REND. SEM. MAT. UNIV. PADOVA, Vol. 107 (2002)

Compact Embedding of a Degenerate Sobolev Space and Existence of Entire Solutions to a Semilinear Equation for a Grushin-type Operator.

FRANCESCA LASCIALFARI - DAVID PARDO (*)

ABSTRACT - We establish a compactness embedding result for suitable Sobolev subspaces naturally arising in the study of a Grushin-type operator in $\mathbb{R}^N$. As an application, we study the solvability of a semilinear problem involving the above operator and a subcritical nonlinear term.

1. Introduction.

It is well known that, under a variational point of view, every positive solution of the problem

$$\text{(1)} \qquad \begin{cases} -\Delta u = f(x, u) \\ u_{|\partial\Omega} = 0 \end{cases} \qquad f \text{ non linear}$$

on a bounded domain $\Omega \subseteq \mathbb{R}^N$, is a critical point of an Euler-type functional associated to (1).

Under a set of assumptions on the function f, such a critical point really exists so that equation (1) is solvable. We point out that a similar result relies on compact embeddings for Sobolev spaces, namely the Rellich-Kondrachov Theorem. This approach does not work in general if Ω is unbounded, say $\Omega = \mathbb{R}^N$, because of the lack of compactness of the above embeddings. Nevertheless, the geometry of the Laplace operator

(*) Indirizzo degli AA.: Dipartimento di Matematica, Università degli Studi di Bologna - P.zza Porta S. Donato 5, 40127 Bologna, Italy. E-mail: lascia@dm.unibo.it, pardo@dm.unibo.it

and the invariance of $\mathbb{R}^N$ under rotations suggest to study the problem in a restricted functional setting.

For instance, Béréstycki and Lions in [1] studied problem (1) on $\mathbb{R}^N$, with a function f which depends on u and $|x|$. The symmetry of f with respect to $|x|$ plays a crucial role since it allows to recover compactness even if the domain is unbounded. The use of subspaces of spherical functions was first introduced by Strauss in [9] where equation (1) is studied in the whole space.

The aim of the present paper is to establish a similar result in a degenerate elliptic case.

Precisely, for $\lambda > 0$ we study the following problem:

$$\begin{cases} -\Delta_G u + \lambda u = u^{q-1} & \text{on } \mathbb{R}^N \\ u > 0 & \text{on } \mathbb{R}^N \end{cases} \tag{2}$$

where Δ_G stands for the operator:

$$\Delta_G = \Delta_x + |x|^{2\alpha} \Delta_y, \quad \alpha > 0, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m, \quad n + m = N, \quad n \geqslant 2,$$

the power q is superlinear and subcritical for Δ_G, that is $2 < q < 2^*_G = \frac{2Q}{Q-2}$, and $Q = n + (\alpha + 1)\, m$. In the sequel we shall refer to Δ_G as the *Grushin operator.*

The Dirichlet problem for equation (2) has been investigated by Tri in [11] in the case of starshaped bounded subsets of $\mathbb{R}^2$ and a zero boundary data. Among the authors who faced similar matters for degenerate operators we mention S. Biagini. In her work [2] she dealt with a semilinear problem involving the Heisenberg operator and used a technique that can be fitted in our context.

Now let us make some remarks and introduce useful notations.

The Grushin operator Δ_G can be written as the divergence of a *modified* gradient, namely

$$\Delta_G = \operatorname{div}(\nabla_G), \quad \nabla_G = (\nabla_x, \, |x|^\alpha \nabla_y), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Moreover, if we denote with A the $N \times N$ matrix

$$A(x) = \begin{pmatrix} I_n & 0 \\ 0 & |x|^\alpha I_m \end{pmatrix}$$

an easy computation shows that

$$\nabla_G = A(x)\begin{pmatrix}\nabla_x \\ \nabla_y\end{pmatrix} = A(x)\,\nabla\,. \tag{3}$$

If $\Omega \subseteq \mathbb{R}^N$, we define the Sobolev space

$$\mathcal{S}^{1,2}(\Omega) = \{u \in L^2(\Omega) : \nabla_G u \in L^2(\Omega)\}\,.$$

We remark that $\mathcal{S}^{1,2}(\Omega)$ endowed with the inner product

$$\langle u, v\rangle_{\mathcal{S}^{1,2}(\Omega)} = \int_\Omega (uv + \nabla_G u \cdot \nabla_G v)\, dx\, dy$$

is a Hilbert space. Besides, if Ω is bounded, the embedding

$$\mathcal{S}^{1,2}(\Omega) \hookrightarrow L^q(\Omega) \tag{4}$$

is compact if $2 < q < \dfrac{2Q}{Q-2}$, whereas the embedding

$$\mathcal{S}^{1,2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \tag{5}$$

for every $q \in \left[2, \dfrac{2Q}{Q-2}\right]$ is only continuous (see [3], [10]).

