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A Note on the Bifurcation of Solutions for an Elliptic Sublinear Problem.

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1. Introduction.

We consider the following semilinear problem in a bounded domain $\Omega \subset \mathbf{R}^N$:

$$\text{(1.1)} \qquad \begin{cases} Lu + u^\theta = \lambda u & \text{in } \Omega, \\ u \geq 0, \quad u \not\equiv 0, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \theta < 1$ and Lu is a linear differential operator defined as $Lu = -\operatorname{div}(A(x)\nabla u)$, $A(x)$ being a symmetric, bounded and coercive matrix, that is such that

$$\text{(1.2)} \qquad \alpha|\xi|^2 \leq A(x)\,\xi\cdot\xi \leq \beta|\xi|^2$$

$$\text{for every } \xi \in \mathbf{R}^N, \text{ a.e. } x \in \Omega, \text{ with } \alpha, \beta > 0.$$

A necessary condition to find nonnegative nontrivial solutions of (1.1) is that $\lambda > \lambda_1$, the first eigenvalue of the operator L on Ω with Dirichlet

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boundary conditions. Notice that (1.1) is the Euler equation of the functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega A(x)\,\nabla u\cdot\nabla u\,dx - \frac{\lambda}{2}\int_\Omega (u^+)^2\,dx + \frac{1}{\theta+1}\int_\Omega |u|^{\theta+1}\,dx \qquad (1.3)$$

$$u \in H_0^1(\Omega),$$

which, since $0<\theta<1$, is not bounded from below if $\lambda>\lambda_1$.

As it was remarked in [8] (see also [6]), the existence of a solution of (1.1) for every $\lambda>\lambda_1$ can be obtained applying the abstract fixed point results by Krasnoselskii ([9]). This approach relies on the definition of the inverse operator of $u \mapsto Lu + |u|^\theta \frac{u}{|u|}$ and makes use of its compactness properties and the computation of its derivative at zero and at infinity. Similarly, using the results of Rabinowitz in [11], in [6] it is proved the existence of a connected branch of solutions (λ, u_λ) bifurcating from infinity at $\lambda=\lambda_1$. The same kind of arguments is used in a more general context in [3] to deal also with sublinear eigenvalue problems, following the method developed in [1] (see also [10]).

As a consequence of the previous results, the bifurcation diagram for solutions of (1.1) reads as follows: there is bifurcation from infinity if and only if $\lambda=\lambda_1$ while no finite value of λ can be a bifurcation point from the zero solution. The aim of this note is to prove that the bifurcation from the trivial solution actually occurs at $\lambda=+\infty$.

More precisely, we prove that there exists a sequence u_λ, $\lambda>\lambda_1$, of Mountain Pass type solutions of (1.1) such that $\|u_\lambda\|_{H_0^1(\Omega)}$ tends to zero as λ goes to infinity. In fact, we do not know whether any sequence of solutions of (1.1) will converge to zero as λ tends to infinity, nevertheless the sublinear behavior of the absorption term both near the origin and at infinity yields a classical Mountain Pass type structure to the functional J_λ which allows to construct such a sequence, obtaining the bifurcation from $u=0$ at $\lambda=+\infty$. This is the result that we prove.

THEOREM 1.1. *Let* $0<\theta<1$, *and let* (1.2) *be satisfied. Then for every* $\lambda>\lambda_1$ *problem* (1.1) *admits a solution* $u_\lambda \in H_0^1(\Omega)$ *which is a critical point of Mountain Pass type of the functional* J_λ *defined in* (1.3). *Moreover this family of solutions* u_λ *satisfies*

$$\lim_{\lambda\to+\infty} \|u_\lambda\|_{H_0^1(\Omega)} = 0\,. \qquad (1.4)$$

The bifurcation from the trivial solution at infinity has already been proved for the semilinear problem

$$\text{(1.5)} \qquad \begin{cases} Lu = \lambda u^p & \text{in } \Omega\,, \\ u \geqslant 0, \quad u \neq 0, & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

