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A Density Property of the Tori and Duality.

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ABSTRACT - In this note, a short proof of a recent theorem of D. Dikranjan and M. Tkachenko is given, and their result is extended.

1. Introduction.

Let us establish notation and terminology. The additive groups of integers and rational numbers taken discrete are denoted by $\boldsymbol{Z}$ and $\boldsymbol{Q}$ respectively. Recall that two nontrivial subgroups of $\boldsymbol{Q}$ are isomorphic if and only if they are of the same type (see [F] Theorem 85.1). The type of $\boldsymbol{Z}$ is written $\boldsymbol{t}_0$. By $\boldsymbol{R}$ we mean the additive topological group of real numbers, and by $\boldsymbol{T}$ we denote the one-dimensional torus with its usual topology. All considered groups are locally compact abelian groups and will be written additively. The subgroup of all compact elements of the group G is denoted by bG, while G_0 stands for the identity component of G. The Pontrjagin dual of G is denoted by $\widehat{G}$. If H is a subset of G, then $(\widehat{G}, H)$ is the annihilator of H in $\widehat{G}$. Throughout this paper, we use the term «isomorphic» for «topologically isomorphic». For details and fundamental results on locally compact abelian groups and Pontrjagin duality, we may refer to the book [HR] of Hewitt and Ross.

For a group G, $a \in G$, and a positive integer n let

$$S_n(a) = \{x \in G : nx = a\}.$$

Notice that if G is divisible, in particular if it is a compact connected group, $S_n(a)$ is nonempty. Following Dikranjan and Tkachenko [DT], we

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say that a compact metrizable group G with an invariant metric d satisfies *condition* ε if for every $a \in G$ and for every positive integer n there is an $\varepsilon_n > 0$ such that $d(y, S_n(a)) \leqslant \varepsilon_n$ for every $y \in G$ and $\varepsilon_n \to 0$ as $n \to \infty$. In [DT], it was shown that the finite-dimensional tori are exactly those compact connected abelian groups having the property that every closed connected one-dimensional subgroup satisfies condition ε (see [DT] Theorem 3.1). In this note, we give a short proof of a more general result.

Pontrjagin duality shows that a nontrivial compact metrizable group satisfies condition ε if and only if its dual group is homogeneous of type $\boldsymbol{t}_0$ (Lemma 2.1). A locally compact abelian group G has the form $\boldsymbol{R}^n \times \boldsymbol{Z}^m \times bG$ exactly if every discrete torsion-free rank-one group G/H is cyclic (see Theorem 2.2). This yields a characterization of finite rank free abelian groups among the class of all discrete torsion-free groups (Corollary 2.3). The groups of the form $\boldsymbol{R}^n \times \boldsymbol{T}^m \times D$ (where D is totally disconnected) are precisely those locally compact abelian groups having the property that every compact connected one-dimensional subgroup satisfies condition ε (see Theorem 2.4). As an immediate consequence, [DT] Theorem 3.1 follows.

2. Condition ε and duality.

The following preliminary lemma will be useful:

LEMMA 2.1. *A nontrivial compact metrizable abelian group satisfies condition ε if and only if its discrete dual group is homogeneous of type $\boldsymbol{t}_0$.*

PROOF. Suppose $G \neq 0$ is a compact metrizable group. By [DT] Lemma 2.4, G satisfies condition ε exactly if for every prime p and every infinite set π of primes, the p-torsion part $t_p(G)$ and the π-socle $\mathrm{soc}_\pi(G) = \bigoplus_{q \in \pi} G[q]$ are dense subgroups of G. Since

$$(\widehat{G}, \overline{t_p(G)}) = p^\omega \widehat{G} \qquad \text{and} \qquad (\widehat{G}, \overline{\mathrm{soc}_\pi(G)}) = \bigcap_{q \in \pi} q\widehat{G},$$

G satisfies condition ε if and only if every nonzero element of $\widehat{G}$ is divisible by only finitely many primes and has finite p-height for every prime p, i.e., if $\widehat{G}$ is homogeneous of type $\boldsymbol{t}_0$. ■

THEOREM 2.2. *The following conditions are equivalent for a locally compact abelian group G*:

(i) *Every discrete torsion-free rank-one group G/H is cyclic;*

(ii) *G is isomorphic to $\boldsymbol{R}^n \times \boldsymbol{Z}^m \times bG$ where n and m are nonnegative integers;*

(iii) *every nontrivial discrete torsion-free group G/H is homogeneous of type $\boldsymbol{t}_0$.*

PROOF. (i) implies (ii). Suppose G satisfies condition (i). Let $G' = G/(G_0 + bG)$ and note that G' is discrete. Any locally compact abelian group is isomorphic to $\boldsymbol{R}^n \times L$ with a nonnegative integer n and L containing a compact open subgroup (see [HR] Theorem 24.30); thus $G' = G/(G_0 + bG) \cong (\boldsymbol{R}^n \times L)/(\boldsymbol{R}^n \times bL) \cong L/bL$ is torsion-free. If G' has infinite rank, it contains a free subgroup Z of infinite rank. Consequently, $\boldsymbol{Q}$ is a homomorphic image of Z. Since $\boldsymbol{Q}$ is injective, it is isomorphic to a factor group of G, a contradiction. Therefore G' has finite rank.

If $\operatorname{rank}(G') = 1$, then G' is isomorphic to $\boldsymbol{Z}$, hence G is isomorphic to $\boldsymbol{R}^n \times \boldsymbol{Z} \times bG$. If $\operatorname{rank}(G') = k > 1$, G' contains a free subgroup F of rank k, say, $F = X_1 \oplus \ldots \oplus X_k$ where each group X_i is isomorphic to $\boldsymbol{Z}$. Let $F_i = \oplus_{j \neq i} X_j$ and Y_i the purification of F_i in G'. Then we have an embedding

$$G' \to G'/Y_1 \oplus \ldots \oplus G'/Y_k,$$

and every group G'/Y_i is isomorphic to $\boldsymbol{Z}$. Consequently, G' is free and therefore G has the form $\boldsymbol{R}^n \times \boldsymbol{Z}^m \times bG$, as desired.

(ii) implies (iii). Suppose G satisfies (ii). If the quotient G/H is discrete and torsion-free, then $bG + G_0$ is a subgroup of H, hence G/H is finitely generated and (iii) follows.

By [F] Theorem 85.1, (iii) implies (i). ■

COROLLARY 2.3. *The following conditions are equivalent for a discrete torsion-free abelian group G:*

(i) *Every torsion-free rank-one group G/H is cyclic;*

(ii) *G is a finite rank free abelian group;*

(iii) *every nontrivial torsion-free group G/H is homogeneous of type $\boldsymbol{t}_0$.*

Locally compact abelian groups whose identity component have the form $\boldsymbol{R}^n \times \boldsymbol{T}^m$, can be characterized in terms of condition ε:

Theorem 2.4. *The following conditions are equivalent for a locally compact abelian group G:*

(i) *Every compact connected one-dimensional subgroup of G satisfies condition ε;*

(ii) *G is isomorphic to* $\boldsymbol{R}^n \times \boldsymbol{T}^m \times D$ *where n and m are nonnegative integers, and D is totally disconnected;*

(iii) *every nontrivial compact connected subgroup L of G is metrizable and* $\widehat{L}$ *is homogeneous of type* $\boldsymbol{t}_0$;

(iv) *every compact connected subgroup of G is metrizable and satisfies condition* ε.

Proof. Using Pontrjagin duality (see [HR] Theorems 24.25, 24.28) (i) implies (ii) because of Lemma 2.1 and Theorem 2.2. Suppose G is a group as in (ii) and L is a compact connected subgroup of G. Then L is a torus, hence $\widehat{L}$ is a free abelian group and therefore (ii) implies (iii). By Lemma 2.1, (iii) implies (iv). Clearly (iv) implies (i). ■

As a corollary, we obtain

Theorem 2.5 (Dikranjan and Tkachenko [DT]). *The following conditions are equivalent for a compact connected abelian group G:*

(i) *Every closed connected one-dimensional subgroup of G satisfies condition* ε;

(ii) *G is a finite-dimensional torus;*

(iii) *every closed connected subgroup of G is metrizable and satisfies condition* ε.

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