We introduce the following closed subspaces of $\mathcal{S}^{1,2}(\mathbb{R}^N)$:

$$\mathcal{S} = \{u \in \mathcal{S}^{1,2}(\mathbb{R}^N) : u(x, y) = \varphi(|x|, y)\}\,,$$

$$\mathcal{S}_r = \{u \in \mathcal{S}^{1,2}(\mathbb{R}^N) : u(x, y) = \varphi(|x|, |y|)\}\,,$$

and the cone

$$\mathcal{V} := \{u \in \mathcal{S}_r(\mathbb{R}^N) : \varphi \text{ is non-increasing in } |y|\}$$

for which the following trivial inclusions holds:

$$\mathcal{V} \subset \mathcal{S}_r \subset \mathcal{S} \subset \mathcal{S}^{1,2}(\mathbb{R}^N)\,.$$

We stress that the requested cylindrical symmetry in the definition of $\mathcal{S}_r$ is suggested by the structure of the Grushin operator.

DEFINITION 1.1. *A function $u \in \mathcal{S}^{1,2}(\mathbb{R}^N)$ will be called a weak solution* of (2) *if it satisfies the following identity:*

$$\int_{\mathbb{R}^N} \nabla_G u \cdot \nabla_G \phi + \lambda \int_{R^N} u\phi = \int_{\mathbb{R}^N} u^{q-1}\phi$$

for every choice of $\phi \in C_0^\infty(\mathbb{R}^N)$.

We are ready to state the main results of this paper:

THEOREM 1.1 (COMPACT EMBEDDING). *If* $2 < q < 2^*_G$, *then the restriction to the cone* $\mathcal{V}$ *of embedding* $S^{1,2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ *is compact.*

THEOREM 1.2 (EXISTENCE AND REGULARITY). *Let* $q \in]2, 2^*_G[$. *Then there exists a weak solution* $u \in C^{\theta}_{\text{loc}}(\mathbb{R}^N) \cap C^{2,\theta}_{\text{loc}}(\mathbb{R}^n \setminus \{0\} \times \mathbb{R}^m)$ *of problem* (2).

Moreover, u *is radially symmetric with respect to each group of variables, that is* $u(x, y) = u(|x|, |y|)$.

The plan of the paper is the following.

In section 2 we prove Theorem 1.1 mainly exploiting a decay Lemma for functions belonging to $\mathcal{V}$.

In section 3 we deal with problem (2). The existence of a positive solution will be achieved by applying Theorem 1.1 and rearrangements techniques. We prove also the regularity properties for the solution.

2. Compact embedding.

The purpose of this section is the proof of Theorem 1.1. We expect that a good decay of functions at infinity may help to recover compactness of embeddings (5), although the domain is unbounded. The structure of $\mathcal{V}$ assures the following result:

LEMMA 2.1. *Let* $n \geq 2$, $m \geq 1$, *and suppose* $u \in \mathcal{V}$. *The following pointwise estimate holds:*

$$u(x, y) \leq \frac{k_N}{|x|^{\frac{n-1}{2}} |y|^{\frac{m}{2}}} \|u\|^{1/2}_{L^2(\mathbb{R}^N)} \|\nabla_x u\|^{1/2}_{L^2(\mathbb{R}^N)}$$

where k_N *is a dimensional constant.*

PROOF. If $u \in \mathcal{V}$, there exists a two-variable function φ such that $u(x, y) = \varphi(|x|, |y|)$. Since u is non-increasing with respect to $|y|$, for a fixed $\sigma > 0$ we have

$$\varphi(\sigma, \tau) \geq \varphi(\sigma, t) \qquad \forall \tau \in [0, t].$$