with $p > 1$. For a certain range of values of p this was obtained in [5] via a priori estimates while a general result (for every subcritical $p > 1$) was proved in [4] using the variational structure of the equation and estimates on the Mountain Pass level of the related functional. Our problem (1.1) is much different with respect to (1.5) since the nonlinearity $\lambda u - u^\theta$ is asymptotically linear at infinity and the Mountain Pass character is due to a sublinear positive perturbation. This implies that the approaches of these previous papers do not work, in particular the standard regularity (bootstrap) argument is not such helpful and the method used in [4] would only give here an estimate on the $L^{\theta+1}(\Omega)$ norm of u. For this reason we use here a refined construction of the Mountain Pass solutions u_λ, which are obtained through a recurrence argument, allowing, at fixed $\overline{\lambda}$, to estimate $J_{\overline{\lambda}}$ on the paths $[0, u_\lambda]$, where $\lambda < \overline{\lambda}$ and u_λ is the Mountain Pass solution previously found.

Let us stress that, in dimension $N = 1$, a full study of the equation (1.1) is carried out in the recent work [7]. In that paper the case of the p-laplace operator is also considered and the set of solutions of the problem is fully characterized, giving multiplicity results as well. While our results can be proved without relevant changes for the p-laplace operator as well, we cannot give here a full picture of the behavior of the solutions in dependence of λ.

2. Proof of Theorem 1.1.

We denote by λ_1 the first eigenvalue of the operator L with Dirichlet boundary conditions on $\partial\Omega$, and by φ_1 the positive eigenfunction such that $L\varphi_1 = \lambda_1 \varphi_1$, $\|\varphi_1\|_{L^2(\Omega)} = 1$.

Proof of Theorem 1.1. The proof of Theorem 1.1 is a consequence of the following two Propositions.

Proposition 2.1. *Let $\lambda > \lambda_1$. Then problem* (1.1) *admits a solution u_λ which is a critical point of Mountain Pass type of the functional J_λ*

defined in (1.3). *Moreover* u_λ *satisfies:*

$$J_\lambda(u_\lambda) \leqslant \left(\frac{1}{\theta+1} - \frac{1}{2}\right) \frac{\left(\int_\Omega |w|^{\theta+1}\,dx\right)^{\frac{2}{1-\theta}}}{\left(\lambda \int_\Omega w^2 - \int_\Omega A(x)\,\nabla w\cdot\nabla w\,dx\right)^{\frac{1+\theta}{1-\theta}}}, \tag{2.1}$$

$$\text{for every } w : \lambda \int_\Omega w^2\,dx - \int_\Omega A(x)\,\nabla w\cdot\nabla w\,dx > 0\,.$$

Proof. We are going to use the classical theorem of Ambrosetti and Rabinowitz ([2]) with a little change in the choice of the paths.

We start by proving that $u = 0$ is a local minimum for the functional J_λ. Precisely, we claim that

$$\exists \varrho > 0 : \quad J_\lambda(u) > 0 \quad \forall u \in \overline{B}_\varrho \setminus \{0\} = \{\|u\|_{H_0^1(\Omega)} \leqslant \varrho,\, u \neq 0\}, \tag{2.2}$$

which will easily imply that

$$\exists \sigma : \quad J_\lambda(u) \geqslant \sigma \quad \forall u : \|u\|_{H_0^1(\Omega)} = \varrho\,. \tag{2.3}$$

In order to prove (2.2) we argue by contradiction. If (2.2) were not true, it would exist a sequence u_n in $H_0^1(\Omega)$ such that:

$$\|u_n\|_{H_0^1(\Omega)} \to 0, \qquad u_n \neq 0, \qquad J_\lambda(u_n) \leqslant 0 \ \ \forall n \in \boldsymbol{N}\,,$$

that is

$$\frac{1}{2}\int_\Omega A(x)\,\nabla u_n\cdot\nabla u_n\,dx + \frac{1}{\theta+1}\int_\Omega |u_n|^{\theta+1}\,dx \leqslant \frac{\lambda}{2}\int_\Omega |u_n|^2\,dx\,. \tag{2.4}$$

