that $L_1 = F^* H_{N'}$. On the other hand, proposition 3.1 says that there exists an unique cover $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$, such that $\Phi^*(H_N) = H'_N$. The theorem follows. ■

Acknowledgments. *I wish to express my entire gratitude to Francesco Baldassarri for introducing me to the subject and to Bruno Chiarellotto for several useful conversations and for pointing out to me his article* [5].

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Manoscritto pervenuto in redazione il 22 ottobre 2001