We now multiply both terms by τ^{m-1} and integrate over $[0, t]$. Then:

$$\int_0^t \varphi(\sigma, \tau)\, \tau^{m-1}\, d\tau \geqslant \int_0^t \varphi(\sigma, t)\, \tau^{m-1}\, d\tau = \frac{t^m}{m}\, \varphi(\sigma, t)\,. \tag{6}$$

Furthermore, an easy computation shows that

$$\frac{d}{d\sigma}\left[\sigma^{n-1}\left(\int_0^t \varphi(\sigma, \tau)\, \tau^{m-1}\, d\tau\right)^2\right] \geqslant$$

$$\geqslant 2\sigma^{n-1}\int_0^t \varphi(\sigma, \tau)\, \tau^{m-1}\, d\tau \cdot \int_0^t \frac{\partial\varphi}{\partial\sigma}(\sigma, \tau)\, \tau^{m-1}\, d\tau\,.$$

Let $s > 0$; then, by integrating with respect to $\sigma \in [s, +\infty]$ we obtain:

$$s^{n-1}\left(\int_0^t \varphi(s, \tau)\, \tau^{m-1}\, d\tau\right)^2 \leqslant$$

$$\leqslant 2\int_s^{+\infty}\left|\int_0^t \varphi(\sigma, \tau)\, \tau^{m-1}\, d\tau\right| \cdot \left|\int_0^t \frac{\partial\varphi}{\partial\sigma}(\sigma, \tau)\, \tau^{m-1}\, d\tau\right| \sigma^{n-1}\, d\sigma$$

$$\leqslant 2\int_0^{+\infty}\left|\int_0^t \varphi(\sigma, \tau)\, \tau^{m-1}\sigma^{\frac{n-1}{2}}\, d\tau\right| \cdot \left|\int_0^t \frac{\partial\varphi}{\partial\sigma}(\sigma, \tau)\, \tau^{m-1}\sigma^{\frac{n-1}{2}}\, d\tau\right| d\sigma\,.$$

Apply Hölder inequality in the $d\sigma$-integral:

$$s^{n-1}\left(\int_0^t \varphi(s, \tau)\, \tau^{m-1}\, d\tau\right)^2 \leqslant 2\left[\int_0^{+\infty}\left(\int_0^t \varphi(\sigma, \tau)\, \tau^{m-1}\, d\tau\right)^2 \sigma^{n-1}\, d\sigma\right]^{1/2} \times$$

$$\times\left[\int_0^{+\infty}\left(\int_0^t \frac{\partial\varphi}{\partial\sigma}(\sigma, \tau)\, \tau^{m-1}\, d\tau\right)^2 \sigma^{n-1}\, d\sigma\right]^{1/2},$$

and the same in the $d\tau$-integral:

$$s^{n-1}\left(\int_0^t \varphi(s,\tau)\,\tau^{m-1}\,d\tau\right)^2 \leqslant$$

$$\leqslant 2\left[\int_0^{+\infty}\left(\int_0^t \varphi^2(\sigma,\tau)\,\tau^{m-1}\,d\tau\right)\left(\int_0^t \tau^{m-1}\,d\tau\right)\sigma^{n-1}\,d\sigma\right]^{1/2}\times$$

$$\times\left[\int_0^{+\infty}\left(\int_0^t\left(\frac{\partial\varphi}{\partial\sigma}(\sigma,\tau)\right)^2\tau^{m-1}\,d\tau\right)\left(\int_0^t \tau^{m-1}\,d\tau\right)\sigma^{n-1}\,d\tau\right]^{1/2}.$$

Then we get:

$$s^{n-1}\left(\int_0^t \varphi(s,\tau)\,\tau^{m-1}\,d\tau\right)^2 \leqslant c_m t^m \|\varphi\|_{L^2(\mathbb{R}^N)}\cdot\|\nabla_x\varphi\|_{L^2(\mathbb{R}^N)}.$$

By extracting the square roots of both members, and taking in account of (6), we find:

$$\varphi(s,t) \leqslant \frac{k_m}{|s|^{\frac{n-1}{2}}\,|t|^{\frac{m}{2}}}\|\varphi\|^{1/2}_{L^2(\mathbb{R}^N)}\cdot\|\nabla_x\varphi\|^{1/2}_{L^2(\mathbb{R}^N)}. \qquad \blacksquare$$