Setting $w_n = \dfrac{u_n}{\|u_n\|_{H_0^1(\Omega)}}$, we have:

$$\int_\Omega |w_n|^{\theta+1}\,dx \leqslant \frac{\lambda(\theta+1)}{2}\|u_n\|_{H_0^1(\Omega)}^{1-\theta}\int_\Omega |w_n|^2. \tag{2.5}$$

Since $\|w_n\|_{H_0^1(\Omega)} = 1$, we have that, up to subsequences, w_n converges to a function $w \in H_0^1(\Omega)$ almost everywhere in Ω, hence we deduce by Fatou's lemma and (2.5) that $w = 0$, so that w_n strongly converges to zero in $L^2(\Omega)$ as well by Rellich theorem. Dividing the equality in (2.4) by $\|u_n\|_{H_0^1(\Omega)}^2$, using the ellipticity of $A(x)$ we conclude that w_n tends to zero in $H_0^1(\Omega)$, contradicting the fact that $\|w_n\|_{H_0^1(\Omega)} = 1$. Thus (2.2) is proved. Now, assume that (2.3) is violated. Then we deduce again the existence of a sequence u_n such that $\|u_n\|_{H_0^1(\Omega)} = \varrho$ and $\limsup_{n\to\infty} J_\lambda(u_n) = 0$. Up to a subsequence we have that there exists a function u in $\overline{B}_\varrho = \{\|u\|_{H_0^1(\Omega)} \leq \varrho\}$ such that u_n weakly converges to u in $H_0^1(\Omega)$. By lower semicontinuity we get that $J_\lambda(u) \leq 0$, so that (2.2) implies that $u = 0$. As a consequence, u_n tends to zero strongly in $L^2(\Omega)$, hence from

$$\frac{\alpha}{2}\int_\Omega |\nabla u_n|^2\,dx \leq \frac{1}{2}\int_\Omega A(x)\,\nabla u_n\cdot\nabla u_n\,dx \leq J_\lambda(u_n) + \frac{\lambda}{2}\int_\Omega |u_n|^2,$$

we conclude, since $\limsup_{n\to\infty} J_\lambda(u_n) = 0$, that u_n strongly converges to zero in $H_0^1(\Omega)$, which is not possible since $\|u_n\|_{H_0^1(\Omega)} = \varrho$ for every n in $\boldsymbol{N}$. Thus (2.3) is proved.

The proof that the functional J_λ satisfies the Palais-Smale condition is standard and the details are left to the reader. Let us just recall that the main tool is, as usual, the proof that a Palais-Smale sequence u_n is bounded, in order to apply the compactness properties of the equation. To obtain the boundedness of the sequence u_n, it is enough to argue by contradiction, assuming that $\|u_n\|_{H_0^1(\Omega)}$ goes to infinity and reasoning on the normalized sequence $\dfrac{u_n}{\|u_n\|_{H_0^1(\Omega)}}$, which can be proved to strongly converge to zero (actually, it will converge to a nonnegative eigenfunction corresponding to the eigenvector λ, since $\lambda > \lambda_1$ it must converge to zero).

We define now the class of paths to which apply the usual deformation lemma. First, let

$$C_\lambda := \left\{ w \in H_0^1(\Omega) : \lambda\int_\Omega w^2 - \int_\Omega A(x)\,\nabla w\cdot\nabla w\,dx > 0 \right\},$$

which is clearly a nonempty subset of $H_0^1(\Omega)$ since $\varphi_1 \in C_\lambda$ for every $\lambda > \lambda_1$. Note that for any $w \in C_\lambda$ we have that there exists only one value $t_w > 0$ such that $J_\lambda(t_w w) = 0$.

Then we define the set of paths:

$$X_\lambda := \{\gamma \in C([0,1], H_0^1(\Omega)) : \exists w \in C_\lambda : \gamma(0) = 0,\ \ \gamma(1) = t_w w\}.$$

We can then prove that

$$c := \inf_{\gamma\in X_\lambda}\ \sup_{[0,1]} J_\lambda(\gamma(t))$$

is a critical value for J_λ. Indeed, let ϱ and σ be given from (2.3). Since $J_\lambda(t_w w) = 0$ it follows from (2.2) that $\|t_w w\|_{H_0^1(\Omega)} > \varrho$, hence there exists a value t such that $\|\gamma(t)\|_{H_0^1(\Omega)} = \varrho$ and thus $c \geqslant \sigma$. Now, if the set $K_c = \{v \in H_0^1(\Omega) : J_\lambda(v) = c, J_\lambda'(v) = 0\}$ were empty, it would exist a continuous deformation $\eta : [0, 1] \times H_0^1(\Omega) \to H_0^1(\Omega)$ and a value $\varepsilon > 0$ such that $\eta(t, v) = v$ for any $v \in H_0^1(\Omega)$ such that $|J_\lambda(v) - c| > \frac{\sigma}{2}$ and $J_\lambda(\eta(1, v)) \leqslant c - \varepsilon$ for every v such that $J_\lambda(v) \leqslant c + \varepsilon$. If γ_ε is a path such that $\sup\limits_{[0,1]} J_\lambda(\gamma_\varepsilon(t)) \leqslant c + \varepsilon$, we have that the path $\eta(1, \gamma_\varepsilon(t))$ belongs to X_λ and $\sup\limits_{[0,1]} J_\lambda(\eta(1, \gamma_\varepsilon(t))) \leqslant c - \varepsilon$, contradicting the definition of c. Therefore c is a critical value and there exists a critical point u_λ in K_c. Clearly, u_λ is a weak solution of (1.1). Moreover, note that the path $\gamma(t) = t t_w w$ belongs to X_λ for any $w \in C_\lambda$, hence we get:

$$J_\lambda(u_\lambda) = c \leqslant \sup_{[0,1]} J_\lambda(t t_w w).$$

A straightforward calculation implies (2.1). ■

PROPOSITION 2.2. *Let u_λ be the solution of* (1.1) *found in Proposition* 2.1. *Then we have:*

$$\lim_{\lambda \to +\infty} \|u_\lambda\|_{H_0^1(\Omega)} = 0\,. \tag{2.6}$$

PROOF. We denote, in what follows, by C all possibly different constants which do not depend on λ. First of all, since $J_\lambda'(u_\lambda) = 0$, we have

$$\left(\frac{1}{\theta+1} - \frac{1}{2}\right) \int_\Omega u_\lambda^{\theta+1}\, dx = J_\lambda(u_\lambda) - \frac{1}{2}\langle J_\lambda'(u_\lambda), u_\lambda\rangle = J_\lambda(u_\lambda),$$

so that using (2.1) we obtain the estimate:

$$\int_\Omega u_\lambda^{\theta+1}\, dx \leqslant \frac{\left(\int_\Omega |w|^{\theta+1}\, dx\right)^{\frac{2}{1-\theta}}}{\left(\lambda \int_\Omega w^2 - \int_\Omega A(x)\, \nabla w \cdot \nabla w\, dx\right)^{\frac{1+\theta}{1-\theta}}}, \quad \text{fore every } w \in C_\lambda. \tag{2.7}$$

In particular, choosing $w = \varphi_1$ in (2.7) we get

$$\int_\Omega u_\lambda^{\theta+1}\, dx \leqslant \frac{C}{\lambda^{\frac{\theta+1}{1-\theta}}}, \qquad \forall \lambda > \lambda_1. \tag{2.8}$$