REMARK 2.1. *Suppose that the function φ is non-increasing with respect to its first variable, that is u non-increasing in $|x|$ instead of $|y|$. If $n\geqslant 1$, $m\geqslant 2$, the same argument of the proof of the previous Lemma leads to the following estimate:*

$$\varphi(s,t) \leqslant \frac{k_n}{|s|^{\frac{n+\alpha}{2}}\,|t|^{\frac{m-1}{2}}}\|\varphi\|^{1/2}_{L^2(\mathbb{R}^N)}\cdot\||x|^\alpha\nabla_y\varphi\|^{1/2}_{L^2(\mathbb{R}^N)}.$$

We point out that in this case we need the additional hypotheses $\alpha < \frac{n}{2}$.

In order to prove compactness of embedding $\mathcal{V}\hookrightarrow L^q(\mathbb{R}^N)$ with $q\in\,]2, 2^*_G[$ we must show that every bounded sequence in $\mathcal{V}$ is precompact.

We divide $\mathbb{R}^N$ into suitable subsets Q_i, $i=1, \ldots, 4$, defined as follows:

$$Q_1 := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : |x| \leqslant R; \ |y| \leqslant R\}$$

$$Q_2 := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : |x| \leqslant R; \ |y| \geqslant R\}$$

$$Q_3 := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : |x| \geqslant R; \ |y| \leqslant R\}$$

$$Q_4 := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : |x| \geqslant R; \ |y| \geqslant R\}$$

where R is a constant to be chosen later.

The convergence of a bounded sequence in Q_1 (enventually up to a subsequence) readily comes from embedding (4); in Q_4 it is a consequence of Lemma 2.1. The main difficulties are represented by the particular unboundedness of Q_2 and Q_3, where one variable is arbitrarily large.

For overcoming this obstacle, we shall take advantage of a technique, used by P. L. Lions in [7], that will be explained during the proof.

PROOF OF THEOREM 1.1. Let $(u_k)_{k \geqslant 1}$ be a bounded sequence in $\mathcal{V}$. Then there exists $u_0 \in L^q$ and a subsequence, that we still denote by (u_k), such that:

$$\begin{array}{ll} u_k \to u_0 & \text{in } L^q_{\mathrm{loc}} \cap L^2 \\ u_k(x, y) \to u_0(x, y) & \text{a.e. in } \mathbb{R}^N. \end{array}$$

Remark that

$$\int_{\mathbb{R}^N} |u_k - u_0|^q = \sum_{j=1}^{4} I_j \qquad \text{where} \qquad I_j := \int_{Q_j} |u_k - u_0|^q.$$

Due to embedding (4), we get $I_1 \to 0$ as $k \to +\infty$.

About I_4 we write:

$$I_4 = \int_{Q_4} |u_k - u_0|^q \leqslant C_q \left(\int_{Q_4} |u_k|^{q-2} |u_k|^2 + \int_{Q_4} |u_0|^{q-2} |u_0|^2 \right).$$

By Lemma 2.1 one gets:

$$I_4 \leqslant C_q \frac{C}{R^{(q-2)(\frac{n-1}{2} + \frac{m}{2})}} \left(\int_{Q_4} |u_k|^2 + \int_{Q_4} |u_0|^2 \right)$$

where C is a constant depending only on the norms of the functions u_k. Then $I_4 \to 0$ as $R \to \infty$, uniformly in $k \in \mathbb{N}$.

It remains to prove the convergence of I_2 and I_3 to 0 as $k \to \infty$. For this purpose we define

$$\phi_k(x) := \int_{|y| \geqslant R} |u_k(x, y)|^q \, dy \, .$$

The sequence $(\phi_k)_{k \geqslant 1}$ is bounded in $L^1(B)$, where $B := \{|x| \leqslant R\} \subset \mathbb{R}^n$; indeed the sequence $(u_k)_{k \geqslant 1}$ is bounded in L^q. In addition, Lemma 2.1 implies that

$$\int_{|y| \geqslant R} |u_k(x, y)|^q \, dy \to \int_{|y| \geqslant R} |u_0(x, y)|^q \, dy =: \phi(x)$$

as $k \to \infty$, by compact embedding in L^q_{loc}. Then $\phi_k(x) \to \phi(x)$ almost everywhere. As B is a compact set, the sequence ϕ_k converges to ϕ in $L^1(B)$. Moreover:

$$\|\nabla_x \phi_k\|_{L^1(B)} = \int_{|x| < R} |\nabla_x \phi_k(x)| \, dx = \int_{Q_2} |\nabla_x (u_k(x, y)|)^q \, dx dy \leqslant$$

$$\leqslant q \int_{Q_2} |u_k(x, y)|^{q-1} |\nabla_x u_k(x, y)| \, dx \, dy \, .$$

Apply Hölder inequality and get:

$$\|\nabla_x \phi_k\|_{L^1(B)} \leqslant C \|\nabla_x u_k\|_{L^2(Q_2)} \|u_k\|^{q-1}_{L^{2(q-1)}(Q_2)} .$$

If $2(q-1) \leqslant \dfrac{2Q}{Q-2}$, then

$$\|u_k\|^{q-1}_{L^{2(q-1)}(Q_2)} \leqslant C \|\nabla_G u_k\|^{q-1}_{L^2(Q_2)},$$

that implies $\|\nabla_x \phi_k\|_{L^1(B)} \leqslant C |\nabla_G u_k\|^{q-1}_{L^2(Q_2)}$. We can conclude that the sequence $(\phi_k)_{k \geqslant 1}$ is bounded in $W^{1,1}(B)$, so that $\|u_k\|_{L^q(Q_2)} \to \|u_0\|_{L^q(Q_2)}$.

Finally, we prove that $I_3 \to 0$. Define, as above:

$$\psi_k(y) := \int_{|x| \geqslant R} |u_k(x, y)|^q \, dx \, .$$

By a similar proceeding, it is enough to observe that, for almost every y,

$$\nabla_y \psi_k(y) = q \int_{|x|>R} \frac{u_k^{q-1}(x, y)}{|x|^\alpha} |x|^\alpha \nabla_y u_k(x, y)\, dx \leqslant$$

$$\leqslant q \left(\int_{|x|>R} \frac{u_k^{2(q-1)}(x, y)}{|x|^{2\alpha}} dx \right)^{1/2} \left(\int_{|x|>R} |x|^{2\alpha} |\nabla_y u_k(x, y)|^2 dx \right)^{1/2} \leqslant$$

$$\leqslant \frac{q}{R^\alpha} \left(\int_{|x|>R} u_k^{2(q-1)}(x, y)\, dx \right)^{1/2} \left(\int_{|x|>R} |\nabla_G u_k(x, y)|^2 \right)^{1/2}.$$

Then the sequence (ψ_k) is bounded in $W^{1,1}(|y| \leqslant R)$ and $\|u_k\|_{L^q(Q_3)} \to \|u_0\|_{L^q(Q_3)}$. Thus Theorem 1.1 is proved.

3. A semilinear problem for the Grushin operator.

This section is devoted to the proof of Theorem 1.2.

As announced in the Introduction, our approach to problem (2) is variational. Therefore we define, for every $u \in \mathcal{S}^{1,2}(\mathbb{R}^N)$, the functional:

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla_G u|^2 + \lambda |u|^2)\, dz\,. \tag{7}$$

Our goal is to find a critical point of J as the minimum on a suitable manifold. We first prove the following

LEMMA 3.1. *Let $v \in \mathcal{S}$ and let v^* denote the spherical decreasing rearrangement of v with respect to the variable y. It holds*

$$J(v^*) \leqslant J(v)\,. \tag{8}$$

PROOF. We first recall some useful properties of the rearrangements we are dealing with. If $v \in L^p$, $u \in L^{p'}$, and $\frac{1}{p} + \frac{1}{p'} = 1$, then:

$$(i) \quad \|v\|_{L^p} = \|v^*\|_{L^p}$$

$$(ii) \quad \int_{\mathbb{R}^m} u(x, y)\, v(x, y)\, dy \leqslant \int_{\mathbb{R}^m} u^*(x, y)\, v^*(x, y)\, dy \ \text{ a.e. } \ x \in \mathbb{R}^n$$

$$(iii) \quad \int_{\mathbb{R}^m} |\nabla_y v^*|^p\, dy \leqslant \int_{\mathbb{R}^m} |\nabla_y v|^p\, dy \ \text{ for all } \ p > 1\,.$$