Moreover, for every $\lambda' < \lambda$ we have that

$$\int_\Omega A(x)\,\nabla u_{\lambda'} \cdot \nabla u_{\lambda'}\,dx - \lambda' \int_\Omega u_{\lambda'}^2\,dx = -\int_\Omega u_{\lambda'}^{\theta+1}\,dx\,,$$

hence

$$\int_\Omega A(x)\,\nabla u_{\lambda'} \cdot \nabla u_{\lambda'}\,dx - \lambda \int_\Omega u_{\lambda'}^2\,dx = (\lambda' - \lambda)\int_\Omega u_{\lambda'}^2\,dx - \int_\Omega u_{\lambda'}^{\theta+1}\,dx < 0\,.$$

Thus $u_{\lambda'} \in C_\lambda$ for every $\lambda' < \lambda$, so that (2.7) implies:

$$\int_\Omega u_\lambda^{\theta+1}\,dx \leqslant \frac{\left(\int_\Omega |u_{\lambda'}|^{\theta+1}\,dx\right)^{\frac{2}{1-\theta}}}{\left(\lambda \int_\Omega u_{\lambda'}^2 - \int_\Omega A(x)\,\nabla u_{\lambda'} \cdot \nabla u_{\lambda'}\,dx\right)^{\frac{1+\theta}{1-\theta}}}\,,$$

which yields, using the equation solved by $u_{\lambda'}$,

$$\int_\Omega u_\lambda^{\theta+1}\,dx \leqslant \frac{\left(\int_\Omega |u_{\lambda'}|^{\theta+1}\,dx\right)^{\frac{2}{1-\theta}}}{\left((\lambda-\lambda')\int_\Omega u_{\lambda'}^2 + \int_\Omega |u_{\lambda'}|^{\theta+1}\,dx\right)^{\frac{1+\theta}{1-\theta}}} \qquad \forall \lambda' < \lambda\,. \tag{2.9}$$

Our main task now is proving that the sequence $\{u_\lambda\}$ is bounded in $H_0^1(\Omega)$. To this purpose, we argue by contradiction. Assume then that $\|u_\lambda\|_{H_0^1(\Omega)}$ is not bounded. Then we can construct a subsequence λ_h, such that:

$$\lambda_h - \lambda_{h-1} \geqslant 1, \quad \|u_{\lambda_h}\|_{H_0^1(\Omega)} \geqslant \|u_{\lambda_{h-1}}\|_{H_0^1(\Omega)}, \quad \lim_{h\to\infty} \|u_{\lambda_h}\|_{H_0^1(\Omega)} = +\infty\,. \tag{2.10}$$

From (2.9) written for u_{λ_h} and $u_{\lambda_{h-1}}$ we obtain (note that $\lambda_h - \lambda_{h-1} \geqslant 1$):

$$\int_\Omega u_{\lambda_h}^{\theta+1}\,dx \leqslant \frac{\left(\int_\Omega |u_{\lambda_{h-1}}|^{\theta+1}\,dx\right)^{\frac{2}{1-\theta}}}{\left(\int_\Omega u_{\lambda_{h-1}}^2 + \int_\Omega |u_{\lambda_{h-1}}|^{\theta+1}\,dx\right)^{\frac{1+\theta}{1-\theta}}}\,.$$

In order to simplify our notations, we denote henceforth u_{λ_h} by u_λ and

$u_{\lambda_{h-1}}$ by $u_{\lambda-1}$, so that we rewrite the previous inequality as

$$(2.11)\qquad \int_\Omega u_\lambda^{\theta+1}\,dx \leqslant \frac{\left(\int_\Omega |u_{\lambda-1}|^{\theta+1}\,dx\right)^{\frac{2}{1-\theta}}}{\left(\int_\Omega u_{\lambda-1}^2 + \int_\Omega |u_{\lambda-1}|^{\theta+1}\,dx\right)^{\frac{1+\theta}{1-\theta}}}\,.$$