Properties (i) and (ii) are a trivial consequence of the fact that, for a fixed x, $u^*(x, \cdot)$, $v^*(x, \cdot)$ are nothing but the Schwartz rearrangements of $u(x, \cdot)$, $v(x, \cdot)$ (see for instance [6]). The third property follows directly from the Polya-Szegö inequality. We also remark that property (iii) implies

$$\int_{\mathbb{R}^N} |x|^{2\alpha}\, |\nabla_y v^*|^2\, dx\, dy \leqslant \int_{\mathbb{R}^N} |x|^{2\alpha}\, |\nabla_y v|^2\, dx\, dy\,.$$

In order to prove (8) it remains to show the following inequality:

$$\|\nabla_x v^*\|_{L^2} \leqslant \|\nabla_x v\|_{L^2}.$$

Let $j = 1, \ldots, n$ and let $h \neq 0$. From the above properties (i) and (ii) we get:

$$\int_{\mathbb{R}^m} \frac{|v^*(x + he_j, y) - v^*(x, y)|^2}{h^2}\, dy =$$

$$= \int_{\mathbb{R}^m} \frac{1}{h^2} \big(|v^*(x + he_j, y)|^2 + |v^*(x, y)|^2 - 2 v^*(x, y)\, v^*(x + he_j, y) \big)\, dy$$

$$\leqslant \int_{\mathbb{R}^m} \frac{1}{h^2} \big(|v(x + he_j, y)|^2 + |v(x, y)|^2 - 2 v(x, y)\, v(x + he_j, y) \big)\, dy$$

$$= \int_{\mathbb{R}^m} \frac{|v(x + he_j, y) - v(x, y)|^2}{h^2}\, dy$$

$$= \int_{\mathbb{R}^m} \left| \int_0^1 \nabla_{x_j} v(x + the_j, y)\, dt \right|^2 dy$$

$$\leqslant \int_{\mathbb{R}^m} \int_0^1 |\nabla_{x_j} v(x + the_j, y)|\, dt^2\, dy\,.$$

Then, by integrating with respect to x:

$$\int_{\mathbb{R}^N} \frac{|v^*(x + he_j, y) - v^*(x, y)|^2}{h^2}\, dx\, dy \leqslant \|\partial_{x_j} v\|^2_{L^2(\mathbb{R}^N)}.$$

This proves that the family

$$w_h := \left(\frac{v^*(\cdot + he_j, \cdot) - v^*(\cdot, \cdot)}{h^2} \right)_h$$

is bounded in L^2 by the L^2-norm of $\partial_{x_j} v$. Then, for a sequence $h_j \searrow 0$, w_{h_j} weakly converges to a certain w in L^2 such that $\|w\|_{L^2} \leqslant \|\partial_{x_j} v\|_{L^2}$. It follows that w is the weak derivative of v^* with respect to x_j and $\|\partial_{x_j} v^*\|_{L^2} \leqslant \|\partial_{x_j} v\|_{L^2}$. Since this inequality holds for any $j = 1, \ldots, n$, we get

$$\|\nabla_x v^*\|_{L^2} \leqslant \|\nabla_x v\|_{L^2}. \qquad \blacksquare$$

Proof of Theorem 1.2.. Let $\mathcal{M} = \{u \in \mathcal{S} : \|u\|_{L^q} = 1\}$ and let J be the functional defined in (7). Since $J > 0$ on $\mathcal{M}$, there exists $\inf_{\mathcal{M}} J(u) = J_0 \geqslant 0$. Our goal is to prove that J_0 actually is a minimum for J, so that $J_0 > 0$. Let $(u_k) \subset \mathcal{M}$ be a minimizing sequence. By definition, $J(u) = J(|u|)$, then we can assume $u_k \geqslant 0$ for every k.

Moreover:

1. $\|u_k\|_{\mathcal{S}^{1,2}(\mathbb{R}^N)} = J(u_k) = J_0 + o(1)$, therefore $(u_k)_k$ is bounded and there exists a function $u_0 \in \mathcal{S}$ such that $u_k \rightharpoonup u_0$ up to a subsequence.