Let us set $v_\lambda = \frac{u_\lambda}{\|u_\lambda\|_{H_0^1(\Omega)}}$. Then we have:

$$(2.12)\qquad \|v_\lambda\|_{H_0^1(\Omega)} = 1, \qquad Lv_\lambda + \frac{v_\lambda^\theta}{\|u_\lambda\|_{H_0^1(\Omega)}^{1-\theta}} = \lambda v_\lambda\,.$$

Moreover, using (2.11) and since $\|u_\lambda\|_{H_0^1(\Omega)} \geqslant \|u_{\lambda-1}\|_{H_0^1(\Omega)}$ we obtain:

$$\int_\Omega v_\lambda^{\theta+1}\,dx \leqslant \frac{1}{\|u_\lambda\|_{H_0^1(\Omega)}^{\theta+1}}\,\frac{\left(\int_\Omega |u_{\lambda-1}|^{\theta+1}\,dx\right)^{\frac{2}{1-\theta}}}{\left(\int_\Omega u_{\lambda-1}^2 + \int_\Omega |u_{\lambda-1}|^{\theta+1}\,dx\right)^{\frac{1+\theta}{1-\theta}}}$$

$$\leqslant \frac{1}{\|u_{\lambda-1}\|_{H_0^1(\Omega)}^{\theta+1}}\,\frac{\left(\int_\Omega |u_{\lambda-1}|^{\theta+1}\,dx\right)^{\frac{2}{1-\theta}}}{\left(\int_\Omega u_{\lambda-1}^2 + \int_\Omega |u_{\lambda-1}|^{\theta+1}\,dx\right)^{\frac{1+\theta}{1-\theta}}}$$

$$\leqslant \frac{\int_\Omega |v_{\lambda-1}|^{\theta+1}\,dx}{\left(1 + \frac{\int_\Omega u_{\lambda-1}^2\,dx}{\int_\Omega |u_{\lambda-1}|^{\theta+1}\,dx}\right)^{\frac{1+\theta}{1-\theta}}}\,,$$

which yields:

$$(2.13)\qquad \int_\Omega v_\lambda^{\theta+1}\,dx \leqslant \frac{\int_\Omega |v_{\lambda-1}|^{\theta+1}\,dx}{1 + \left(\frac{\int_\Omega u_{\lambda-1}^2\,dx}{\int_\Omega |u_{\lambda-1}|^{\theta+1}\,dx}\right)^{\frac{1+\theta}{1-\theta}}}\,.$$

We claim now that there exists a constant $\varepsilon > 0$ such that:

$$(2.14)\qquad \liminf_{\lambda\to\infty} \frac{\int_\Omega u_\lambda^2\,dx}{\int_\Omega |u_\lambda|^{\theta+1}\,dx} \geqslant \varepsilon > 0\,.$$

Indeed, assume that (2.14) is not true: then we get that, for a subsequence (not relabeled)

$$\lim_{\lambda \to \infty} \frac{\int_\Omega u_\lambda^2 \, dx}{\int_\Omega |u_\lambda|^{\theta+1} \, dx} = 0 ,$$

so that, using (2.8),

$$\int_\Omega v_\lambda^2 \, dx = \frac{1}{\|u_\lambda\|^2_{H_0^1(\Omega)}} \int_\Omega u_\lambda^2 \, dx \leqslant \frac{C}{\|u_\lambda\|^2_{H_0^1(\Omega)} \lambda^{\frac{\theta+1}{1-\theta}}} .$$

But, using (2.10) (recall that here $u_\lambda = u_{\lambda_h}$), this implies that

$$\lim_{\lambda \to \infty} \lambda \int_\Omega v_\lambda^2 \, dx = 0 ,$$

which yields, from the equation solved by v_λ (2.12):

$$\lim_{\lambda \to \infty} \int_\Omega |\nabla v_\lambda|^2 \, dx = 0 .$$

This can not hold because of (2.12), hence we deduce that (2.14) must be true. Now, (2.13) and (2.14) together imply that there exists $\bar{\lambda} > \lambda_1$ and a constant $\delta > 0$ such that:

$$\int_\Omega v_\lambda^{\theta+1} \, dx \leqslant \frac{1}{1+\delta} \int_\Omega v_{\lambda-1}^{\theta+1} \, dx \qquad \forall \lambda > \bar{\lambda} .$$

Therefore, we also have that there exists a constant $C > 0$:

$$\int_\Omega v_\lambda^{\theta+1} \, dx \leqslant C \left(\frac{1}{1+\delta} \right)^\lambda . \tag{2.15}$$

Now, (2.12) implies, on account of standard regularity results (see [12]), that, for a value $p > \frac{N}{2}$:

$$\|v_\lambda\|_{L^\infty(\Omega)} \leqslant C \|\lambda v_\lambda\|_{L^p(\Omega)} ,$$

which yields, thanks to (2.15) (take also $p > \theta + 1$):

$$1 \leqslant C\lambda \left\| \frac{v_\lambda}{\|v_\lambda\|_{L^\infty(\Omega)}} \right\|_{L^p(\Omega)} \leqslant C\lambda \left\| \frac{v_\lambda}{\|v_\lambda\|_{L^\infty(\Omega)}} \right\|_{L^{\theta+1}(\Omega)}^{\frac{\theta+1}{p}} \leqslant C\lambda \left(\frac{1}{1+\delta} \right)^{\frac{\lambda}{p}} \frac{1}{\|v_\lambda\|^{(\theta+1)/p}_{L^\infty(\Omega)}} .$$

Then we have

$$\|v_\lambda\|_{L^\infty(\Omega)} \leqslant C\lambda^{\frac{p}{\theta+1}} \left(\frac{1}{1+\delta}\right)^{\frac{\lambda}{\theta+1}},$$

so that $\|v_\lambda\|_{L^\infty(\Omega)}$ converges to zero as λ tends to infinity. Then

$$\lambda \int_\Omega v_\lambda^2 \, dx \leqslant C\lambda \int_\Omega v_\lambda^{\theta+1} \, dx \leqslant C\lambda \left(\frac{1}{1+\delta}\right)^{\lambda}.$$

Since (2.12) implies (recall (1.2))

$$\alpha \int_\Omega |\nabla v_\lambda|^2 \, dx \leqslant \int_\Omega A(x)\, \nabla v_\lambda \cdot \nabla v_\lambda \, dx \leqslant \lambda \int_\Omega v_\lambda^2 \, dx\,,$$

we conclude that v_λ strongly converges to zero in $H_0^1(\Omega)$, which contradicts the fact that $\|v_\lambda\|_{H_0^1(\Omega)} = 1$. The contradiction proves that no subsequence u_{λ_h} satisfying (2.10) can exist, and then that $\|u_\lambda\|_{H_0^1(\Omega)}$ is bounded.

It is now easy to repeat the same arguments to prove that u_λ converges to zero in $H_0^1(\Omega)$. Indeed, if this were not true it would exist a value $\bar\varepsilon > 0$ and a subsequence u_{λ_h} such that:

$$\lambda_h - \lambda_{h-1} \geqslant 1, \qquad \|u_{\lambda_h}\|_{H_0^1(\Omega)} \geqslant \bar\varepsilon \quad \forall h > 0\,. \tag{2.16}$$

In this case (2.9) still implies that

$$\int_\Omega u_{\lambda_h}^{\theta+1} \, dx \leqslant \frac{\left(\int_\Omega |u_{\lambda_{h-1}}|^{\theta+1} \, dx\right)^{\frac{2}{1-\theta}}}{\left(\int_\Omega u_{\lambda_{h-1}}^2 + \int_\Omega |u_{\lambda_{h-1}}|^{\theta+1} \, dx\right)^{\frac{1+\theta}{1-\theta}}}\,,$$

and then

$$\int_\Omega u_{\lambda_h}^{\theta+1} \, dx \leqslant \frac{\int_\Omega |u_{\lambda_{h-1}}|^{\theta+1} \, dx}{1 + \left(\dfrac{\int_\Omega u_{\lambda_{h-1}}^2 \, dx}{\int_\Omega |u_{\lambda_{h-1}}|^{\theta+1} \, dx}\right)^{\frac{1+\theta}{1-\theta}}}\,. \tag{2.17}$$