2. Lemma 3.1 allows us to assume u_k radially symmetric and decreasing with respect to $|y|$ without loss of generality.

3. The sequence $(u_k)_k$ is precompact in L^q by Theorem 1.1. Hence $u_k \to u_0$ in L^q, and $\|u_0\|_{L^q} = 1$.

On the other hand, by the semicontinuity of J with respect to the weak convergence, we have

$$J(u_0) \leqslant \liminf_{k \to \infty} J(u_k) = J_0 .$$

Then $J(u_0) = J_0$. In addition, by Lagrange-Lusternik multiplier Theorem, u_0 verifies the following integral identity:

$$\int_0^{+\infty} \int_{\mathbb{R}^m} \left(\frac{\partial u_0}{\partial r}(r, y) \frac{\partial \varphi}{\partial r}(r, y) - \frac{n-1}{r} \frac{\partial u_0}{\partial r}(r, y)\, \varphi(r, y) \right) r^{n-1} dy\, dr +$$

$$- \int_0^{+\infty} \int_{\mathbb{R}^m} \left(r^{2\alpha} \langle \nabla_y u_0(r, y), \nabla_y \varphi(r, y) \rangle + \lambda u_0(r, y)\, \varphi(r, y) \right) r^{n-1} dy\, dr =$$

$$= \mu \int_0^{+\infty} \int_{\mathbb{R}^m} [u_0(r, y)]^{q-1} \varphi(r, y)\, r^{n-1} dy\, dr$$

for every $\varphi = \varphi(r, y) \in C_0^\infty(]0, +\infty[\times \mathbb{R}^m)$. Here μ is a Lagrange multiplier. Since $J(u_0) > 0$ it must be $\mu > 0$. In a standard way, we can rescale u_0 in order to get $\mu = 1$. Then u_0 is a weak solution in $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^m$ to the elliptic equation

$$- \frac{\partial^2 \varphi(r, y)}{\partial r^2} - \frac{n-1}{r} \frac{\partial}{\partial r} \varphi(r, y) - r^{2\alpha} \Delta_y \varphi(r, y) + \lambda \varphi = \varphi^{q-1} .$$

A classical bootstrap argument shows that is u_0 is a pointwise solution of

$$-\Delta_G u + \lambda u = u^{q-1} \qquad \text{in } (\mathbb{R}^m \setminus \{0\}) \times \mathbb{R}^n .$$

Now we are going to prove that u_0 is a weak solution on the whole space, i.e. u_0 is a weak solution of problem (2).

It suffices to establish that for all $\phi \in C_0^\infty(\mathbb{R}^N)$ one can find a sequence $\varepsilon_j \searrow 0$ verifying

$$\int_{\mathbb{R}^N \setminus C_{\varepsilon_j}} \nabla_G u_0 \cdot \nabla_G \phi - \int_{\mathbb{R}^N \setminus C_{\varepsilon_j}} (u_0^{q-1} \phi - \lambda u_0 \phi) \to 0 \qquad \text{as } j \to \infty \tag{9}$$

where

$$C_{\varepsilon_j} := \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m : |x| \leqslant \varepsilon_j ; \ |y| \leqslant R \}$$

and R must be chosen in a way that supp $\phi \subset \mathbb{R}^n \times B_R^m$. Here we denote by B_R^m the set $\{y \in \mathbb{R}^m \colon |y| \leq R\}$.

Now fix $\varepsilon > 0$. Since u_0 is a classical solution in $\mathbb{R}^N \backslash C_\varepsilon$ and $\phi \equiv 0$ when $|y| = R$, by divergence Theorem we obtain

$$\int_{\mathbb{R}^N \backslash C_\varepsilon} (\nabla_G u_0 \cdot \nabla_G \phi - u_0^{q-1} \phi - \lambda u_0 \phi) = \int_{\Gamma_\varepsilon} \phi A \nabla u_0 \cdot n \, d\sigma(x) \tag{10}$$

where we denoted with Γ_ε the set

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \colon |x| = \varepsilon ;\ |y| \leq R\}$$

and with $\underline{n} = \left(\frac{x}{\varepsilon}, 0 \right)$.