This is the same inequality as (2.13) previously obtained on v_λ. Thus, the same arguments can now be applied; indeed, we have the alternative that either there exists a constant ε such that

$$\liminf_{h\to\infty} \frac{\int_\Omega u_{\lambda_h}^2\,dx}{\int_\Omega |u_{\lambda_h}|^{\theta+1}\,dx} \geqslant \varepsilon > 0 \qquad \forall h>0\,,$$

or there exists a further subsequence (not relabeled) such that

$$\lim_{h\to\infty} \frac{\int_\Omega u_{\lambda_h}^2}{\int_\Omega |u_{\lambda_h}|^{\theta+1}\,dx} = 0\,.$$

In both cases, the same arguments used before for v_λ allow to conclude that a subsequence of u_{λ_h} strongly converges to zero in $H_0^1(\Omega)$, which contradicts (2.16). This concludes the proof that u_λ strongly converges to zero in $H_0^1(\Omega)$ ■

REMARK 2.3. Note that, at least if $\frac{N-2}{N+2} < \theta < 1$, we can also argue from the proof of Theorem 1.1 that

$$\lim_{\lambda\to\infty} \|u_\lambda\|_{L^\infty(\Omega)} = 0\,.$$

In fact, recall that we have the estimate:

$$\int_\Omega u_\lambda^{\theta+1}\,dx \leqslant \frac{C}{\lambda^{\frac{\theta+1}{1-\theta}}} \qquad \forall\lambda>\lambda_1.$$

If $\frac{N-2}{N+2} < \theta$, then $\frac{\theta+1}{1-\theta} > \frac{N}{2}$, hence we get from the equation solved by u_λ,

$$\|u_\lambda\|_{L^\infty(\Omega)} \leqslant C\lambda \|u_\lambda\|_{L^{\frac{\theta+1}{1-\theta}}(\Omega)}.$$

Hence, using that $\frac{\theta+1}{1-\theta} > \theta+1$,

$$1 \leqslant C\lambda \left\| \frac{u_\lambda}{\|u_\lambda\|_{L^\infty(\Omega)}} \right\|_{L^{\frac{\theta+1}{1-\theta}}(\Omega)} \leqslant C\lambda \left\| \frac{u_\lambda}{\|u_\lambda\|_{L^\infty(\Omega)}} \right\|_{L^{\theta+1}(\Omega)}^{1-\theta} \leqslant C \frac{1}{\|u_\lambda\|_{L^\infty(\Omega)}^{1-\theta}}\,.$$

We first deduce from this inequality that $\|u_\lambda\|_{L^\infty(\Omega)}$ is bounded, then we obtain, for a value p such that $\max\left\{\theta+1,\, \frac{N}{2}\right\} < p < \frac{\theta+1}{1-\theta}$:

$$\lim_{\lambda\to\infty} \lambda^p \int_\Omega u_\lambda^p\,dx \leqslant C\lambda^p \int_\Omega u_\lambda^{\theta+1}\,dx \leqslant C \frac{1}{\lambda^{\frac{\theta+1}{1-\theta}-p}}\,,$$

so that λu_λ strongly converges to zero in $L^p(\Omega)$ with $p > \frac{N}{2}$. Classical regularity results (see [12]) then imply that u_λ converges to zero in $L^\infty(\Omega)$. In particular, this proves the convergence of u_λ in $L^\infty(\Omega)$ for any value of θ at least in dimension $N \leqslant 2$.

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