Since u_0 is symmetric in $|x|$ and $|y|$, we have

$$u_0(x, y) = \nu(|x|, |y|)$$

for a suitable 2-variable function ν. Hence $\nabla u_0 = \left(\nu_x \cdot \frac{x}{|x|}, \nu_y \cdot \frac{y}{|y|} \right)$ and it turns out that $A \nabla u_0 \cdot n = \nu_x(|x|, |y|)$ on Γ_ε. So

$$\int_{\Gamma_\varepsilon} \phi A \nabla u_0 \cdot n \, d\sigma(x) = \int_{\Gamma_\varepsilon} \phi \nu_x(|x|, |y|) \, d\sigma(x) \, dy .$$

Since ϕ is bounded

$$\left| \int_{\Gamma_\varepsilon} \phi A \nabla \nu \cdot n \, d\sigma(x) \, dy \right| \leq \sup |\varphi| \int_{\Gamma_\varepsilon} |\nu_x| d\sigma(x) \, dy . \tag{11}$$

The Hölder inequality yields:

$$\int_{\Gamma_\varepsilon} |\nu_x(|x|, |y|)| d\sigma(x) \, dy \leq k \left(\int_{\Gamma_\varepsilon} |\nu_x|^2 d\sigma(x) \, dy \right)^{1/2} \varepsilon^{\frac{n-1}{2}} . \tag{12}$$

On the other hand, since ν_x belongs to L^2

$$\int_0^1 \left(\int_{\Gamma_\varepsilon} |\nu_x|^2 d\sigma(x) \, dy \right) d\varepsilon = \int_{|y| < R} \left(\int_{|x| < 1} |\nu_x|^2 d\sigma(x) \right) dy$$

$$\leq \int_{\mathbb{R}^{2n+1}} |\nu_x|^2 dx \, dy < +\infty .$$

Then there exists a sequence $\varepsilon_j \searrow 0$ such that

$$\varepsilon_j \int_{\Gamma_{\varepsilon_j}} |\nu_x|^2 \, d\sigma(x)\, dy \to 0 \qquad \text{as } j \to \infty .$$

Using this result in (12), (11) and (10), we get (9) completing the proof. Finally, the local Hölder regularity of u_0 can be proved applying the Moser iteration technique as presented in [5] and extended to the Grushin-type operators in [3] and [4].

Acknowledgements. The authors thank E. Lanconelli for suggesting the problem and for the hints raised up during the lively meetings.

REFERENCES

[1] H. Berestycky - P. L. Lions, *Nonlinear Scalar Field Equations I,* Arch. Rational Mech. Anal., **82** (1983), pp. 313-345.
[2] S. Biagini, *Positive Solutions for a Semilinear Equation on the Heisenberg Group*, Boll. UMI-B, **9** (7) (1995), pp. 901-918.
[3] B. Franchi - E. Lanconelli, *An Embedding Theorem for Sobolev Spaces related to Non-Smooth Vector Fields and Harnack Inequality*, Comm. in Part. Diff. Eq., **9** (13) (1984), pp. 1237-1264.
[4] B. Franchi - R. Serapioni, *Pointwise Estimates for a Class of Strongly Degenerate Elliptic Operators: a Geometrical Approach*, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV Ser. 14, N. **4** (1987), pp. 527-568.
[5] D. Gilbarg - N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Second Edition, Springer Verlag, New York, 1983.
[6] B. Kawohl, *Rearrangements and Convexity of Level Sets in PDE*, Lecture Notes in Mathematics **1150**, Springer-Verlag, Berlin (1985).
[7] P. L. Lions, *Symétrie et Compacité dans les Espaces de Sobolev,* J. Funct. Anal., **49** (1982), pp. 315-334.
[8] L. P. Rotschild - E. M. Stein, *Hypoelliptic differential operators on nilpotent groups,* Acta Math., **137** (1977), pp. 247-320.
[9] W. A. Strauss, *Existence of Solitary Waves in Higher Dimensions*, Comm. Math. Phys., **55** (1977), pp. 149-152.
[10] N. M. Tri, *Critical Sobolev Exponent for Degenerate Elliptic Operators*, Acta Math. Viet., **23** (1) (1998), pp. 83-94.
[11] N. M. Tri, *On Grushin's Equation,* Math. Notes, **63** (1) (1998), pp. 84-93 (Transl. from Mmat. Zametki, **63** (1) (1998), pp. 95-105).